Extending non-commutative Schatten norms to product spaces

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Assume $1 \leqslant q \leqslant p \leqslant \infty$ and let r be such that $1/r = 1/q - 1/p$. Define

$$\|Y\|_{(q,p)} := \inf\left\{\sum_i \|A_i\|_{2r}\|Z_i\|_p\|B_i\|_{2r} : Y = \sum_i (A_i \otimes I)Z_i(B_i \otimes I)\right\}.$$

If $p \leqslant q$, define

$$\|Y\|_{(p,q)} := \sup_{A,B}\left\{\frac{\|(A\otimes I)Y(B\otimes I)\|_q}{\|A\|_{2r}\|B\|_{2r}}\right\}.$$

Theorem. Let $Y, W \in M_n \otimes M_m$ and p', q' be such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then

a) triangle inequality: $\|Y + W\| \leqslant \|Y\| + \|W\|$;

b) Hölder's inequality: $|\operatorname{tr}(YW)| \leqslant \|Y\|_{(p,q)}\|W\|_{(p',q')}$;

c) duality:

$$\begin{aligned}\|Y\|_{(p,q)} &= \sup\left\{|\operatorname{tr}(YW)| : \|W\|_{(p',q')} \leqslant 1\right\}\\ \|Y\|_{(q,p)} &= \sup\left\{|\operatorname{tr}(YW)| : \|W\|_{(q',p')} \leqslant 1\right\};\end{aligned}$$

d) $\|Y_1 \otimes Y_2\|_{(p,q)} = \|Y_1\|_p\|Y_2\|_q$;

e) $\|Y\|_{(p,p)}=\|Y\|_p$;

f) if Y is diagonal, then $\|Y\|_{(p,q)} = (\operatorname{tr}_1(\operatorname{tr}_2|Y|^q)^{p/q})^{1/p}$.

Proof. a) If $p \leqslant q$,

$$\begin{aligned}\|Y+W\|_{(p,q)} &= \sup_{A,B}\left\{\frac{\|(A\otimes I)(Y+W)(B\otimes I)\|_q}{\|A\|_{2r}\|B\|_{2r}}\right\} = \\ &= \sup_{A,B}\left\{\frac{\|(A\otimes I)Y(B\otimes I)+(A\otimes I)W(B\otimes I)\|_q}{\|A\|_{2r}\|B\|_{2r}}\right\} \leqslant \\ &\leqslant \sup_{A,B}\left\{\frac{\|(A\otimes I)Y(B\otimes I)\|_q+\|(A\otimes I)W(B\otimes I)\|_q}{\|A\|_{2r}\|B\|_{2r}}\right\} \leqslant \\ &\leqslant \|Y\|_{(p,q)}+\|W\|_{(p,q)}.\end{aligned}$$

If $q \leqslant p$, $\exists A_i, B_i, Z_i$ such that

$$Y = \sum_i (A_i \otimes I) Z_i (B_i \otimes I),$$

then

$$\|Y\|_{(q,p)} + \epsilon \geqslant \sum_i \|A_i\|_{2r}\|B_i\|_{2r}\|Z_i\|_p =: S_1.$$

Similarly $\exists C_i, D_i, X_i$ such that

$$W = \sum_i (C_i \otimes I) X_i (D_i \otimes I),$$

then

$$\|W\|_{(q,p)} + \epsilon \geqslant \sum_i \|C_i\|_{2r}\|D_i\|_{2r}\|X_i\|_p =: S_2.$$

Hence

$$\|Y\|_{(q,p)} + \|W\|_{(q,p)} + 2\epsilon \geqslant S_1 + S_2 \geqslant \|Y+W\|_{(q,p)}.$$

b) Let $q \leqslant p$. For any decomposition

$$W = \sum_i (A_i \otimes I) Z_i (B_i \otimes I),$$

we have

$$\begin{aligned}
|\operatorname{tr}(YW)| &= \left|\operatorname{tr}\left(\sum_i Y(A_i\otimes I)Z_i(B_i\otimes I)\right)\right| \leqslant \\
&\leqslant \sum_i |\operatorname{tr}(Y(A_i\otimes I)Z_i(B_i\otimes I))| = \\
&= \sum_i |\operatorname{tr}((B_i\otimes I)Y(A_i\otimes I)Z_i)| \leqslant \\
&\leqslant \sum_i \|(B_i\otimes I)Y(A_i\otimes I)\|_q\|Z_i\|_{q'} \leqslant \\
&\leqslant \sum_i \|A_i\|_{2r}\|B_i\|_{2r}\|Y\|_{(p,q)}\|Z_i\|_{q'}.
\end{aligned}$$

