# Central simple modules of a Gorenstein Artin algebra

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**Abstract:** Let $R$ be the polynomial ring in $r$ variables over a field $k$. Let $A = R/I$ be a standard Artinian algebra quotient of $R$, not necessarily graded. For A graded and a given linear form $z$ in $A_1$, T Harima and J. Watanabe in a series of papers studied the "central simple modules" (CSM) of the pair $(A, z)$: these are the nonzero factor modules (here $c$ is the socle degree of $A$) of the sequence

$$(0 : z^c) + (z) \supset (0 : z^{c-1}) + (z) \supset \cdots \supset (0 : z) + (z).$$

They related these to the Lefschetz properties of $(A, z)$. The latter have to do with the Jordan block decomposition of the multiplication map $m_z$ "multiply by $z$" on $A$. In particular Harima and Watanabe showed that the CSM's have symmetric Hilbert functons when $A$ is Gorenstein graded. We will describe these concepts, give examples, and suggest how they might generalize to non-graded $A$ that are Gorenstein. We also state some open questions.

## 1 Introduction: Gorenstein Artin algebras

$k =$ alg. closed field. $R = k[x, y]$ or $k[x, y, z]$ polynom, ring.

Let $I=$ ideal primary to the maximal ideal $m = (x, y)$ or $(x, y, z)$ : so $I \supset m^{c+1}, I \not\supset m^c$ some $c =$ *socle degree* of $A$.

An *Artin algebra* $A = R/I, \dim_k A = n < \infty$.

$\mathrm{Soc}(A) = (0 : m) = \{a \in A \mid ma = 0\}$

$A$ is *Gorenstein Artinian* (GA) if $\dim_k \mathrm{Soc}(A) = 1$. Then

$\mathrm{Soc}A = A_c$. Let $\phi : A_c \to k$. Then $< \cdot, \cdot >_\phi$: $A \times A \to k$, $< a, b >_\phi = \phi(ab)$ is a non-degenerate bilinear form.

Let $S = k[X, Y]$ or $k[X, Y, Z]$. $R$ acts on $S$ as PDO w.o. coeffs (contraction): $x^i \circ X^u = X^{u-i}$ if $u \geq i$, or 0 otherwise.

Let $F \in S$, set Ann $(F) = \{f \in R \mid f \circ F = 0\}$.

Let $I \subset R$ ideal. Set $I^\perp = \{F \in S \mid I \circ F = 0\}$

**Thm** (F.H.S. Macaulay, 1915). $A$ is Gorenstein Artin $\Leftrightarrow$ $\exists F \in S \mid A = R/I, I =$ Ann $(F)$.

Associated graded alg. $A^* = Gr_m(A) = A_0 \oplus A_1 \oplus \cdots \oplus A_c$,

$$A_i = m^i A/m^{i+1} \cong (I \cap m^i + m^{i+1})/(m^{i+1}).$$

Hilbert functon $H(A) = (h_0, h_1, \ldots, h_c), h_i = \dim_k A_i$.

**Ex 1**. Let $I = (y^2, x^3)$. Then $A = \langle\overline{1, x, x^2, y, yx, yx^2}\rangle$,

$\mathrm{Soc}(A) \cong A_3 = \langle yx^2\rangle$. Here $I = \mathrm{Ann}\,\langle X^2Y\rangle$. $\quad H = (1, 2, 2, 1)$.

**Ex 2**. Let $I = \mathrm{Ann}\,\langle X^2Y + Y^2\rangle = (y^2 - x^2y, x^3)$ "

**Ex 3**. Let $I = \mathrm{Ann}\,\langle X^4 - Y^2\rangle = (xy, y^2 + x^4)$.

$$H = (1, 2, 1, 1, 1) \quad A = \langle\overline{1, x, y, x^2, x^3, x^4}\rangle.$$

**Thm.**(F.H.S. Macaulay), $R = k[x, y]$: $A$ Gorenstein $\Leftrightarrow$ $A$ CI.

($I = (f, g)$. two generators. Analogue not true for $k[x, y, z]$.)

**Ex 4.** $F = X^3 + Y^3 + Z^3, I = (xy, xz, yz, x^3 - y^3, x^3 - z^3)$.

Question: How can you tell when you have all the generators of the ideal?

