Total Positivity and cluster Algebras.

reference :

Total positivity : Teses and parametrizations. (Fomin - Zelevinsky)

Def①: A matrix $X \in Gl_n(\mathbb{C})$ is called totally positive if all its minors (determinets of submatrixes) are positive.
i.e $\Delta_{IJ}(X) > 0$ $\quad I, J \subset [1, \cdots n]$ $\quad |I| = |J|$

Facts①: An nxn matrix has $\binom{2n}{n} - 1$ minors.

Ex①

$$X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \text{minors: } a, b, c, d, ad - bc = \det(X)$$

Question : Can we test as few as possible of the minors to tell a matrix is TP (totally positive) ?

Ex②: By observe ex①, we know TP testing set $\{a, b, c, \det(X)\}$ is enough. Since $d = \dfrac{\det(X) + cb}{a}$

Prop①: A minimum TP testing set has size n^2 for nxn matrix.

In order to show this is true, we need the following :

Double Wiring Diagrams and total positivity criteria (DWD)

We will develope this DWD by showing an example:

Ex③. Still $X \in GL_3(\mathbb{C})$ $(n=3)$

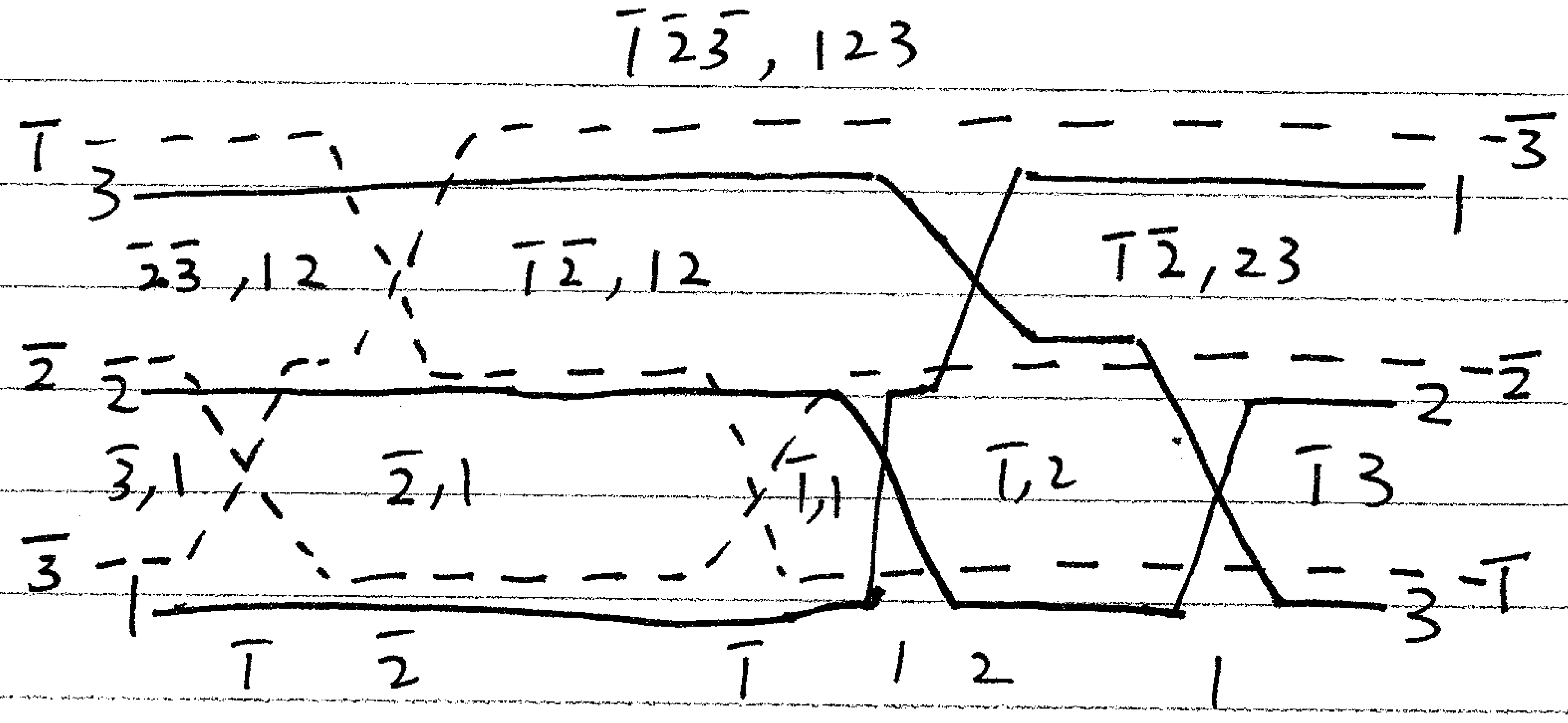


fig #1

step 1: writing down 3 to 1 on left hand side and 1 to 3 on right hand side
$\bar{1}$ to $\bar{3}$ — // — and $\bar{3}$ to $\bar{1}$ — // —

step 2: Connect numbers correspondingly using straight line and slash line.
Here, to distinguish numbers, we use line for 1, 2, 3
and we use dote line for $\bar{1}, \bar{2}, \bar{3}$

step 3: We can see that several chambers has fromed.
and we will mark the chambers by the lines below it.
The numbers with bar come first and numbers without bar come second.

