Cluster algebras of geometric type (Fomin / Zelevinsky)

- $m \geq$ n positive integers

- ambient field $\mathcal{F}$ of rational functions over $\mathbb{Q}(x_{n+1}, \ldots, x_m)$ in n independent variables

Definition: A **seed** in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}}, \tilde{B})$ where

1. $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ form a free generating set for $\mathcal{F}$. ($\tilde{\mathbf{x}}$ is the **extended cluster**.)

2. $\tilde{B} = (b_{ij})$ is an $m \times n$ integer matrix such that the submatrix B consisting of the top n rows of $\tilde{B}$ (called the **principal part**) is skew-symmetrizable.

$\tilde{B}$ is the **exchange matrix**, $x_1, \ldots, x_n$ are the **cluster variables** in the seed, $x_{n+1}, \ldots, x_m$ are the **coefficient variables**.

Definition: $(\tilde{\mathbf{x}}, \tilde{B})$ seed, $\tilde{B} = (b_{ij})$. For $k \in [1, n]$, a **seed mutation** in direction k transforms $(\tilde{\mathbf{x}}, \tilde{B})$ into the pair $(\tilde{\mathbf{x}}', \mu_k(\tilde{B}))$ given by:

1. $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}} - \{x_k\} \cup \{x'_k\}$, where

$$x'_k = x_k^{-1} \left(\prod_{\substack{i \in [m] \\ b_{ik} > 0}} x_i^{b_{ik}} + \prod_{\substack{i \in [m] \\ b_{ik} < 0}} x_i^{-b_{ik}} \right)$$

   This is an **exchange relation**.

2. (matrix mutation) $\mu_k(\tilde{B}) = (b'_{ij})$ is an $m \times n$ matrix, where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + |b_{kj}|b_{ik}}{2} & \text{otherwise.} \end{cases}$$

Let $\mathbb{T}_n$ be the n-regular tree with edges labeled by $1, \ldots, n$ so that the n edges incident to a vertex receive different labels.

Start with an initial seed $(\tilde{x}_0, \tilde{B}^0)$ at $t_0 \in \mathbb{T}_n$, and mutate to every vertex $t \in \mathbb{T}_n$.

Denote the seed at $t \in \mathbb{T}_n$ by $(\tilde{x}_t, \tilde{B}^t)$, where

$$\tilde{x}_t = (x_{1;t}, \ldots, x_{n;t}, x_{n+1;t}, \ldots, x_{m;t})$$

$$\tilde{B}^t = (b_{ij}^t)$$

The **cluster algebra** $\mathcal{A}(\tilde{x}_0, \tilde{B}^0) = \mathcal{A}(\tilde{B}^0)$ is the $\mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables.

Quantum Cluster Algebras (Berenstein / Zelevinsky)

A **quantum cluster algebra** is a certain noncommutative deformation of a cluster algebra using an additional variable q.

Ambient field is $\mathcal{F}_q$, the skew field of fractions of $\mathbf{Z}[q^{\pm 1}, Y_1, \ldots, Y_m]$, where $Y_1, \ldots, Y_m$ are algebraically independent variables which are quasi-commutative, i.e. $Y_i Y_j = q^{\lambda_{ij}} Y_j Y_i$ for some integers λ_{ij}.

Definition: A **quantum seed** in $\mathcal{F}_q$ is a pair $(\tilde{\mathbf{X}}, \tilde{B})$, where

1. $\tilde{\mathbf{X}} = (X_1, \ldots, X_m)$ is a free generating set such that $X_i X_j = q^{\lambda_{ij}} X_j X_i$ for some $\lambda_{ij} \in \mathbb{Z}$.

2. $\tilde{B}$ is an $m \times n$ integer matrix which is compatible with $\Lambda = (\lambda_{ij})$, i.e. $\tilde{B}^T \Lambda = (D|0)$, where D is a diagonal matrix with positive diagonal entries.

