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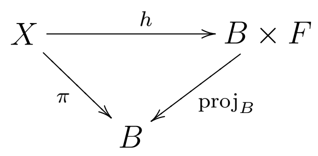
Why use perverse sheaves to study hypersurface singularities?

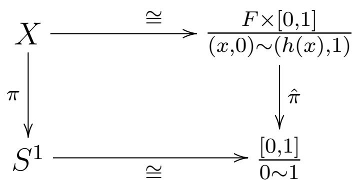
1 Basic Topology

Definition: A fibration $\pi : X \to B$ is *trivial* if it "looks like" a projection from $B \times F$; that is, if there is a homeomorphism $h : X \to B \times F$ making the following diagram commute:



Definition: A fibration $\pi : X \to B$ is *locally trivial* if for all $b \in B$ there is an open neighborhood U of b so that $\pi : \pi^{-1}(U) \to U$ is a trivial fibration.

Locally trivial fibrations over S^1 are particularly well-understood. Let $\pi : X \to S^1$ be locally trivial, and let $F = \pi^{-1}((1,0))$; since S^1 is path connected, all fibers of π are homeomorphic to F. Let $h : F \to F$ be a homeomorphism. Then we have the following commuting diagram:



In other words, all locally trivial fibrations over S^1 have the form on the right. For a given fibration, the map h is called a *characteristic homeomorphism* for the fibration. In general, the choice of h is not unique, but the induced map on cohomology $h^* : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ is unique.

2 Hypersurfaces

In this section, we will be looking at hypersurfaces in $\mathbb{C}^{n+1}$. Let $U \subset \mathbb{C}^{n+1}$ be open with $\underline{0} \in U$. Let $f : (U, \underline{0}) \to (\mathbb{C}, \underline{0})$ be a complex analytic map that fixes the origin and is not locally constant. Let $V(f) = f^{-1}(\underline{0})$. Then $V(f)$ has complex dimension n, so it is a hypersurface in $\mathbb{C}^{n+1}$. For our purposes, we'll focus on hypersurfaces of this type.

Example: $f = xy$. Then $V(f) = V(x) \cup V(y)$, which is not locally Euclidean at the origin. So $V(f)$ is not a manifold.

Example: $f = y^2 - x^3$. When we draw the picture "over $\mathbb{R}$", there is a cusp at the origin. So it looks like we get a topological manifold, but not a differentiable manifold.

In addition to the native topological structure of $V(f)$, we'd like to know the local ambient topology near the origin (where the singularity is). In other words, how does $V(f)$ imbed in U?

Milnor proved that for $\epsilon \ll 1$, we have $\text{Cone}(S_\epsilon, S_\epsilon \cap V(f)) \cong (B_\epsilon, B_\epsilon \cap V(f), \underline{0})$ if $V(f)$ has isolated singularities. Let $K = S_\epsilon \cap V(f)$. Then for the first example above ($f = xy$), we have K is the disjoint union of two circles (actually a Hopf link). For the second example ($f = y^2 - x^3$), we have $K \cong S^1$, but the embedding is nontrivial; in particular, it's a trefoil knot in S^3_ϵ.

By the implicit function theorem, if $\frac{\partial f}{\partial \underline{z}}|_p \neq \underline{0}$, then $f^{-1}(f(p)) = V(f - f(p))$ is a smooth (analytic) submanifold of U near p. Let Σf be the critical locus of f; i.e., $\Sigma f = V(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$. So the only chance $V(f - f(p))$ has to be non-smooth is at points in Σf. Of course, this condition isn't sufficient!

Our general question is: how can we work with singular spaces? Can we translate questions about them to questions about manifolds? A good way to approach this is by looking at $S_\epsilon - (S_\epsilon \cap V(f))$. By removing the points at which f is $\underline{0}$, we obtain a well-defined map $f/|f| : S_\epsilon - (S_\epsilon \cap V(f)) \to S^1 \subset \mathbb{C}$. Milnor proves that this is a locally trivial fibration, so we can apply the analysis we did at the beginning. We call the fiber the *Milnor fiber* $F_{f,\underline{0}}$. The unique map induced on $H^*(F_{f,\underline{0}}; \mathbb{Z})$ by a characteristic homeomorphism for this fiber is called the *Milnor monodromy*. Note that $F_{f,\underline{0}}$ is a smooth real $2n$- manifold.

It ends up being a little difficult to work with this fibration as written, but for $\delta \ll \epsilon$, this fibration is diffeomorphic to $f : \overset{\circ}{B}_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \to \partial \mathbb{D}_\delta$, where now the fiber is a complex n-manifold. In particular, this formulation compactifies nicely: just change $\overset{\circ}{B}_\epsilon$ to B_ϵ!