Answer: The dual module $\hat{A} = R \circ F$ has the same Hilbert function as $A$. Here $R \circ F = 1; X, Y, Z; X^2, Y^2, Z^2, F$ of Hilbert function $H(\hat{A}) = (1, 3, 3, 1) = H(A)$.. One checks that $J = (xy, xz, yz, x^3 - y^3, x^3 - z^3) \in \mathrm{Ann}\ (F)$. Then, since $H(R) = (1, 3, 6, 10, \ldots)$, and $H(R/J) = (1, 3, 3, 1)$ can be seen directly, we have $J \subset I$ but $H(R/J) = H(R/I) \Rightarrow I = J$. The calculation is made in algebra programs.

Question: And when $A$ is not graded? Answer: Go from $\hat{A}$ to $\hat{A}^*$ (top degree forms)

and from $I$ to $I^*$ (initial - low degree form) and verify $H(\hat{A}^*) = H(R/J^*)$. See [I]

**Thm.**(F.H.S. Macaulay). $A$ graded GA $\Rightarrow H(A)$ symmetric and $\langle \cdot, \cdot \rangle_\phi : A_i \times A_{j-i} \to k$ is an exact duality.

**Thm.** (J. Watanabe [W], also [I, Prop 1.7]) Assume $A$ GA. Then $H(A)$ symmetric $\Leftrightarrow$ A$^*$ Gorenstein.

In Ex 2, $A^*$ same as in Ex. 1, $H(A) = (1, 2, 2, 1)$, symmetric.

In Ex. 3, $A^* \cong R/(y^2, xy, x^5)$, $H(A) = (1, 2, 1, 1, 1)$.

Here $A^*$ is *not* Gorenstein, as $H$ is *not* symmetric.

**Q1.** What symmetry or duality property does $A^*$ inherit?

**Ans.** $A^*$ has a descending series of ideals whose successive quotients are reflexive modules [I]. (see below).

Let $z \in A, \overline{z} \neq 0 \in A_1$. $m_z : A \to A$, $m_z(a) = z \cdot a$.

Let $P_z =$ partition of Jordan blocks of $m_z$.

**Q2.** Given $A$ what $P_z$ are possible? What is $P_z$ for $z$ generic?

**Lefschetz property**: $(A, z)$ is *strong Lefschetz* (SL) if $m_z$ has partition $H^\vee$ (dual to $H$). $A$ has SLP if $\exists z \mid (A, z)$ is SL.

(See $m_z$ in Ex. 1$^*$, or $m_x$ in Ex. 3$^*$ below.)

## 2 Invariant spaces for $m_z$

**Ex 1***. $A = R/I, I = (y^2, x^3)$. $H = (1, 2, 2, 1)$.

$m_x$ invariant subspaces $\langle\overline{1, x, x^2}\rangle, \langle\overline{y, yx, yx^2}\rangle$ $P_x = (3, 3)$,

$k[x]$ generators $\langle\overline{1, y}\rangle$.

$m_y$: invariant subspaces $\langle\overline{1, y}\rangle, \langle\overline{x, yx}\rangle, \langle\overline{x^2, yx^2}\rangle$. $P_y = (2, 2, 2)$.

$k[y]$ generators $\langle\overline{1, x, x^2}\rangle$.

$m_z, z = x + y$: invt.sp. $\langle\overline{1, z, z^2, z^3}\rangle, \langle\overline{x - y, x^2}\rangle$. $P_z = (4, 2)$.

$k[z]$ generators $\overline{1}$ and $\overline{x - y}$. $P_z = H^\vee$, so $(A, z)$ is SL.

**Ex 3***. $A = R/I, I = (xy, y^2 + x^4)$, H=(1,2,1,1,1).

$m_y$ invt. spaces $\langle\overline{1, y, y^2}\rangle; \overline{x}; \overline{x^2}; \overline{x^3}$, $P_x = (3, 1, 1, 1)$.

$m_x$ invariant subspaces $\langle\overline{1, x, x^2, x^3, x^4}\rangle, \langle\overline{y}\rangle$. $P_x = (5, 1)$.

**Ex 3*** cont.: $P_x = (5, 1) = H^\vee$. So (A,x) is SL.

**Thm**. $A$ CI quotient of $R = k[x.y] \Rightarrow A$ is SL.

**Q.** (open) $A$ height three CI or Gorenstein $\Rightarrow A$ is SL?

T. Harima and J. Watanabe consider for $z \in m - m^2$ in $A$,

$$(0 : z^c) + (z) \supset (0 : z^{c-1}) + (z) \supset \cdots \supset (0 : z) + (z) \qquad (**)$$