Now observe that

$$\frac{1}{r}=\frac{1}{q}-\frac{1}{p}=1-\frac{1}{q'}-\left(1-\frac{1}{p'}\right)=\frac{1}{p'}-\frac{1}{q'}$$

and $q\leqslant p$ implies $p'\leqslant q'$; therefore

$$|\operatorname{tr}(YW)|\leqslant\|Y\|_{(p,q)}\|W\|_{(p',q')}.$$

There is no need to check the case $p\leqslant q$ since both norms appear in the inequality.

c) Let $p\leqslant q$. It follows immediately from the definition that

$$\|Y\|_{(p,q)}=\sup_{A,B}\{\|(A\otimes I)Y(B\otimes I)\|_q : \|A\|_{2r}\leqslant 1, \|B\|_{2r}\leqslant 1\}.$$

Using duality for the q norm, we get

$$\begin{aligned}
\|Y\|_{(p,q)} &= \sup_{A,B,Z}\{|\operatorname{tr}((A\otimes I)Y(B\otimes I)Z)| : \|Z\|_{q'}\leqslant 1, \|A\|_{2r}\leqslant 1, \|B\|_{2r}\leqslant 1\} \leqslant \\
&\leqslant \sup_{A,B,Z}\{|\operatorname{tr}(Y(A\otimes I)Z(B\otimes I))| : \|A\|_{2r}\|B\|_{2r}\|Z\|_{q'}\leqslant 1\} = \\
&= \sup_{A,B,Z}\{|\operatorname{tr}(YW)| : W=(A\otimes I)Z(B\otimes I), \|A\|_{2r}\|B\|_{2r}\|Z\|_{q'}\leqslant 1\} \leqslant \\
&\leqslant \sup_{W}\{|\operatorname{tr}(YW)| : \|W\|_{(p',q')}\leqslant 1\}
\end{aligned}$$

and the last inequality holds since the condition involving the (p',q') norm allows for more decompositions of W. Now, from Hölder's inequality, we have

$$|\operatorname{tr}(YW)|\leqslant\|Y\|_{(p,q)}\|W\|_{(p',q')}$$

hence

$$\sup_W \{ |\operatorname{tr}(YW)| : \|W\|_{(p',q')} \leqslant 1 \} \leqslant \|Y\|_{(p,q)}.$$

Therefore, combining the two inequalities, we obtain

$$\|Y\|_{(p,q)} = \sup_W \{ |\operatorname{tr}(YW)| : \|W\|_{(p',q')} \leqslant 1 \}.$$

Now consider the case $q \leqslant p$. Again using Hölder's inequality, we deduce

$$\sup_W \{ |\operatorname{tr}(YW)| : \|W\|_{(q',p')} \leqslant 1 \} \leqslant \|Y\|_{(q,p)}.$$

Thus it is sufficient to find W such that

$$|\operatorname{tr}(YW)| = \|Y\|_{(q,p)} \|W\|_{(q',p')}.$$

Consider the space M_{mn} with the $\|\cdot\|_{(q,p)}$ norm. $Y \in M_{mn}$ so, by a corollary of Hahn-Banach theorem, there exists a linear functional $f : M_{mn} \to \mathbb{C}$ such that $f(Y) = \|f\| \|Y\|_{(q,p)}$. But every linear functional f on M_{mn} can be written as $f(Z) = \operatorname{tr}(ZW)$ for some $W \in M_{mn}$. Hence we have

$$\begin{aligned} \|f\| &= \sup_Z \{ |f(Z)| : \|Z\|_{(q,p)} \leqslant 1 \} = \\ &= \sup_Z \{ |\operatorname{tr}(ZW)| : \|Z\|_{(q,p)} \leqslant 1 \} = \|W\|_{(q',p')}, \end{aligned}$$

where the last equality holds by the previous duality statement. Finally observe that

$$|\operatorname{tr}(YW)| = |f(Y)| = \|f\| \|Y\|_{(q,p)}$$

to get the desired equality.

e) Consider the sup norm first.

$$\|Y\|_{(p,p)} = \sup_{A,B} \left\{ \frac{\|(A \otimes I) Y (B \otimes I)\|_p}{\|A\|_\infty \|B\|_\infty} \right\}.$$

