An introduction to deformation theory in algebraic geometry

All rings here are assumed to be finite generated algebra over $\mathbb{C}$.
All schemes are Noetherian schemes of finite type over $\mathbb{C}$, unless otherwise assumed.

Basic idea

A point on a manifold is more than a point, it has

- the germ of C^∞ functions. C_p^∞.
- tangent space T_pX

If we choose a chart around p, tangent vectors are just an n-vector $(v_1, \cdots, v_n)$ acting on C_p^∞ in the following way:

$\forall f \in C_p^\infty$. by Taylor expansion, $f(x) = a_1x_1 + \cdots + a_nx_n + \cdots$,

$f \mapsto a_1v_1 + \cdots + a_nv_n$.

T_pX is the 1st order approximation to X at p.

FACT: The tangent space is the dual of m/m^2 where m is the maximal ideal in the ring C_p^∞.

For an algebraic variety, we can consider the n-th order approximation around p for $p \in X$.

Let $\mathcal{O}_{X,x}$ be the germ of regular functions around p, m_x be its only maximal ideal.

We consider $\mathcal{O}_{X,x}/m_x^n$. This is an Artin, local ring.

Geometrically there's only one point, but it contains a lot of information.

E.g. ① For $n \geq 2$. $\mathcal{O}_{X,x}/m_x^n$. The max ideal of this ring is $\overline{m_x} = m_x/m_x^n$. $\overline{m}_x^2 = m_x^2/m_x^n$. This means $\overline{m}_x/\overline{m}_x^2 = m_x/m_x^2$. The tangent space is the same as the tangent space at x of X.

② The complete local ring at x is defined to be

$$\widehat{\mathcal{O}}_{X,x} = \varprojlim_n \mathcal{O}_{X,x}/m_x^n.$$

This ring contains information about the tangent space, the dimension of X at x, and more.

③ The smoothness (singularity) around x.

In this sense, $\mathcal{O}_{X,x}$, which is hard to recover from $\mathcal{O}_{X,x}/m_x^n$, is less well-behaved than $\hat{\mathcal{O}}_{X,x}$.

Examples ① $R = \mathbb{C}[x,y]$. $m = (x,y)$.

$$\hat{R} = \mathbb{C}[[x,y]]$$

Thm: For X any algebraic variety, $x \in X$, x is a smooth point iff $\hat{\mathcal{O}}_{X,x} = \mathbb{C}[[x_1, \cdots, x_n]]$ where $n = \dim X$ at x.

② $R = \mathbb{C}[x,y]/(x^2-y^3)$. $m = (x,y)$

$\hat{R} = \mathbb{C}[[x,y]]/(x^2-y^3)$

③ $R = \mathbb{C}[x,y]/(x^2-x^3+y^2)$ $m = (x,y)$

$\hat{R} = \mathbb{C}[[s,t]]/(st)$.

But in this example, R_m is a normal ring.

Functors of Artin local rings

Interesting philosophy: rather than study the object itself, we study how the other objects acting on it.

Thm (Yoneda's lemma) (roughly)

If $\mathrm{Hom}(X,A) \cong \mathrm{Hom}(X,B)\ \forall X$, then $A \cong B$.

Let X be a scheme, $\mathrm{Spec}\,\mathbb{C} \xrightarrow{\eta_0} X$ be a fixed point.

Def $\mathrm{Def}_{\eta_0} : \mathbb{C}\text{-Art} \longrightarrow \mathrm{Set}$ ($\mathbb{C}$-Art is the category of $\mathbb{C}$ Artin local algebras)

$A \longmapsto$ the set of all maps $\eta : \mathrm{Spec}\,A \to X$

s.t.
$$\begin{tikzcd} \mathrm{Spec}\,\mathbb{C} \arrow[r, "\eta_0"] \arrow[d] & X \\ \mathrm{Spec}\,A \arrow[ur, "\eta"'] & \end{tikzcd}$$
commutes.

FACT: (Because for any Artin local ring (A, n), $\exists\, k$ s.t. $n^k = 0$)

$$\mathrm{Hom}_{\eta_0}(R, A) = \varinjlim_i \mathrm{Hom}_{\eta_0}(R/m^i, A) = \mathrm{Hom}_{\eta_0}(\varprojlim_i R/m^i, A)$$

($\varprojlim_i R/m^i$ is acturally not in $\mathbb{C}$-Art but still a $\mathbb{C}$ algebra)

This means $\mathrm{Hom}_{\eta_0}(\hat{R}, A) = \mathrm{Def}_{\eta_0}(A)$,
$\hat{R}$ represents the functor Def_{η_0}. (Or regorously, pro-represents this functor)

The study of Def_{η_0} is the same as studying $\hat{R}$.
Why don't we just compute $\hat{R}$?
For most of the time, deformation theory is used to study Moduli problems.
Rather than studying a scheme, we can study general functors $\mathcal{F}: Sch^{op} \to set$, and try to find a scheme X s.t.

$$\mathrm{Hom}(Y, X) \cong \mathcal{F}(Y) \quad \forall \text{ scheme } Y.$$

Such scheme may not exist.

- study the functor by thinking it as a geometric object.
- If this functor comes from problem, we change this problem a bit s.t. this functor is "trying hard" to be $\mathrm{Hom}(-, X)$ for some X.

E.g., In the study of moduli problems, people are interested in the study of moduli functors $\mathcal{F}: sch \to set$ sending X to the set of all maps $\begin{matrix} * \\ \downarrow \\ X \end{matrix}$ which are nice (flat, geometricly connected fibers and more).