**Def.** A *central simple* module $U_i(z)$ for $(A, z)$ is a nonzero quotient of successive terms in (**). [1]

$$U_i(z) = ((0 : z^i) + (z))/((0 : z^{i-1} + (z)),$$

**Ex 1****. $A = R/(y^2, x^3)$, The $m_x$ generators are CS module: $U_3(x) = \langle\overline{1, y}\rangle = A/(x) = (0 : x^3) + (x)/(0 : x^2) + (x)$. Recall that $P_x = (3, 3)$. Writing the multiples of $U_3(x)$ by $\{1, x, x^2\}$, and (on the right) their Hilbert functions, we find

$$\begin{array}{rlllll}
U_3(x) = & \langle\overline{1, y}\rangle & 1 & 1 & & \\
xU_3(x) = & \langle\overline{x, xy}\rangle & 0 & 1 & 1 & \\
x^2U_3(x) = & \langle\overline{x^2, x^2y}\rangle & 0 & 0 & 1 & 1 \\
\hline
 & H(A) = & 1 & 2 & 2 & 1
\end{array}$$

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[1] We here use a different notation than [HW], where the $i$-th nonzero $U_i$ is corresponds to a part $f_i$ of $P_z$.

The $m_y$ generators are the CS-module $U_2(y)$

$$U_2(y) = \langle\overline{1, x, x^2}\rangle = A/(y) = (0 : y^2) + (y)/(0 : y) + (y).$$

Recall that $P_y = (2, 2, 2)$. Writing the multiples of $U_2(y)$ by $\{1, y\}$, and their Hilbert functions, we find

$$\begin{array}{rlllll}
U_2(y) = & \langle 1, x, x^2\rangle & 1 & 1 & 1 & \\
yU_2(y) = & \langle y, xy, x^2y\rangle & 0 & 1 & 1 & 1 \\
\hline
 & H(A) = & 1 & 2 & 2 & 1
\end{array}$$

The $m_z, z = x + y$ generators are *two* CS modules:

$$U_4(z) : \langle 1\rangle = A/(z^4) + (z) = ((0 : z^4) + (z))/((0 : z^3) + (z)).$$

$$U_2(z) = \langle x - y\rangle = ((0 : z^2) + (z))/((0 : z) + (z)).$$ par

Writing the corresponding submodules of $A$, we have

$$\begin{array}{rlllll}
k[z]U_4(z) = & \langle 1, z, z^2, z^3\rangle & 1 & 1 & 1 & 1 \\
k[z]U_2(z) = & \langle x - y, z(x - y)\rangle & 0 & 1 & 1 & \\
\hline
 & H(A) = & 1 & 2 & 2 & 1
\end{array}$$

### 2.1 Dual sequence $W_i$ to $U_i$

We have, dually, a sequence

$$(0 : z) \supset 0 : z \cap (z) \supset (0 : z) \cap (z^2) \supset \dots$$

whose successive quotients are

$$W_i(z) = \big(0 : z) \cap (z^{i-1})\big) \,/\, \big((0 : z) \cap (z^i)\big) .$$

**Thm** (Harima-Watanabe) $A$ graded Artinian Gorenstein $\Rightarrow$

a. Each $U_i(z)$ has symmetric (palindrome) Hilbert function.

b. $W_i(z) = z^{i-1}U_i(z)$.

c. A SL$\Leftrightarrow$ $\exists z \in A_1 \mid$ each $U_i(z)$ is SL.

d. A SL $\Leftrightarrow$ for generic $z \in A_1$, each $U_i(z)$ is concentrated in a single degree.

e. The pairing $\langle \cdot, \cdot \rangle_\phi$ defines an exact pairing $U_i \times W_i \to k$.[2]

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[2] This is implicit in [HW], and is generalized in [BoI]

**Ex**. $A = R/(y^2, x^3)$, $\quad \mathrm{Soc}A = \langle \overline{x^2 y} \rangle$. $\quad \phi(\overline{x^2 y}) = 1$.

$U_3(x) = \langle 1, y \rangle$, $W_3(x) = \langle x^2, yx^2 \rangle$.

Then $\langle \cdot, \cdot \rangle_\phi : U_3 \times W_3 \to k$ is exact. $1 = (yx^2)^\vee, y = x^2)^\vee$.

**Q.** Why is $U_i(z)$ called a "central simple" module?[3]

**Ans**. Let $\mathfrak{C}(z)$ = centralizer of $m_z$ in End$(A) \cong Mat_n(k)$,

Then End $U_i$ is a simple $\mathfrak{C}(z)$ module, and

$$\pi : \mathfrak{C}(z) \to \prod_i End(U_i)$$

is the canonical map to a semisimple $\mathfrak{C}(z)$ module, with kernel the Jacobson radical of $\mathfrak{C}(z)$.