Notation: If $X_1, \ldots, X_m$ satisfy $X_i X_j = q^{\lambda_{ij}} X_j X_i$, then for $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$, let

$$X^{\mathbf{c}} = (q^{\frac{1}{2}\sum_{i>j} \lambda_{ij} c_i c_j}) X_1^{c_1} \cdots X_m^{c_m}$$

Definition: $(\tilde{\mathbf{X}}, \tilde{B})$ quantum seed. For $k \in [1, n]$, the **quantum seed mutation** in direction k transforms $(\tilde{\mathbf{X}}, \tilde{B})$ into $(\tilde{\mathbf{X}}', \mu_k(\tilde{B}))$, where $\tilde{\mathbf{X}}' = \tilde{\mathbf{X}} - \{X_k\} \cup \{X_k'\}$, and

$$X_k' = X^{-e_k + \sum_{b_{ik}>0} b_{ik} e_i} + X^{-e_k - \sum_{b_{ik}<0} b_{ik} e_i}$$

The quantum cluster algebra $\mathcal{A}$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}, Y_{n+1}^{\pm 1}, \ldots, Y_m^{\pm 1}]$-subalgebra of $\mathcal{F}_q$ generated by all cluster variables.

Note: Setting $q = 1$ in the quantum cluster algebra $\mathcal{A}$ with initial seed $(\tilde{\mathbf{X}}^0, \tilde{B}^0)$ yields the cluster algebra $\mathcal{A}(\tilde{B}^0)$.

F-polynomials (Fomin / Zelevinsky)

Fix an $n \times n$ skew-symmetrizable matrix $B^0 = (b_{ij})$

Assume any cluster algebra below has the property that the exchange matrix $\tilde{B}^0 = (b_{ij})$ at t_0 has principal part B^0.

Definition: The cluster algebra with **principal coefficients** is the cluster algebra with initial seed $(\tilde{x} = (x_1, \ldots, x_n, y_1, \ldots, y_n), \tilde{B}^0)$, where

$$\tilde{B}^0 = \begin{pmatrix} B^0 \\ I_n \end{pmatrix}$$

This cluster algebra is denoted by $\mathcal{A}_\bullet = \mathcal{A}_\bullet(B^0, t_0)$ or $\mathcal{A}_\bullet(B^0)$.

Definition: In $\mathcal{A}_\bullet$, any cluster variable $x_{j;t}$ can be expressed as a subtraction-free rational function in $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Denote this rational expression by

$$X_{j;t} = X_{j;t}^{B^0;t_0} \in \mathbb{Q}_{\mathsf{sf}}(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Let $F_{j;t} = F_{j;t}^{B^0;t_0} := X_{j;t}(1, \ldots, 1, y_1, \ldots, y_n) \in \mathbb{Q}_{\mathsf{sf}}(y_1, \ldots, y_n)$. $F_{j;t}$ is an F-**polynomial**.

Proposition: $F_{j;t} \in \mathbb{Z}[y_1, \ldots, y_n]$.

$\mathbb{Z}^n$-**grading on** $\mathcal{A}_\bullet \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]$:

$$deg(x_i) = \mathbf{e}_i$$

$$deg(y_i) = -\mathbf{b}_i^0$$

(where $\mathbf{b}_i^0$ denotes the ith column of B^0).

Proposition. Under this grading, all cluster variables in $\mathcal{A}_\bullet$ are homogeneous elements.

Write $\mathbf{g}_{j;t}$ for the $\mathbb{Z}^n$ degree of $x_{j;t}$. This is called the **g-vector.**

Notation: In any cluster algebra $\mathcal{A}$ with initial extended cluster $(x_1, \ldots, x_m)$, write

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}}.$$

$$\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}}.$$

If P is a subtraction-free polynomial in n variables, then let

$$P|_{Trop(x_{n+1},\ldots,x_m)}(y_1, \ldots, y_n)$$

denote the monomial in $x_{n+1}, \ldots, x_m$ obtained by plugging $y_1, \ldots, y_n$ into P, and replacing regular addition $(+)$ by "tropical addition" $(\oplus)$ defined on pairs of Laurent monomials in the variables $x_{n+1}, \ldots, x_m$:

$$\left(\prod_{i=n+1}^{m} x_i^{a_i}\right) \oplus \left(\prod_{i=n+1}^{m} x_i^{b_i}\right) = \prod_{i=n+1}^{m} x_i^{min(a_i,b_i)}$$

