Enumerative geometry of Brill-Noether loci in Jacobians

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Abstract

This talk consists of an glimpse of applications of enumerative geometry of degeneracy loci into the study of Brill-Noether loci in Jacobians which provide a lot of information about special linear systems on algebraic curves. After a brief review of intersection theory, I will state several results about the enumerative geometry of degeneracy loci, e.g., the Chern class formulas, Euler characteristic formulas. Then, after introducing the basic notions about Brill-Noether loci in Jacobians of algebraic curves, we will see what the results about degeneracy loci tells us in this case and what are the geometric meanings. Nothing in this talk will be original.

1 Enumerative geometry of degeneracy loci

1.1 A basic example

Let E, F be two vector spaces over $\mathbb{C}$ of dim n, m resp. with $n \leq m$. Let $\gamma : \mathbb{C} \to End_{\mathbb{C}}(E, F)$ be a morphism, i.e., a family of matrices parameterized by $t \in \mathbb{C}$.

We want to know in how many points $\gamma(t)$ has rank $< n$(fail to be non-degenerate), in how many points $\gamma(t)$ has rank $< n-1$, etc.

We can do it in the following way:

$$Z_0 \subset \cdots Z_{n-1} \subset Z_n \subset Z = End_{\mathbb{C}}(E, F)$$

where $Z_i = V(I_i)$ with I_i the ideal generated by minors of rank i.

Then γ gives a line in Z. It may intersects Z_i in some points. Since the loci Z_i's are exactly where the rank of matrices have dim $< i$, the $t \in \mathbb{C}$ where the curve intersects Z_i are the points we are interested in.

This basic example gives us the ideas about what we want to study in general and how we study it.

This suggests us, we have to know in how many points two subvarieties intersects.

1.2 How do we do intersection of subvarieties

In topology, we do it like this. Let M be a smooth manifold of dim n, V, $W \subset M$ are two closed submanifolds of codim r, s respectively. There submanifolds represent their homology classes $[V]$ and $[W]$ in $H_*(M)$. We can take the Poincare dual of $[V]$, which we denote it by $\alpha_V \in H^r_c(M)$. The class α_V gives a map $H_i(M) \to H_{i-r}(M)$. The image of $[W]$ under this map is the intersection of these two submanifolds. If V and W intersects transversally in M, the intersection we defined in this way is the same as the homology class of their intersection as a submanifold.

Geometrically, let $\pi : N_{V/X} \to V$ be the normal bundle of the submanifold $V \subset X$. This can be identified with the contraction of the tubular neighborhood on to V in side of X. The poincare dual of $[V]$ in the tubular neighborhood is the same as the poincare dual in M. We consider it as the former one which can be identified as a cohomology class of $N_{V/X}$ considered as a manifold. The pullback of this class by the zero section is a cohomology class of V, called the Euler class of the normal bundle. Note that the definition generalizes directly into arbitrary vector bundles. The Euler class of a vector bundle of rank r is in $H^r_c(M)$.

Now we will see the definition of Chern classes of vector bundles. For line bundles, we define the zero's Chern class to be $1 \in H^0(M)$, the first Chern class c_1 to be the Euler class which is in $H^1(M)$, $c_i = 0$ for higher i. The Chern classes for higher rank bundles can be defined by a splitting principle. The characteristic classes of vector bundles generalizes into algebraic geometry.

In algebraic geometry, the analogue of homology is given by the Chow groups. For a variety X, the Chow group $A_i(X)$ is defined by the free abelian group generated by the subvarieties of dim i quotient the linear equivalence relation, i.e., $\sum V_i$ and $\sum W_j$ with $\dim V_i = \dim W_j = i$ are the same if they differers by a rational functions on a subvariety of dim $i + 1$.

The A_* is functors in two senses. For a morphism between algebraic varieties, $f : X \to Y$, if f is flat, the pull-back map $f^* : A_k(Y) \to A_{k+n}(X)$ is well-defined. If f is proper, the push-forward $f_* : A_k(X) \to A_k(Y)$.

For a vector bundle E, the $c_i(E) : A_*(X) \to A_{*-i}(X)$ are defined roughly in the same way as in topology, i.e., first define it for line bundles and then using splitting principle to extend it to vector bundles.