Remarks: ① Chambers. can be defined as horizontal segments between consecutive crossings at the same level.
② There will always be a unbounded chamber $(\bar{1}\bar{2}\bar{3}, 123)$.
③ Each pair of lines of same kind (line, dote line) intersect exactly once.

In general, # of ④ with ③ the intersection is $2\binom{n}{2} = n(n-1)$

So the total number of chambers is $n(n-1) + \underset{\text{top}}{\underline{1}} + \underset{\text{left hand}}{\underline{(n-1)}} = n^2$.

Step 4: Encode a DWD by the word of length $n(n-1)$ in the alphabet $\{1, \cdots n-1\}, \bar{1}, \cdots, \overline{n-1}\}$

Write down the number below each intersect.

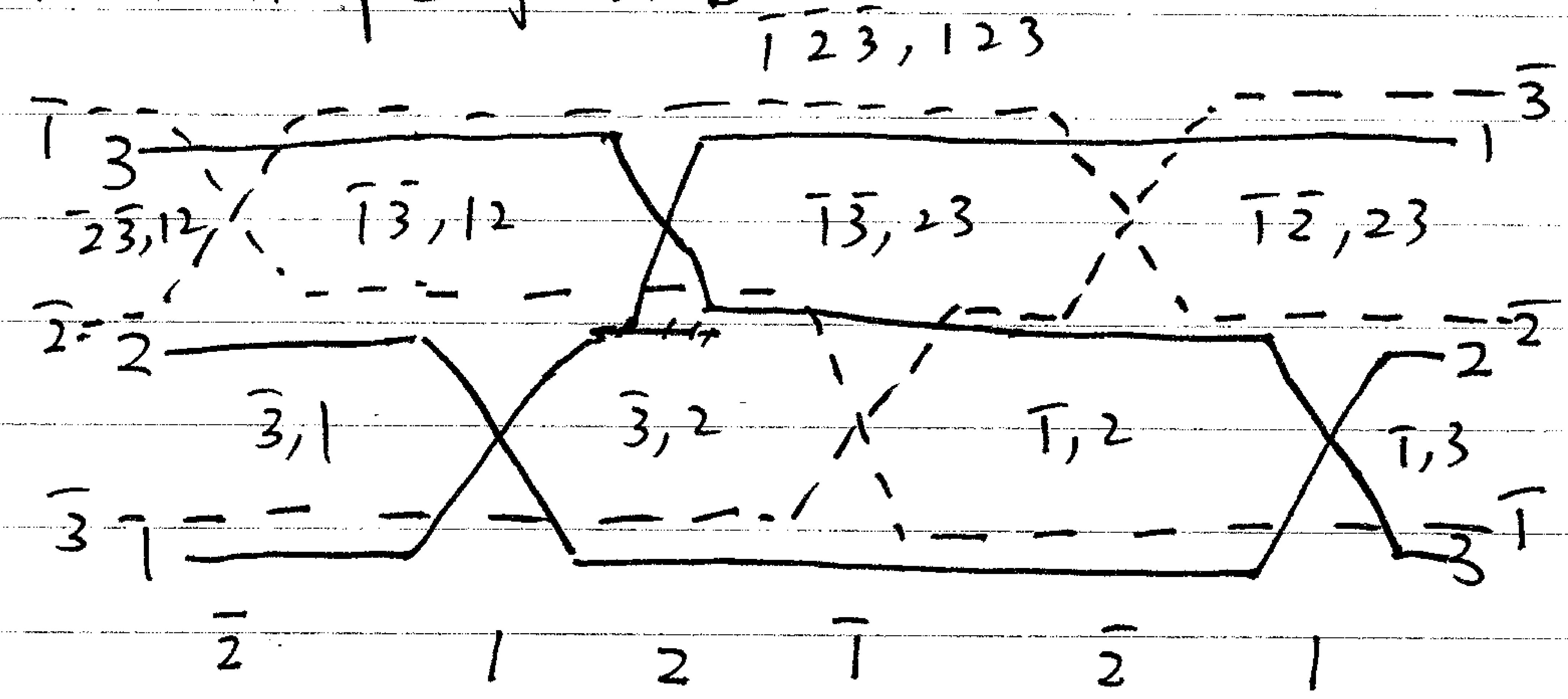
Step 5: Construct minors from chambers, For chamber $\bar{1}\bar{2}\bar{3}, 123 \Rightarrow \Delta_{\bar{1}\bar{2}\bar{3}, 123}$

For chamber $\bar{2}\bar{3}, 12 \Rightarrow \Delta_{\bar{2}\bar{3}, 12}$

etc.