For $1/s = 1/q + 1/p$, we have $\|YW\|_s \leqslant \|Y\|_q \|W\|_p$. We use this property for the three matrices in the numerator of the fraction above:

$$\|Y\|_{(p,p)} \leqslant \sup_{A,B} \left\{ \frac{\|A \otimes I\|_\infty \|Y\|_p \|B \otimes I\|_\infty}{\|A\|_\infty \|B\|_\infty} \right\} = \|Y\|_p.$$

On the other hand, choosing $A = I$ and $B = I$ yields

$$\|Y\|_{(p,p)} \geqslant \frac{\|Y\|_p}{1 \cdot 1} = \|Y\|_p,$$

therefore we have equality. Now consider the inf norm.

$$\|Y\|_{(p,p)} = \inf \left\{ \sum_i \|A_i\|_\infty \|Z_i\|_p \|B_i\|_\infty : Y = \sum_i (A_i \otimes I) Z_i (B_i \otimes I) \right\}.$$

For any decomposition of Y

$$\begin{aligned} \|Y\|_p &\leqslant \sum_i \|(A_i \otimes I) Z_i (B_i \otimes I)\|_p \leqslant \\ &\leqslant \sum_i \|A_i\|_\infty \|Z_i\|_p \|B_i\|_\infty \leqslant \|Y\|_{(p,p)}. \end{aligned}$$

On the other hand, for the decomposition $Y = (I \otimes I) Y (I \otimes I)$ we get

$$\|Y\|_{(p,p)} \leqslant 1 \cdot \|Y\|_p \cdot 1 = \|Y\|_p,$$

therefore we have equality.

□

Let $p \geqslant q$. Using the equality $\|Y\|_{(p,p)} = \|Y\|_p$ from the theorem above, we can rewrite the (p,q) norm as

$$\|Y\|_{(p,q)} := \sup_{A,B} \left\{ \frac{\|(A \otimes I) Y (B \otimes I)\|_{(q,q)}}{\|A\|_{2r} \|B\|_{2r}} \right\};$$

similarly, the (q,p) norm becomes

$$\|Y\|_{(q,p)} := \inf \left\{ \sum_i \|A_i\|_{2r} \|B_i\|_{2r} \|Z_i\|_{(p,p)} : Y = \sum_i (A_i \otimes I) Z_i (B_i \otimes I) \right\}.$$

Notice that, in the first case, the pair (p,q) on the left hand side is decreased to (q,q) in the definition; in the second case, (q,p) is increased to (p,p). We now proceed to generalize the norm to the product of three spaces using the observation above. Consider the space $M_n \otimes M_m \otimes M_k$ and set $\|Y\|_{(p,p,p)} := \|Y\|_p$. Let $p \geqslant q$. If we change (q,q,p) to (p,p,p), we set

$$\|Y\|_{(q,q,p)} := \inf \left\{ \sum_i \|A_i \otimes B_i\|_{2r} \|C_i \otimes D_i\|_{2r} \|Z_i\|_{(p,p,p)} : Y = \sum_i (A_i \otimes B_i \otimes I) Z_i (C_i \otimes D_i \otimes I) \right\};$$

if we change (p,p,q) to (q,q,q), we set

$$\|Y\|_{(p,p,q)} := \sup_{A,B,C,D} \left\{ \frac{\|(A\otimes B\otimes I)Y(C\otimes D\otimes I)\|_{(q,q,q)}}{\|A\otimes B\|_{2r}\|C\otimes D\|_{2r}} \right\};$$

if we increase q to p in (q,p,s), we set

$$\|Y\|_{(q,p,s)} := \inf \left\{ \sum_i \|A_i\|_{2r}\|B_i\|_{2r}\|Z_i\|_{(p,p,s)} : Y = \sum_i (A_i\otimes I\otimes I)Z_i(B_i\otimes I\otimes I) \right\};$$

finally, if we decrease p to q in (p,q,s), we set

$$\|Y\|_{(p,q,s)} := \sup_{A,B} \left\{ \frac{\|(A\otimes I\otimes I)Y(B\otimes I\otimes I)\|_{(q,q,s)}}{\|A\|_{2r}\|B\|_{2r}} \right\}.$$

This process gives a norm that satisfies Hölder's inequality and duality. In the same way one can generalize to a product of more than three spaces.