<u>Example</u>: $\mathcal{M}_g(T)$ for any scheme T is the set of all maps $\begin{matrix} \mathcal{C} \\ \downarrow \\ T \end{matrix}$ which is proper, smooth, with each fiber geometrically connected curve of genus $g \geq 2$.

How's deformation theory helping us in this picture?

<u>Def</u> A functor $\mathcal{F}: \mathcal{C}\text{-Art} \to Set$ is called a pre-deformation functor if $\mathcal{F}(\mathbb{k})$ is a singleton.

<u>Two important examples</u>

1. If $\mathcal{F}$ is a moduli functor, fix an object η_0 in $\mathcal{F}(\mathbb{C})$, we define a pre-deformation functor $Def_{\eta_0}: \mathbb{C}\text{-Art} \to Set$ by
$$Def_{\eta_0}(A) = \{ \eta \in \mathcal{F}(A) : \eta|_{\mathbb{C}} = \eta_0 \}$$

2. Let X_0 be a variety $/\mathbb{C}$, we define a functor $Def_{X_0}: \mathbb{C}\text{-Art} \to set$ by
$$Def_{X_0}(A) = \left\{ \begin{array}{c} X \\ \downarrow \\ \operatorname{spec} A \end{array} \text{flat} \;\middle|\; \begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec} A \end{tikzcd} \right\}$$

3. If $\mathcal{F}(Y) = Hom(Y, X)$. We fix a point $\operatorname{Spec}\mathbb{C} \xrightarrow{\eta_0} X$ and define $Def_{\eta_0}: \mathbb{C}\text{-Art} \to Set$ by
$$Def_{\eta_0}(A) = \left\{ \eta : \operatorname{Spec} A \to X \;\middle|\; \begin{tikzcd} \operatorname{spec}\mathbb{C} \arrow[r, "\eta_0"] \arrow[d] & X \\ \operatorname{Spec} A \arrow[ur, "\eta"'] \end{tikzcd} \right\}$$

As has been remarked, $Def_{\eta_0}(A) \cong Hom(\hat{\mathcal{O}}_{X,x}, A)\ \forall A$.

<u>Def</u> • For a pre-deformation functor D, the tangent space T_D is defined to be $D(\mathbb{C}[\varepsilon]/\varepsilon^2)$

• D is said to be smooth if $\forall A' \twoheadrightarrow A$ with kernel square zero ideal, $\forall \eta \in D(A)$, $\exists \eta' \in D(A')$ with $\eta'|_A = \eta$

<u>Rmk</u> If $D(A) \cong Hom_{\eta_0}(\operatorname{Spec} A, X)$ for some scheme X and $D = Def_{\eta_0}$.

• $D(\mathbb{C}[\varepsilon]/\varepsilon^2) = Hom(\hat{\mathcal{O}}_{X,x}, \mathbb{C}[\varepsilon]/\varepsilon^2) = T_x X$

• Thm (formal criterion for smoothness) X is smooth at x iff D is smooth as a functor.

<u>Tangent spaces & obstruction theory for deformations of smooth varieties</u>

If X_0 is a smooth variety, the tangent sheaf $X_0/\mathbb{C}$, $T_{X_0/\mathbb{C}}$ is defined by $\mathcal{H}om_{\mathcal{O}_{X_0}}(\Omega^1_{X_0/\mathbb{C}}, \mathcal{O}_{X_0})$

We are studying $Def_{X_0}: \mathbb{C}\text{-Art} \to Set$. A lot is known for this case.

• Tangent space. Every vector in this space is called a 1st order deformation of X_0.

<u>Thm</u> For X_0 a smooth variety over $\mathbb{C}$, $\mathcal{U} = \{U_i\}$ any open affine covering of X_0. Then all the 1st order deformations of X_0 are in correspondence with $\check{H}^1(\mathcal{U}, T_{X_0/\mathbb{C}})$.

• Next question: given an local artin algebra A, and a map
$$A \twoheadrightarrow \mathbb{C}[\varepsilon]/\varepsilon^2$$
for a given 1st order deformation, can we lift it to $\operatorname{Spec} A$?

$$\begin{tikzcd} \exists ? X & X_1 \arrow[l] & X_0 \arrow[l] \\ \operatorname{Spec} A \arrow[u, leftarrow] & \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[l] \arrow[u, leftarrow] & \operatorname{Spec} \mathbb{C} \arrow[l] \arrow[u, leftarrow] \end{tikzcd}$$

We are doing this by factoring $A \twoheadrightarrow \mathbb{C}[\varepsilon]/\varepsilon^2$ into a sequence of maps s.t. for each $A_i \twoheadrightarrow A_j$, the kernel is a principle square zero ideal. Then studying the lifting for maps of this type $A \twoheadrightarrow \Lambda$.

<u>Thm</u> For any $A \overset{f}{\twoheadrightarrow} \Lambda$ with kernel principle square zero ideal and a deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \operatorname{Spec} \Lambda \arrow[r] & \operatorname{Spec} \mathbb{C} \end{tikzcd}$$

$\exists$ a canonical $o(f) \in \check{H}^2(\mathcal{U}, T_{X/\Lambda})$ s.t. $o(f) = 0$ iff $\begin{matrix} X \\ \downarrow \\ \operatorname{Spec} \Lambda \end{matrix}$ can be lift to $\operatorname{Spec} A$.