Def: Two DWD are isotopic if they have the same set of Chamber minors.

Ex④: Another example of DWD



Prop②: Two DWD are isotopic iff their encodings can be transfered by the following three movements:

① $\cdots\, i\, j \cdots \longleftrightarrow \cdots\, j\, i \cdots \qquad |i-j| \geq 2$

② $\cdots\, \bar{i}\, \bar{j} \cdots \longleftrightarrow \cdots\, \bar{j}\, \bar{i} \cdots \qquad |\bar{i}-\bar{j}| \geq 2$

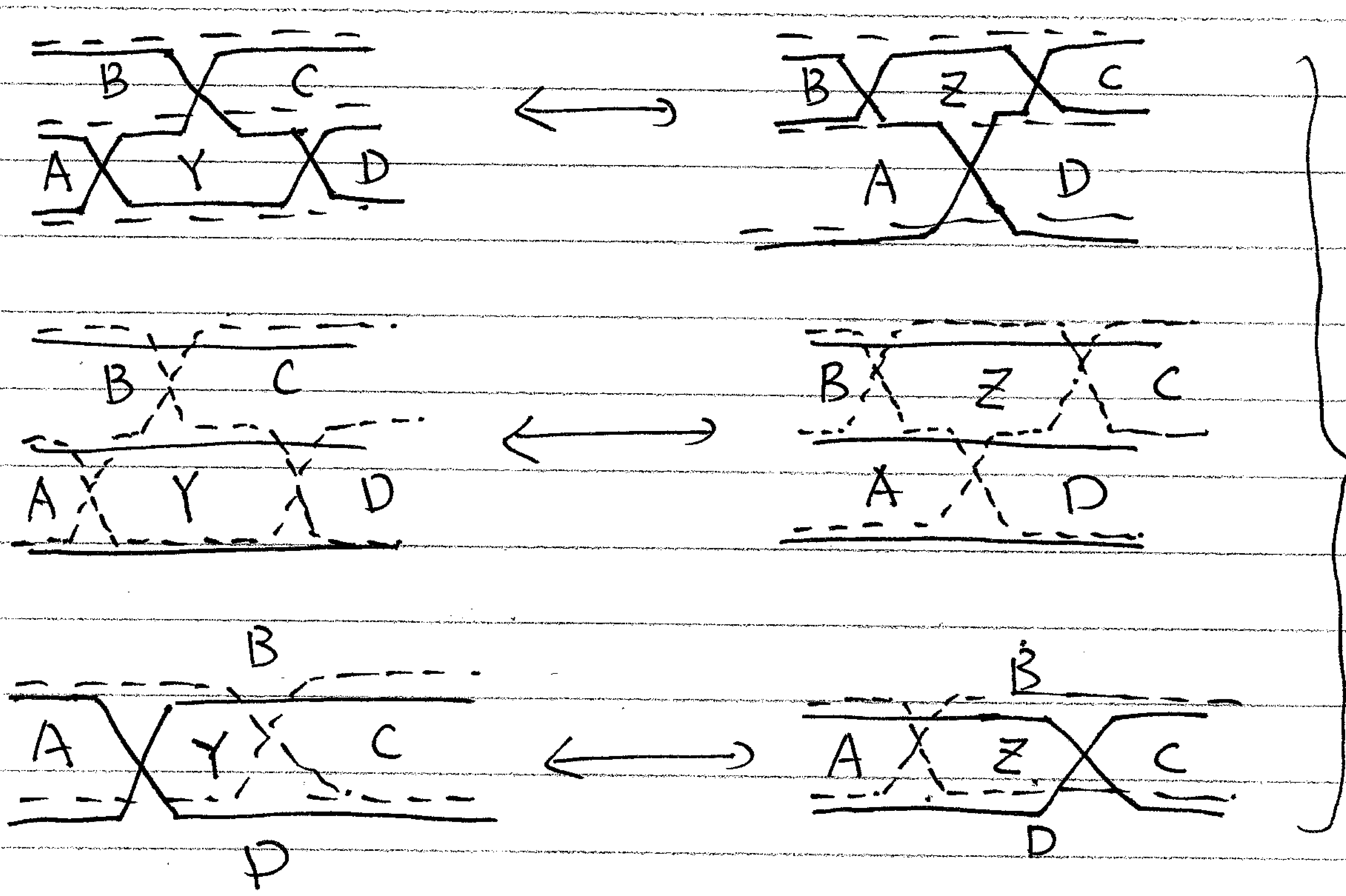
③ $\cdots\, i\, \bar{j} \cdots \longleftrightarrow \cdots\, \bar{j}\, i \cdots \qquad i \neq j$

Thm①: For Every double wiring diagram, the collection of its chamber minors is a TP-test.