We introduce some notations, the total Chern class $c(E) = 1 + C_1(E) + \cap$, the Gysin classes $s(E) = 1 + s_1(E) + \cap = c(E)^{-1}$.

The intersections on varieties are defined roughly in the following more technical way. For $i : V \to X$ local complete intersection morphism, $f : W \to X$ any morphism, we take the fibered diagram.

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \downarrow & & \downarrow \\ V' & \longrightarrow & W \\ \downarrow & & \downarrow f \\ V & \xrightarrow{i} & X \end{array}$$

There is a map $i^! : A_k W \to A_{k-d} V'$ called Gysin homomorphism, defined by $[Z] \mapsto c(p^* N_{X/V}) \cdot (c(N_{Z/Z'})^{-1} \cdot [Z'])$. Intuitively, this map send any Z to its intersection with the subvariety V.

In particular, if we replace $f : W \to X$ by the identity map $X \to X$, we get $i^! : A_k X \to A_k V$.

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle id} \\ V & \xrightarrow{i} & X \end{array}$$

1.3 Formulation of the general problem

We have seen in the basic example at the beginning that we want to do intersection of the degeneracy loci Z_i and an arbitrary subvariety. This suggests us to find the corresponding classes of the degeneracy loci in the Chow group of $Hom(E, F)$.

More generally, we formulate the problem as follows.

Let E and F be two vector bundles over a variety X of rank n and m respectively, $\phi : E \to F$ be a morphism of vector bundles. The sets $\{D_r(\phi) : x \in X \text{ with } \phi(x) < r\}$ are the counterparts of the sets Z_i in the basic example. We want to know the corresponding classes of all such loci D_r in the Chow groups of X. This can be expressed as a polynomial of Chern classes of E and F in some sense which will be explained.

Without proof, we introduce the Chern class formulas and Euler characteristic formulas. (Proofs can be found in [2], [4])

Theorem 1.1 (Porteous formula)**.** *Notations as above,*

$$D_r = \Delta(c(F-E)) := det \begin{pmatrix} h_{n-r} & h_{n-r+1} & \cdots & h_{n-r+(m-r)} \\ h_{n-r-1} & h_{n-r} & \cdots & h_{n-r+(m-r)-1} \\ \vdots & \vdots & & \vdots \\ & & \cdots & h_{n-r} \end{pmatrix}$$

where h_i is defined by $c(F)/c(E) = \sum h_i$, $h_0 = 1$, $h_i = 0$ for negative i.

The left-hand side and right-hand side are really apples and oranges, we have to say what is it meant that they are equal. Let $H = Hom(E, F)$ be the bundle of homs, $\widetilde{D}_r \subset H$ be the universal degeneracy loci. There's a map $s : X \to H$. We know that $D_r = s^{-1}(\widetilde{D}_r)$. We define $\mathbb{D}_r \in A_{n-r}(D_r)$ by $\mathbb{D}_r = s^{-1}[\widetilde{D}_r]$. The formula means

$$i_* \mathbb{D}_r = \Delta(c(F-E)) \cap [X].$$

Theorem 1.2. *Notations as above, there is a class $\mathbb{D}_r \in A_{n-r}(D_r)$ satisfying the following there properties and is uniquely characterized by 2, 3.*

1. *Each irreducible components of D_r has codim $\leq r$ in X. If $codim D_r = r$, then $\mathbb{D}_r$ is a positive cycle supported on D_r.*

2. *If $codim D_r = r$ and X is Cohen-Macaulay, then D_r is Cohen-Macaulay and $\mathbb{D}_r = [D_r]$.*

3. *The formation of $\mathbb{D}_r$ commutes with Gysin maps and proper push-forward.*

In this theorem, 3 can be made more precise by for all $f: X' \to X$, $g: D' \to D$ induced by f, then,

1. if f is flat, then $g^*\mathbb{D} = \mathbb{D}'$;
2. if f is lci, then $f^!\mathbb{D} = \mathbb{D}'$;
3. if f is proper, X and X' are properties, then $g_*\mathbb{D}' = deg(X'/X)\mathbb{D}$.

