Hamiltonian Dynamics and Computer Vision

Jayant Shah

Tapas Seminar, June 18, 2009

Suppose we have several pictures of different human brains. We'd like to be able to determine what an "average brain" should look like, and how large the variance is. This problem generalizes nicely to arbitrary manifolds. In order to average two manifolds, we want to find some shortest path from one to the other; that is, we want a way to smoothly interpolate between the two manifolds. We will consider this interpolation problem here, focusing on closed loops.

Our loops will be functions $c(\theta) : S^1 \to \mathbb{R}^2$ that are C^∞ immersions; the space of all such loops is $\mathrm{Imm} := \mathrm{Imm}(S^1, \mathbb{R}^2)$. The space $\mathrm{Diff} := \mathrm{Diff}(S^1)$ of diffeomorphisms of S^1 acts on Imm on the right in the obvious way: $c\phi(\theta) = c(\phi(\theta))$. In other words, $\phi \in \mathrm{Diff}$ reparameterizes the curve c. For our application, we really only care about the image of c, so instead of working in Imm, we want to work in $B_i := \mathrm{Imm}/\mathrm{Diff}$. However, for most of the constructions to follow, we will stay in Imm whenever possible and occasionally descend to B_i.

Having identified our space, we now need to pick our Riemannian metric. Furthermore, we want it to be invariant under the action of Diff; in other words, we want it to only depend on the image of the curve, not on the parameterization. Now, for the tangent bundle $T\mathrm{Imm}(S^1, \mathbb{R}^2)$, we have $T\mathrm{Imm}(S^1, \mathbb{R}^2) \simeq \mathrm{Imm}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2)$. Now, we have many choices of metric; for example, we can take

$$G_c(h, k) = \int_s (h \cdot k + A D_s^n h \cdot D_s^n k) ds,$$

the Sobolev metric of order n, where n is a nonnegative integer, A is some constant, and we are integrating with respect to arc length. There are other choices of metric as well. Here, our concern is not with which metric to use – a question whose answer may depend on the application – but rather, on how to calculate the geodesic between two curves. Our goal is to set up an appropriate momentum map to make this possible.

We've already noted the structure of the tangent bundle to Imm; the cotangent bundle is $\mathrm{Imm} \times \mathcal{D}^2(S^1)$, where the latter part is the space of distributions on S^1. There is a natural map $G : T(\mathrm{Imm}) \to T^*(\mathrm{Imm})$ by $G(c, h) = (c, G_c(h, \cdot))$. Note that since our manifold is infinite-dimensional, this map is never surjective. This map induces a map

$DG : T(T(\text{Imm})) \to T(T^*(\text{Imm}))$. The space on the left is

$$\text{Imm} \times C^\infty(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2),$$

and the space on the right is

$$\text{Imm} \times \mathcal{D}^2(S^1) \times C^\infty(S^1, \mathbb{R}^2) \times \mathcal{D}^2(S^1).$$

The map DG sends $(c, h; k, \ell)$ to $(c, G_c(h, \cdot); k, d_{k,\ell}G_c(h, \cdot))$, where $d_{k,\ell}G_c(h, \cdot) = D_{c,k}G_c(h, \cdot) + G_c(\ell, \cdot)$; the first component is the change in G due to deformation of c in the direction of k, and the second component is the change in G due to deformation of h in the direction of ℓ.

We always have a natural symplectic form on the cotangent bundle. We start with the natural 1-form Θ on $T^*(\text{Imm})$ defined by $\Theta(c, \alpha, h, \beta) = \langle \alpha, h \rangle$; i.e., we evaluate the distribution α on h. Now, we can pull this back to $G^*(\Theta)$ on $T(\text{Imm})$ by $G^*(\Theta)(c, h, k, \ell) = G_c(h, k)$. And now we get a symplectic 2-form $\omega = -dG^*(\Theta)$, where

$$\omega_{c,h}((k_1, \ell_1)(k_2, \ell_2)) = -d_{(k_1,\ell_1)}G_c(h, k_2) + d_{(k_2,\ell_2)}G_c(h, k_1).$$

Since we have a symplectic structure, we can get Hamiltonian flow and a momentum map, as we shall see shortly.