Prop③ Every minor can be written as a subtraction-free rational expression in the chamber minors at a given DWD.

Prop④. Any two DWD can be transformed into each other by a sequence of "local moves"

Three local moves : $n=3$



$YZ = AC + BD$

From these we can construct cluster algebras by the following graph :

34-vertex (non-isotopic DWD)

18-vertices of deg 4 (DWDs that allow 4 local moves)

16 vertices of deg 3 (~~—~~ = — 3 ~~— —~~)

8 8

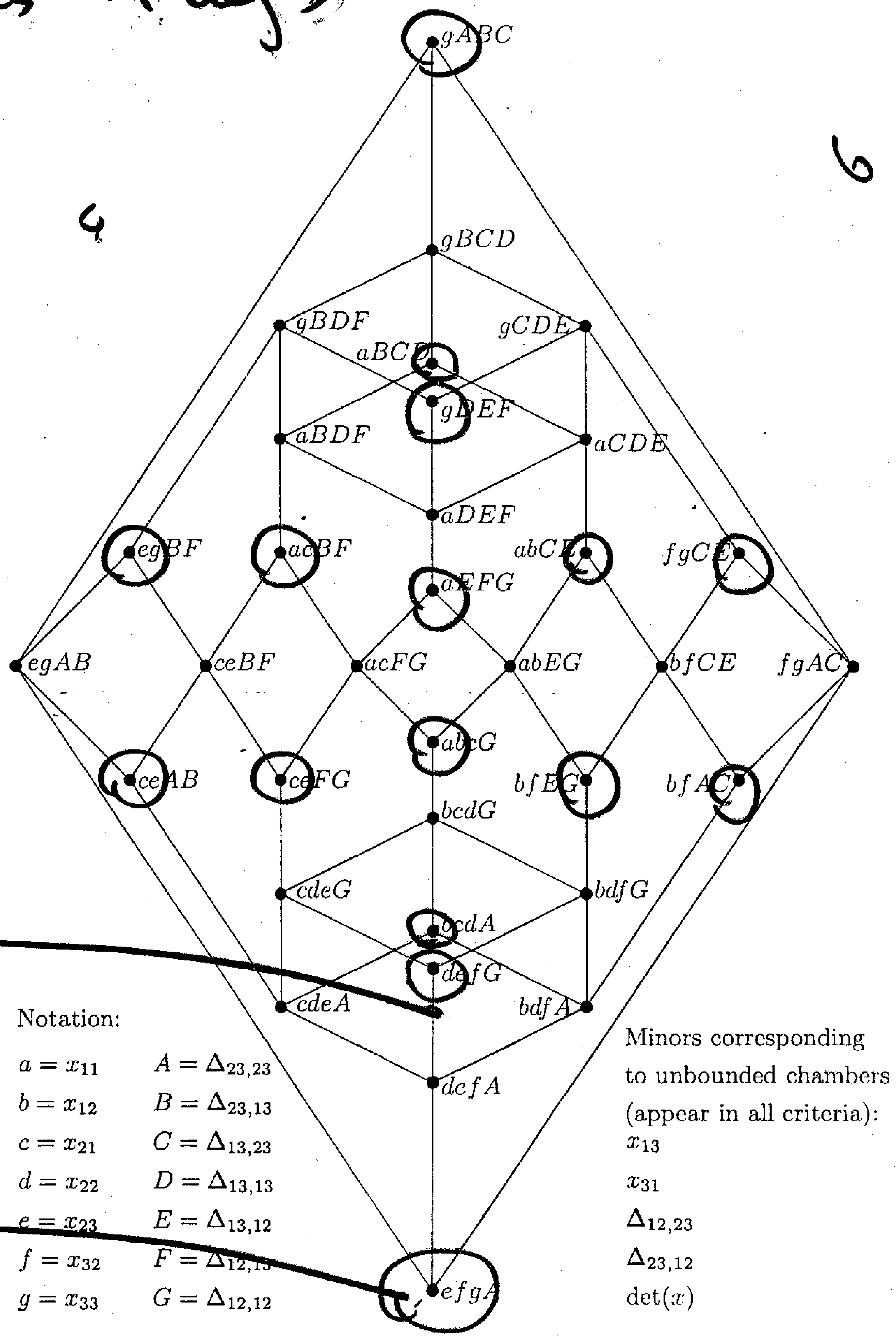


FIGURE 8. Total positivity criteria for GL_3

Two "hidden" variables:

α: $x_{11} \cdot \Delta_{23,23} - \det(x)$

β: $x_{33} \cdot \Delta_{12,12} - \det(x)$