Theorem: Let $\mathcal{A}$ be a cluster algebra such that the extended cluster in the initial seed at t_0 is given by $(x_1, \ldots, x_m)$. Then the cluster variable $x_{j;t} \in \mathcal{A}$ can be given by

$$x_{j;t} = \frac{F_{j;t}(\widehat{y}_1, \ldots, \widehat{y}_n)}{F_{j;t}|_{Trop(x_{n+1},\ldots,x_m)}(y_1, \ldots, y_n)} x_1^{g_1} \ldots x_n^{g_n},$$

where $\mathbf{g}_{j;t} = (g_1, \ldots, g_n)$.

Remark: If $\mathcal{A} = \mathcal{A}_\bullet$, then $F_{j;t}|_{Trop(x_{n+1},\ldots,x_m)}(y_1, \ldots, y_n) = 1$.

Quantum F-polynomials

Fix the following:

- a diagonal matrix D with positive diagonal entries $d_1, \ldots, d_n$ such that DB^0 is skew-symmetric

- an $n \times n$ skew-symmetric matrix $\Lambda_0 = (\lambda_{ij})$

Assume that all quantum cluster algebras below satisfy:

- the exchange matrix $\tilde{B}^0$ at t_0 has principal part B^0.

- the initial extended cluster is $X_1, \ldots, X_m$, and $X_i X_j = q^{\lambda_{ij}} X_j X_i$ for all $1 \le i, j \le n$

Proposition: There exists a unique quantization of $\mathcal{A}_\bullet = \mathcal{A}_\bullet(B^0, t_0)$ such that the quasi-commutation relations of the initial cluster variables are given by Λ_0 and the compatibility condition is satisfied.

Call this quantum cluster algebra $\mathcal{A}_\bullet = \mathcal{A}_\bullet(B^0, D, \Lambda_0, t_0)$.

Notation: For any quantum cluster algebra, define $\widehat{Y}_j = \mathbf{X}^{\mathbf{b}^j}$, where $\mathbf{b}^j$ is the jth column of $\tilde{B}^0$. These elements satisfy the following quasi-commutation relations: $\widehat{Y}_i\widehat{Y}_j = q^{d_i b_{ij}}\widehat{Y}_j\widehat{Y}_i$

We will consider rational functions in quasi-commuting variables $Z_1, \ldots, Z_n$, with the quasi-commutation relations given by:

$$Z_i Z_j = q^{d_i b_{ij}} Z_j Z_i$$

Theorem / Definition: For each $j \in [1, n]$, $t \in \mathbb{T}_n$, there exists a unique rational function $F_{j;t}$ in variables $Z_1, \dots, Z_n$ with coefficients in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ such that in $\mathcal{A}_\bullet$,

$$X_{j;t} = F_{j;t}(\widehat{Y}_1, \dots, \widehat{Y}_n)\mathbf{X}^{\mathbf{g}_{j;t}}.$$

We call $F_{j;t}$ the **quantum F-polynomial**.

Theorem. In any quantum cluster algebra $\mathcal{A}$, the cluster variables $X_{j;t}$ can be expressed in the form

$$X_{j;t} = q^{\lambda_{j;t}} F_{j;t}(\widehat{Y}_1, \dots, \widehat{Y}_n)\mathbf{X}^{\mathbf{h}_{j;t}},$$

for some $\mathbf{h}_{j;t} \in \mathbb{Z}^m$, $\lambda_{j;t} \in \frac{1}{2}\mathbb{Z}$.

F-polynomials in classical types

Let B^0 be an acyclic $n \times n$ exchange matrix of type A_n, B_n, C_n, or D_n.

Theorem (Fomin, Zelevinsky) The cluster variables not in the initial cluster are in bijective correspondence with the positive roots.

Φ_+: Positive roots in classical types

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots.