Suppose we have a Lie group with Lie algebra $\mathfrak{g}$ acting on Imm from the right by isometries (for example, Diff). Let r^g denote the action by $g \in \}$. Let $\chi(\text{Imm})$ be the collection of tangent vector fields to Imm. Then we get a Lie algebra homomorphism $\zeta : \} \to \chi(\text{Imm})$ by sending x to ζ_x defined by

$$\zeta_x(c) = \partial_t \mid_{t=0} r^{\exp(tx)}(c).$$

Now, using this we can define a momentum map $j : \} \to T^*(\text{Imm})$ by sending x to j_x, where $j_x(c, h) = G_c(\zeta_x, h)$. By the conservation of momentum principle, we have that along any geodesic $c(t)$ in Imm, the momentum is constant; in other words, that $G_c(\zeta_x, c(t))$ does not depend on t.

A primary application of this approach is to consider plane curves modulo rotations, translations, and scales. It is convenient then to think of our curves $c \in \text{Imm}$ as being in the complex plane $\mathbb{C}$ instead of $\mathbb{R}^2$. Our vertical vectors then are of the form $\alpha c + \beta$, with $\alpha, \beta \in \mathbb{C}$. The horizontal vectors are those h such that

$$\int h \, ds = \int h_s \cdot v \, ds = \int h_s \cdot n \, ds = 0,$$

where v is the unit tangent vector field on c and n the unit normal vector field. Then a metric on the horizontal vectors will induce a metric on the quotient (of plane curves modulo rotations, translations, and scales).

Given our curve $c : S^1 \to \mathbb{C}$, we have $c_s = v$ and $ic_s = n$. Our horizontal vectors have a decomposition

$$h_s = (h_s \cdot v)v + (h_s \cdot n)n = [(h_s \cdot v) + i(h_s \cdot n)]c_s,$$

and for convenience, we will write h_s/c_s to mean $(h_s \cdot v) + i(h_s \cdot n)$. Now we define our bilinear form on horizontal vectors by

$$\langle m, h \rangle = \ell \int_c \left(\frac{m_s}{c_s}\right)_s \cdot \left(\frac{h_s}{c_s}\right)_s \, ds.$$

This vanishes on vertical vectors, but is nondegenerate on horizontal vectors, so we get a metric on the quotient, as desired.

Now, given a curve c, its energy is

$$E(c) = \frac{1}{2} \int_0^1 \langle c_t, c_t \rangle \, dt.$$

To find a geodesic, we set the first derivative of this equal to 0, getting

$$D_m E(c) = -\int_0^1 \langle m, \gamma \rangle \, dt,$$

where γ is geodesic curvature. This gives us a complicated equation to solve, but we can rewrite it using momentum.

Starting with

$$\langle m, h \rangle = \int_c \left(\frac{m_s}{c_s}\right)_s \cdot \left(\frac{h_s}{c_s}\right)_s \, ds,$$

we can integrate by parts to obtain

$$\int_c m \cdot [(m_s \cdot v)_{ss} v + (h_s \cdot n)_s s n]_s \, ds.$$

Now, let x be in the Lie algebra of $\text{Diff}(S^1)$. We can write $x = bv$, and we see that h is horizontal if and only if $\langle bv, h \rangle = 0$.

We now have

$$G_c(\zeta_x, c_t) = \int_c \langle x, c_t \rangle \, ds = \int_c bv \cdot u \, ds,$$

where the u is the momentum vector along the path:

$$[(c_{ts} \cdot v)_{ss} v + (c_{ts} \cdot n)_{ss} n]_s.$$

So our path is horizontal with respect to Diff if and only if $v \cdot u = 0$, which is to say that $u = a(t)n$. So we end up with the following geodesic equation:

$$a_t + 2(c_{ts} \cdot v)a = (cts \cdot v)_{ss}(c_{ts} \cdot n)_s - (cts \cdot n)_{ss}(c_{ts} \cdot v)_s + \frac{\kappa}{2}[(c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2]^0.$$