Suppose that the origin is an isolated critical point. Let $s = \dim_{\underline{0}} \Sigma f$. Milnor proves that if $s = 0$, then $F_{f,\underline{0}}$ has homotopy type of $\vee_\mu S^n$, the wedge of finitely many n-spheres. So the reduced cohomology of the fiber is 0 except in degree n. That is: $\tilde{H}^k(F_{f,\underline{0}}) = 0$ unless $k = n$, and $H^n(F_{f,\underline{0}}) \cong \mathbb{Z}^\mu$. What determines μ? μ is the degree of

$$\epsilon(\partial f/\partial \underline{z})/|(\partial f/\partial \underline{z})| : S_\epsilon \to S_\epsilon,$$

which is the same as $\dim_{\mathbb{C}} \frac{\mathbb{C}\{\underline{z}\}}{\langle \partial f/\partial z_0, \ldots, \partial f/\partial z_n \rangle}$.

Example: $f = y^2 - x^3$. Then

$$\mu_{\underline{0}}(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle -3x^2, 2y \rangle} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle x^2, y \rangle} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x\}}{\langle x^2 \rangle} = 2.$$

Example: $f = y^2 - x^3 - x^2$. Then

$$\mu_{\underline{0}}(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle -3x^2 - 2x, 2y\rangle} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x\}}{\langle 3x^2 + 2x\rangle} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x\}}{\langle (3x+2)x\rangle} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x\}}{\langle x\rangle} = 1.$$

What have we done so far? We see that if f has an isolated critical point at the origin, then there is an integer $\mu = \mu_{\underline{0}}(f)$ such that the Milnor fiber $F_{f,\underline{0}}$ is homotopic to a wedge of μ n-spheres. Furthermore, this fiber provides a lot of information about how $V(f)$ imbeds in S_ϵ. How can we extend this to the case when $s > 0$? Kato and Matsumoto proved in 1973 that $F_{f,\underline{0}}$ is $(n-s-1)$ connected; in other words, $\tilde{H}^{n-s}(F_{f,\underline{0}}), \ldots, \tilde{H}^n(F_{f,\underline{0}})$ are the only nonzero cohomology groups. Nevertheless, we have almost no hope of calculating these in general.

Now we'd like to find a way to deal with the case $s > 0$. For example, consider $f = y^2 - x^2 - tx^2$. Then its critical locus Σf is $V(-3x^2 - 2tx, 2y, -x^2) = V(x, y)$, which is the t-axis. So $s = \dim_{\underline{0}} \Sigma f = 1$. We can also think of this as a 1-parameter family of hypersurfaces f_t instead of a single hypersurface. From the examples above, we have $\mu_{\underline{0}}(f_t)$ is 1 if $t \neq 0$, and 2 if $t = 0$.

For any f, we can split $V(f)$ into strata: pieces where the topology is locally constant. For example, using the f above, we can split $V(f)$ into ($V(f)$ - t-axis) $\cup$ (t-axis - $\underline{0}$) $\cup$ ($\underline{0}$). In general, in a small enough neighborhood of $\underline{0}$, there are finitely mana strata, and each stratum is a manifold or can be analyzed using knowledge of isolated singularities.

To understand what happens when $s > 0$, we'll focus on the case $s = 1$ which will illustrate most of the pertinent issues. In this case, the only nonzero cohomology groups are $\tilde{H}^{n-1}(F_{f,\underline{0}})$ and $\tilde{H}^n(F_{f,\underline{0}})$. But even in this case, we can't calculate these groups in general. What we'd like to know is whether the Milnor fiber $F_{f_{t_0},\underline{p}}$ has any relation to $F_{f,\underline{0}}$, where $\underline{p} = (0, 0, t_0)$.

In other words, we're interested in how local data patches together into global data. This suggests that sheaves may be useful! In particular, we want a sheaf whose stalkes give us the cohomology of the Milnor fiber at each point. But usually you get a single ring or group for each stalk. So what we actually want is a differential complex of sheaves:

$$\underline{A}^\bullet = \cdots \to^{\partial^{-2}} \underline{A}^{-1} \to^{\partial^{-1}} \underline{A}^0 \to^{\partial^0} \underline{A}^1 \to^{\partial^1} \underline{A}^2 \to^{\partial^2} \cdots .$$

Whether we take cohomology first and then stalks, or vice versa, we get isomorphic groups. Let $\underline{H}^k(\underline{A}^\bullet)_x$ be the stalk cohomology at x. (From now on, unless said otherwise, we'll fix the base ring as $\mathbb{Z}$.)

We want our (complex of) sheaves $\underline{A}^\bullet$ to be bounded and constructible. In other words, we want

1. $\underline{A}^p = 0$ for $|p| \gg 0$
2. $H^i(\underline{A}^\bullet)_x$ is a finitely-generated module
3. $H^i(\underline{A}^\bullet)$ is locally constant along some complex analytic stratification of X.

Example: $\underline{\mathbb{Z}}^\bullet_x$, where the only nonzero stalk is $\underline{\mathbb{Z}}_x$ in degree 0.