**Ex.** $A = R/(y^2, x^3), P_x = (3, 3)$. $M \in \mathfrak{C}(x), U_3 = \langle$1,y$\rangle$

$$M = \begin{array}{r|cccccc} & 1 & x & x^2 & y & yx & yx^2 \\ \hline 1 & a & b & c & d & e & f \\ x & 0 & a & b & 0 & d & e \\ x^2 & 0 & 0 & a & 0 & 0 & d \\ y & g & h & i & j & k & l \\ yx & 0 & g & h & 0 & j & k \\ yx^2 & 0 & 0 & g & 0 & 0 & j \end{array}, \quad M \to \pi(M) = \begin{pmatrix} a & d \\ g & j \end{pmatrix} \in End(U_3)$$

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[3] Notation introduced by T. Harima and J. Watanabe.

## 3 Descending sequence of ideals in $A^*$.

We now consider nongraded Gorenstein Artinian $A$.

**Thm.[I]** $Q(a)$ **decomposition of** $A^*$**.** Let $A$ be Gor. Artin. Then $A^*$ has a stratification by a descending sequence of ideals

$$A^* = C(0) \supset C(1) \supset \cdots \supset C(c-1) = 0, \qquad (3.1)$$

whose successive quotients $Q(a) = C(a)/C(a+1)$ are reflexive $A^*$ modules; with "reflection degree" $(j-a)/2$). Also $Q(0)$ is a graded Gorenstein algebra. The ideal $C(a)$ satisfies,

$$C(a)_i = \left(m^i \cap (0 : m^{j+1-a-i})\right) / \left(m^{i+1} \cap (0 : m^{j+1-a-i})\right). \quad (3.2)$$

The pairing $\langle \cdot, \cdot \rangle_\phi$ on $A$ induces an exact pairing

$$\langle \cdot, \cdot \rangle_{\phi,a} : Q(a)_\nu \times Q(a)_{j-a-\nu} \to k,$$

**Ex.3**** Let $A = R/I, I = \text{Ann}\ \langle X^4 - Y^2 \rangle = (xy, y^2 + x^4)$.

Recall $\quad A^* \cong R/(y^2, xy, x^5)$, $H(A) = (1, 2, 1, 1, 1)$.

$Q(0) = R/Ann(X^4) = R/(y, x^5); \quad H(Q(0)) = (1, 1, 1, 1, 1)$.

$Q(2) = \overline{y}; \quad H(Q(2)) = (0, 1, 0)$.

**Thm** [BoI] Let $A$ be Gorenstein Artin. Then

(a.) $W_i(z) = z^{i-1}U_i(z)$.

(b.) The pairing $\langle \cdot, \cdot \rangle_\phi$ defines an exact pairing $U_i \times W_i \to k$.

**Q.** Does the palindrome property extend? We think so.

Proof idea: We may replace $m$ by an ideal $J \subset m$ in (3.2) to define $Q^J(a) = C^J(a)/C^J(a+1)$ over $Gr_J(A)$.

We may also doubly grade by two ideals $J \subset m, K \subset m$, obtaining $Q^{J,K}(a,b)$, subquotients of of $Gr_{J,K}(A)$.

**Thm. [BoI] $Q^J(a)$ and $Q^{J.K}(a,b)$ decomposition.** $Q^J(a)$ and $Q^{J,K}(a,b)$ are reflexive $Gr_J(A)$, or $Gr_{J,K}(A)$ modules.

Take $J = L$ and $A$ graded to obtain Harima-Watanabe reflexivity. Or take $J = L$ and $K = m$ to study the Hilbert function $H_m(U_i)$ (grade with respect to powers of $m$).

Let $M$ be a $Gr_J(A)$ module; we denote by $H_J(M)$ the $J$-graded Hilbert function

$$H_J(M)_i = \dim_k(J^iM/J^{i+1}M) = \dim_k Gr_J(M)_i. \qquad (3.3)$$

**Cor.** We have that $H_J(Q^J(a))$ is symmetric about $(j-a)/2$.

Furthermore,

$$H_m(A) = \sum_{a=0}^{j-2} H_m(Q^J(a)). \tag{3.2}$$

**Ex**. Let $F = x^5y + x^2z^3 + xy^3z$, $H = (1,3,6,6,3,2,1)$, $j = 6$.

$I = \text{Ann } F = (z^3 - x^3y, yz^2, y^2z - x^4, xz^2 - y^3, x^2yz, x^2y^2, x^3z)$

$I^* = (z^3, yz^2, y^2z, y^3 - xz^2, x^2yz, x^2y^2, x^3z, x^6)$.