There's a formula expresses the Euler characteristic of D_r in terms of the Chern classes of E and F,

$$\chi(D_r) = \sum_{k=0}^{r}(-1)^k \binom{n-r+k-1}{k} \int_X \psi(r-k)$$

where $\psi(i) = P_i \cdot c(X) = \sum_{\lambda,\mu \vdash n-r}(-1)^{|\lambda|+|\mu|} D^{n-r;m-r}_{\lambda;\mu} s_{((m-r)^{n-r}+\lambda,\overline{\mu})}(E-F)$, $D^{a;b}_{\lambda;\mu} = det[\binom{\lambda_i+\mu_j+a+b-i-j}{\lambda_i+a-i}]$.

1.4 Recent progress in this branch

Classical enumerative geometry is called Schubert calculus which is about the intersection theory of grassmannians. One generalization is to the case of G/P. A lot of results are due to Prof. Lakshmibai and her coauthors many years ago.

Another angle of generalization is to the case of quiver varieties. The study of this field was started by a paper of Fulton and Buch based on some works by Prof. Lakshmibai and Prof. Zelevinsky about 10 years ago. Many questions are still open today.

2 Brill-Noether loci in Jacobians of algebraic curves

Let C be an algebraic curve over $k = \overline{k}$ with a fixed base point P_0. The Jacobians of this curves parameterizes the line bundles over C. They are defined by taking homology of the short exact sequence

$$0 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0.$$

There's a long exact sequence,

$$\cdots \to H^1(C,\mathbb{Z}) \to H^1(C,\mathscr{O}) \to H^1(C,\mathscr{O}^*) \to H^2(C,\mathbb{Z}) \to cdots$$

from which we know that $H^1(C,\mathscr{O}^*) \cong \mathbb{Z} \oplus \frac{H^1(C,\mathscr{O})}{H^1(C,\mathbb{Z})}$. Each of these torus is a Jacobian. In particular, the zero's Jacobian is $\frac{H^1(C,\mathscr{O})}{H^1(C,\mathbb{Z})} =: J$.

Geometrically,this can be done through periodic integral. Take any basis of the space of holomorphic forms $H^1(C,\mathscr{O})$, $\omega_1, \cdots, \omega_g$, for any divisor D of degree 0, we pair it's points $D = \sum(p_i - q_i)$, then $(\sum_i \int_{q_i}^{p_i} \omega_1, \cdots, \sum_i \int_{q_i}^{p_i} \omega_g)$ is well-defined modulo the periodic lattice. This map also factors trough the divisor class group. The $\mathbb{C}/\Lambda$ gives the torus structure for J.

The zero's Jacobian parameterizes the line bundles of degree 0 on C which coincide with the group of divisors on C of degree 0 quotient the subgroup of linear equivalences. The configuration space of all effective divisors of degree d is $C^{(d)}$, the d-th symmetric product of C. There's a map $\mu_d : C^{(d)} \to J$ which sent a divisor D to the class $[D - dP_0]$. Geometrically this map can be described by $p_1 + \cdots + p_d \mapsto (\sum_i \int_{P_0}^{P_i} \omega_1, \cdots, \sum_i \int_{P_0}^{P_i} \omega_g)$.

There are some classical facts about the map μ_d ([1]).

Theorem 2.1. *Notations as above,*

1. *the fibers of this map are the linear systems* $|D| \cong \mathbb{P}^r$*;*
2. *for* $d > 2g - 2$*, this map makes* $C^{(d)}$ *a projective bundle over* J*;*
3. *for* $1 \leq d \leq g$*, this map is birational onto its image* W_d*.*

We denote $w_i = [W_{g-i}]$.

The Brill-Noether loci W_d^r for $1 \leq d \leq g$ are defined to be the image of the linear systems of degree d and dimension $\geq r$ under the map μ_d.

These loci tell us a lot about the linear systems on the curve C. For example, if W_d^r are none empty, then we know that there are always line bundles on C of degree d, with linear systems of dim $\geq r$.

If there are other family of line bundles of degree d we are interested in, which could be defined by a set of polynomials on the Jacobians, we can use the intersection theory with the W_d^r's to find out whither there are line bundles in this family whose linear systems have dim $\geq r$.

3 Enumerative geometry of Brill-Noether loci in Jacobians

3.1 Interpretation of the Brill-Noether loci as degeneracy loci

With all the notations as before. We take the product $J \times C$ and denote the projections onto J and C by p and q respectively. We fix the base point of C to be P_0.

Over the product $J \times C$ there is a line bundle called Poincare bundle, denoted by $\mathscr{L}$. It is characterized by the property $\mathscr{L}|_{\{L\}\times C} \cong L$ over $\{L\} \times C \cong C$.