Type A_n: $\alpha_i + \ldots + \alpha_j$ $(1 \le i \le j \le n)$

Type B_n: $\alpha_i + \ldots + \alpha_j$ $(1 \le i \le j \le n)$,
$\alpha_i + \ldots + \alpha_n + \ldots + \alpha_j$ $(1 \le i \le j \le n)$

Type C_n: $\alpha_i + \ldots + \alpha_j$ $(1 \le i \le j \le n)$,
$\alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_n$ $(1 \le i < j \le n)$

Type D_n:

$\alpha_i + \ldots + \alpha_j$ $(1 \le i \le j \le n, (i,j) \ne (n-1,n))$,
$\alpha_i + \ldots + \alpha_{n-2} + \alpha_n$ $(1 \le i \le n-2)$,
$\alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
$(1 \le i < j \le n-2)$

Fix $\alpha = \sum_{i=1}^{n} a_i \alpha_i \in \Phi_+$. Write F_α for the F-polynomial corresponding to α.

Let Q^0 be the quiver on the vertices $[1, n]$ with $i \to j$ iff $b_{ij} < 0$ (no multiple arrows).

Let $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ such that $0 \leq e_i \leq a_i$ for all i.

Say an arrow $i \to j$ in Q^0 is **acceptable** if $e_i - e_j \leq \max(a_i - a_j, 0)$.

An arrow $i \to j$ is **critical** if either $(a_i, e_i, a_j, e_j) = (2, 1, 1, 0)$ or $(2, 1, 1, 1)$.

Let S be the induced subgraph of Q^0 on the vertices $\{i : (a_i, e_i) = (2, 1)\}$.

For a component C of S, let $\nu(C)$ be the number of critical arrows with a vertex in C.

Theorem 1 *Let* $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$. *The coefficient of* $u_1^{e_1} \ldots u_n^{e_n}$ *in* F_α *is nonzero if and only if*

1. $0 \leq e_i \leq a_i$ *for all* i;

2. *all arrows in* Q^0 *are acceptable;*

3. $\nu(C) \leq 1$ *for all components* C *of* S;

4. *If* B^0 *is of type* B_n, *then*

   (a) $e_n = 1$ *and* $n \to n-1$ *in* Q^0 *implies* $e_{n-1} = 2$;

   (b) $e_{n-1} \geq 1$ *and* $n-1 \to n$ *in* Q^0 *implies* $e_n = 1$.

In this case, the coefficient of $u_1^{e_1} \ldots u_n^{e_n}$ *is* 2^c, *where* c *is the number of components* C *of* S *such that* $\nu(C) = 0$.

Theorem 2 *The* **g***-vector corresponding to* α *is given by*

$$-\sum_{i\in[1,n]} a_i \mathbf{e}_i + \sum_{i\in[1,n]} \sum_{j\in[1,n]} a_i [-b_{ji}]_+ \mathbf{e}_j .$$

Notation: For $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$, define

$$Z^{\mathbf{e}} = q^{\frac{1}{2}(\sum_{1 \le i < j \le n} d_j b_{ji} e_i e_j)} Z_1^{e_1} \cdots Z_n^{e_n}.$$

Theorem 3 *Let B^0 be of type A_n or D_n, and let $D = dI_n$. Then $Z^{\mathbf{e}}$ occurs with nonzero coefficient in the quantum F-polynomial F_α if and only if conditions (1)-(4) above are satisfied. In this case, the coefficient is*

$$q^{\frac{d}{2}(-\mathbf{g}_\alpha \cdot \mathbf{e})}(q^{\frac{d}{2}} + q^{-\frac{d}{2}})^c = q^{\frac{d}{2}(-\mathbf{g}_\alpha \cdot \mathbf{e} - c)}(1 + q^d)^c,$$

where $\mathbf{g}_\alpha = \mathbf{g}_\alpha^{B^0;t_0}$ is the $\mathbf{g}$-vector corresponding to α.

If the $a_i \in \{0, 1\}$ for all i, then $c = 0$ and $-\mathbf{g}_\alpha \cdot \mathbf{e}$ is equal to the number of components in the subgraph of Q^0 induced by $\{i \in [1, n] : e_i = 1\}$.