$Q(0) \cong R/\text{Ann}\,(x^5y) \cong k[x,y]/(y^2, x^6)$; $\quad H(0) = (1,2,2,2,2,2,1)$.

Here $H(1) = (0,1,4,4,1)$, and $Q(1)$ has representatives,

$$z; z^2, yz, y^2, xz; y^3, xyz, xy^2, x^2z; xy^3.$$

Here $P_y = (4,4,4,4,2,2,1,1)$.

The duality between $U_i, W_i$, is the duality between the top and bottom of the reflexive module $Q^{(y)}(j+1-i)$.

We have $U_4(y) = Q^{(y)}(3)_0$ and $W_4(y) = Q^{(y)}(3)_3$ satisfy

$$U_4 = \big((0:y^4)+(y)\big) \,/\, \big((0:y^3)+(y)\big) = \langle \begin{array}{ccc} 1 & x & \\ & & \\ & z & xz \end{array} \rangle.$$

$$W_4 = (y^3) = \langle \begin{array}{ccc} & y^3z & xy^3z \\ & & \\ y^3 & xy^3 & \end{array} \rangle = \langle \begin{array}{ccccc} & & & x^4y & x^5y \\ & & & & \\ y^3 & xy^3 & & & \end{array} \rangle$$

We have

$$U_4 \times W_4 \to k: \quad 1 = (x^5y)^\vee, x = (x^4y)^\vee, z = (xy^3)^\vee, xz = (y^3)^\vee.$$

Notice also that $\times(x+z)$ acting on $U_1 = Q^{(y)}(3)_0$ takes $1 \to x+z \to 2xz \to 0$, and $x-z \to 0$, so has partition $(3,1)$, and $x+z$ is a strong Lefschetz element for $U_1$.

We have $Q^{(y)}(5)$ comprised of $U_2 \cong Q^{(y)}(5)_0$ and $W_2 \cong Q^{(y)}(5)_1$:

$$U_2 = \langle x^2, x^3 \rangle \in Q(0);\ W_2 = \langle yx^2, yx^3 \rangle \in Q(0).$$

Also, $Q^{(y)}(6) \cong U_1 = W_1$ satisfies

$$U_1 = \langle z^2, x^2z \rangle \in Q(1).$$

Also $(A, x+y+z)$ is SL, as $P_{x+y+z} = (7,5,4,2,2,2) = H^\vee$.

The convention we use in Table 3.1 is to exhibit the new elements compared to the adjacent two smaller squares: thus in the box for $((y^2)+(0:y^3)$ we'll write representatives in $Q(0)$ or $Q(1)$ for elements of $(y^2)+(0:y^3)$ not seen in the adjacent smaller submodules of $A^*$, $((y^3)+(0:y^3))$ or $((y^2)+(0:y))$.

| $+$ | $1$ | $(y)$ | $(y^2)$ | $(y^3)$ | $0$ | |
|---|---|---|---|---|---|---|
| $0$ | $1, x, x^2, x^3, x^4, x^5$<br>$z, xz$ | $y, xy, x^2y, x^3y$<br>$yz, xyz$ | $x^4, x^5$<br>$y^2, xy^2$ | $zy^3, xzy^3$<br>$y^3, xy^3$ | $0$ | $Q(0)$<br>$Q(1)$ |
| $(0:y)$ | | $-$ | $-$ | $-$ | $x^2y, x^3y, x^4y, x^5y$<br>$z^2, y^3, x^2z, xy^3$ | $Q(0)$<br>$Q(1)$ |
| $(0:y^2)$ | | $-$ | $-$ | $-$ | $x^2, x^3, x^4, x^5,$<br>$y^2, xy^2$ | $Q(0)$<br>$Q(1)$ |
| $(0:y^3)$ | | $-$ | $-$ | $-$ | $y, xy$<br>$yz, xyz$ | $Q(0)$<br>$Q(1)$ |
| $A^*$ | | | | | $1, x$<br>$z, xz$ | $Q(0)$<br>$Q(1)$ |

Table 3.1: $(0:y^a)+(y^b)$ for $I = \mathrm{Ann}\,(F), F = x^5y + x^2z^3 + xy^3z$

**Apply?** What is the Jordan block matrix of a generic element of $\mathfrak{N}(z) \subset \mathfrak{C}(z)$, nilpotents in the centralizer of $z$? One finds $B$ generic, $B \in \pi^{-1}(\prod(\text{nilpotents}) \subset \prod End(U_i).)$. What are Jordan blocks for $B$? (open, partial results: P. Oblak, T. Košir, G. McNinch,D. I. Panyushev, R. Basili and I.- [Bas2, Ob1, Ob2, KO, Pan, McN]).