The original definition of the Brill-Noether loci is $W_d^r = \{L \in J : h^0(C, L \otimes \mathscr{O}(dP_0)) \geq r + 1\}$. By the definition of the Poincare bundle, $p_*(\mathscr{L} \otimes q^*(dP_0))|_L = H^0(C, L \otimes \mathscr{O}(dP_0))$.

Because of this,

$$W_d^r = \{L \in J : h^0(C, L \otimes \mathscr{O}(dP_0)) \geq r + 1\} = \{L \in J : dim(p_*(\mathscr{L} \otimes q^*(dP_0)|_L)) \geq r + 1\}.$$

Over the curve C, there is an exact sequence of sheaves

$$0 \to \mathscr{O}(dP_0) \to \mathscr{O}(mP_0) \to \mathscr{O}_{tP_0} \to 0$$

where $\mathscr{O}_{tP_0}$ is the skyscraper sheaf, $t = m - d$. Now we apply the functor $p_*(\mathscr{L} \otimes -)$ to the sequence above and look at each tern closely.

The sheaf $p_*(\mathscr{L} \otimes \mathscr{O}_{tP_0})$ is easy to describe. It is free of rank t.

Note that the functor $\mathscr{L} \otimes -$ is exact because vector bundles are locally free. But the functor p_* is only left exact. It's hard for us to say more about the sheaf $p_*(\mathscr{L} \otimes \mathscr{O}(mP_0))$, Fortunately, there's a standard Riemann-Roch argument telling us that for $t \geq 2g - 1$, this sheaf is locally free of rank $m + 1 - g$.

We introduce the notations $E_m = p_*(\mathscr{L} \otimes \mathscr{O}(mP_0))$, $F_t = p_*(\mathscr{L} \otimes \mathscr{O}_{tP_0})$.

Applying the functor $p_*(\mathscr{L} \otimes -)$ to the sequence $0 \to \mathscr{O}(dP_0) \to \mathscr{O}(mP_0) \to \mathscr{O})_{tP_0} \to 0$, we get an exact sequence

$$0 \to E_d \to E_m \to F_t$$

on J. We call the morphism $\eta : E_m \to F_t$. The locus

$$W_d^r = \{L \in J : dim(E_d|_L)) \geq r + 1\} = D_k(\eta)$$

where $k = m - r - g$.

3.2 Applying the formulas to this situation

According to the Porteous Formula, there's a class $\mathbb{W}_d^r \in A_\rho(W_d^r)$ where $\rho = g - (r + 1)(g - d + r)$. It is characterized by the fact that it's image in $A_\rho(J)$ is $\Delta(c(F_t - E_m)) \cap [J]$. We have to compute $c(F_t)$, $c(E_m)$.

Since F_t is a successive extension of trivial bundle, $c(F_t) = 1$.

The Chern classes of E_d, for $d > 2g - 2$, are computed through the geometry of E_d. We have $\mathbb{P}(E_d) = C^{(d)}$. The line bundle $\mathscr{O}_{E_d}(1)$ corresponds to the divisor $C^{(d-1)}$. From this we can get that $s_i(E_m) = w_i$.

Therefore,

$$\mathbb{W}_d^r = det \begin{pmatrix} w_{g-d+r} & \cdots & w_{g-d+2r} \\ \vdots & & \vdots \\ w_{g-d} & \cdots & w_{g-d+n} \end{pmatrix}$$

in $A_\rho(J)$. In particular, dim $W_d \geq \rho$. If dim $W_d^r = \rho$, then $[W_d^r] = \mathbb{W}_d^r$.

If $\rho \geq 0$, $W_d^r \neq \emptyset$. We consider some special cases. If we take $r = 0$, $d > 0$, we have $\rho = g - (g - d) = d > 0$. For any fixed positive d, there are line bundles of degree d with linear systems.

Fixing d, letting $\rho \geq 0$, solve for r, we can get the max dim of the linear systems of line bundles of such degree.

Applying the Euler characteristic formula to this case involves more computation ([3]). But $\chi(W_d^r(C)) = \sum_{k \leq r} (-1)^{k-r} \binom{k}{k-r} \phi(g, d, k)$, where $\phi(g, d, k)$ is purely dependent on g, d, r which is topological invariant.

References

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