Example: $\underline{\mathbb{Z}}^\bullet_x[n]$, where for any complex $\underline{A}^\bullet$, we can shift it to $\underline{A}^\bullet[d]$ defined by $(\underline{A}^\bullet[d])^i = \underline{A}^{i+d}$. So this example has the nonzero stalk in degree $-n$.

There are two complexes of sheaves that are particularly useful for our purposes. Let $\underline{A}^\bullet$ be a bounded, constructible complex of sheaves of $\mathbb{Z}$-modules on X. Let $f : X \to \mathbb{C}$ be complex analytic and not locally constant. Then we have $\psi_f[-1]\underline{A}^\bullet$, the "sheaf of nearby cycles", and $\phi_f[-1]\underline{A}^\bullet$, the "sheaf of vanishing cycles". Both of these are complexes of sheaves on $V(f)$. (No definition today!) These sheaves satisfy the hyper-cohomological relations:

$$H^i(\psi_f[-1]\underline{A}^\bullet)_x \cong \mathbb{H}^{i-1}(F_{f,x}; \underline{A}^\bullet)$$

and

$$H^i(\phi_f[-1]\underline{A}^\bullet)_x \cong \mathbb{H}^i(\overset{\circ}{B}_\epsilon(x), F_{f,x}; \underline{A}^\bullet).$$

If $\underline{A}^\bullet = \underline{\mathbb{Z}}^\bullet_x$, then $\mathbb{H}^{i-1}(F_{f,x}; \underline{A}^\bullet)$ is just the usual cohomology of the Milnor fiber, and $H^i(\phi[-1]\underline{A}^\bullet)_x \cong \tilde{H}^{i-1}(F_{f,x}; \mathbb{Z})$, so that the sheaf of vanishing cycles is just the reduced version of the sheaf of nearby cycles.

Definition: The i^{th} support of $\underline{A}^\bullet$, written $\text{supp}^i(\underline{A}^\bullet)$, is defined to be the closure of $\{x \in X \mid H^i(\underline{A}^\bullet)_x \neq 0\}$. The support of $\underline{A}^\bullet$, written $\text{supp}(\underline{A}^\bullet)$, is $\cup_i \text{supp}^i(\underline{A}^\bullet)$.

Definition: The i^{th} cosupport of $\underline{A}^\bullet$, written $\text{cosupp}^i(\underline{A}^\bullet)$, is the closure of $\{x \in X \mid H^i(\overset{\circ}{B}_\epsilon(x), \overset{\circ}{B}_\epsilon(x) - \{x\}; \underline{A}^\bullet) \neq 0\}$. Similarly, $\text{cosupp}(\underline{A}^\bullet) = \cup_i \text{cosupp}^i(\underline{A}^\bullet)$.

Definition: $\underline{A}^\bullet$ is a *perverse sheaf* if for every i, $\dim \text{supp}^{-i}(\underline{A}^\bullet) \leq i$ and $\dim \text{cosupp}^i(\underline{A}^\bullet) \leq i$.

Example: $\underline{\mathbb{Z}}^\bullet_x[n]$ is perverse provided X is a local complete intersection (where $n = \dim X$).

Perverse sheaves have a couple of nice properties:

1. If $\underline{\mathcal{P}}^\bullet$ is perverse, $H^i(\underline{\mathcal{P}}^\bullet)_x = 0$ unless $-\dim(\text{supp}\underline{\mathcal{P}}^\bullet) \leq i \leq 0$.
2. If $\underline{\mathcal{P}}^\bullet$ is perverse, then $\psi_f[-1]\underline{\mathcal{P}}^\bullet$ and $\phi_f[-1]\underline{\mathcal{P}}^\bullet$ are perverse on $V(f)$. We strongly prefer the latter, since it is smaller.

Finally, let's return to our function $f : U \to \mathbb{C}$. Let's assume $s = 1$, and let $\underline{\mathcal{P}}^\bullet = \phi_f[-1]\underline{\mathbb{Z}}^\bullet_\mu[n+1]$; this is a perverse sheaf. Then:

1. $\mathrm{supp}\underline{\mathcal{P}}^\bullet = \Sigma f$.
2. $\tilde{H}^i(F_{f,\underline{0}}; \underline{\mathbb{Z}}^\bullet[n+1]) \cong \tilde{H}^{i+n}(F_{f,\underline{0}})$.
3. The cosupport condition implies $\dim \mathrm{cosupp}^{-1}\underline{\mathcal{P}}^\bullet \leq -1$, so $\mathrm{cosupp}^{-1}\underline{\mathcal{P}}^\bullet = \emptyset$.

This last property implies that $\tilde{H}^{n-1}(F_{f,\underline{0}}; \mathbb{Z})$ sits in $\ker(\mathrm{id} - h^* : \tilde{H}^{n-1}(F_{f_{t_0}}; \mathbb{Z}) \to \tilde{H}^{n-1}(F_{f_{t_0}}; \mathbb{Z}))$.