**Q.** What is the goal of this work? Do you expect to characterize Gorenstein Artin algebras?

**Ans.** We wish to characterize more properties of the associated graded algebras, and also the Hilbert functions for Gorenstein (non-graded) algebras. The invaiants we discuss can also be deformatiion invarants within the family Gor(H) parametrizing AG algebras of Hilbert function $H$ [I].

Artinian Gorenstein algebras can occur in singularity theory: for a finite mapping germ, the isomorphism class of the associated Artinian algebra is a right-left invariant.

Homogeneous AG algebras are understood for three variables, due to the Buchsbaum-Eisenbud Pfaffian structure theorem, and in height threee, for symmetric Gorenstein sequences $H$, the family $Gor(H)$ is irreducible; but in height four, $Gor(H)$ may have many irreducible components. The possible $H$ are known in heights no greater than three, but have not been characterized in height four (See [MNZ]).

For non-graded GA algebras, even in height three the possible $H$ are not known, despite the presence of a structure theorem (see [I]).

# References

[Bar2001] V. Baranovsky: *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups 6 (2001), no. 1, 3–8.

[Bas1] R. Basili: *On the irreducibility of varieties of commuting matrices*, J. Pure Appl. Algebra 149(2) (2000), 107–120.

[Bas2] ______: *On the irreducibility of commuting varieties of nilpotent matrices.* J. Algebra 268 (2003), no. 1, 58–80.

[BI1] ______ and A. Iarrobino : *Pairs of commuting nilpotent matrices, and Hilbert functions*, J. Algebra 320, 2008, 1235–1254. ArXiv math.AC: 0709.2304.

[BoI] M. Boij and A. Iarrobino : *Duality for central simple modules of Gorenstein algebras*, in preparation.

[HW] T. Harima and J. Watanabe: *The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras*, preprint, (2005, revised 2007), to appear, J. Algebra.

[HW2] ______ and ______: *The central simple modules of Artinian Gorenstein algebras*, J. Pure and Applied Algebra 210(2) (2007), 447–463.

[I] A. Iarrobino : *Associated Graded Algebra of a Gorenstein Artin Algebra*, Memoirs Amer. Math. Society, Vol 107 #514, (1994), Amer. Math. Soc., Providence.

[KO] T. Košir and P. Oblak: *A note on commuting pairs of nilpotent matrices*, to appear, Transformation Groups. preprint, 2007, ArXiv Math.AC/0712.2813.

[Mac] F.H.S. Macaulay: *The algebraic theory of modular systems* (Revised reprint of 1016 original. With an introduction by Paul Roberts.) Camridge Mathematical Library, Cambridge University Press, Cambridge, 1994. MR1281612 (95i:13001).

[McN] G. McNinch: *On the centralizer of the sum of commuting nilpotent elements*, J. Pure and Applied Alg. 206 (2006) # 1-2, 123–140.

[MNZ] J. Migliore, U. Nagel, F. Zanello: *A characterization of Gorenstein Hilbert functions in codimension four with small initial degree.* Math. Res. Lett. 15 (2008), no. 2, 331–349.

[NS] M. Neubauer and D. Saltman: *Two-generated commutative subalgebras of $M_nF$*, J. Algebra **164** (1994), 545–562.

[NSe] ——— and B.A. Sethuraman: *Commuting pairs in the centralizers of 2-regular matrices*, J. Algebra 214 (1999), 174–181.

[Ob1] ———: *Jordan forms for mutually annihilating nilpotent pairs*, Linear Algebra and Applications, 428, no. 7 (2008), 1476–1491. (ArXiv: math.AC/0612300)

[Ob2] P. Oblak: *The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix*, Linear and Multilinear Algebra (electronic 9/2007). Slightly revised in ArXiv: math.AC/0701561.

[Pan] D. I. Panyushev: *Two results on centralisers of nilpotent elements*, J. Pure and Applied Algebra, 212 no. 4 (2008), 774–779.

[W] J. Watanabe: *The Dilworth number of Artinian Gorenstein rings*, Advances in Math. 76 (1989), 194–199.