

# Introduction to toric topology

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## 1 Toric varieties

**Definition:** A *toric variety* (TV) is a normal algebraic variety  $X$  containing the algebraic torus  $(\mathbb{C}^*)^n$  as a dense open subset, such that the natural action of  $(\mathbb{C}^*)^n$  on itself extends to an action on  $X$ .

**Definition:** A *cone* in  $\mathbb{R}^n$  is the positive span of finitely many vectors. If the vectors are rational, we say that the cone is *rational*, or a *lattice cone*.

**Definition:** A *fan*  $\Sigma$  is a collection of rational cones such that

1. If  $\sigma, \tau \in \Sigma$ , then  $\sigma \cap \tau \in \Sigma$ ,
2. If  $\tau \in \Sigma$  and  $\sigma$  is a cone such that  $\sigma \leq \tau$ , then  $\sigma \in \Sigma$ .

(Note the similarity with the definition of a simplicial complex!)

To each cone  $\sigma$  of  $\Sigma$ , we can associate an affine variety  $U_\sigma$ , and then if  $\sigma \cap \tau \neq \emptyset$ , we glue  $U_\sigma$  and  $U_\tau$  together along  $U_{\sigma \cap \tau}$ . This gives us

$$X(\Sigma) = \bigcup_{\sigma \in \Sigma} U_\sigma,$$

the toric variety of the fan  $\Sigma$ .

How do we build the affine variety  $U_\sigma$  from  $\sigma$ ? Let  $(\mathbb{R}^n)^\vee \cong \mathbb{R}^n$  be the dual vector space to  $\mathbb{R}^n$ ; that is, the vector space of linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}$ . Then given a rational cone  $\sigma$ , we do the following:

1. Take the dual cone  $\sigma^\vee = \{u \in (\mathbb{R}^n)^\vee \mid \langle u, v \rangle \geq 0 \ \forall v \in \sigma\}$ .
2. The dual cone is again rational, so take  $\sigma^\vee \cap \mathbb{Z}^n$ .
3.  $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$  is a finitely generated  $\mathbb{C}$ -algebra, so its spectrum is an affine variety.
4. Take  $U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$ .

**Example:** Let  $\sigma$  be the positive span of  $(1, 0)$  and  $(0, 1)$  in  $\mathbb{R}^2$ . Then  $\sigma^\vee$  has the same shape in  $\mathbb{R}^2$ , and  $\mathbb{C}[\sigma \cap \mathbb{Z}^n]$  is just a polynomial in two variables  $\mathbb{C}[t, u]$ . Then  $U_\sigma = \text{Spec}(\mathbb{C}[t, u]) = \mathbb{C}^2$ , the affine plane. If we take the fan  $\Sigma$  consisting of cones generated by two of the three vectors  $(1, 0)$ ,  $(0, 1)$ , and  $(-1, -1)$ , then each of these three cones yields a copy of  $\mathbb{C}^2$ , and these are glued along their intersections, which are copies of  $\mathbb{C}$ . Then the variety  $X(\Sigma)$  is  $\mathbb{C}P^2$ .

**Facts:**

1.  $X(\Sigma)$  is compact  $\iff \Sigma$  covers all of  $\mathbb{R}^n$ .
2.  $X(\Sigma)$  is smooth  $\iff$  every cone of  $\Sigma$  is spanned by part of a basis of  $\mathbb{Z}^n$ .
3.  $X(\Sigma)$  is projective  $\iff \Sigma$  admits a strictly convex support function  $\psi : \Sigma \rightarrow \mathbb{R}^n$ .

Our main result for this section is that *every toric variety is  $X(\Sigma)$  for some fan  $\Sigma$* .

## 2 Toric varieties from polytopes

Given a convex rational polytope  $P$ , there is a canonical way to associate a fan  $\Sigma_P$  to it, essentially from the boundary complex of the dual polytope. Then we use  $X_P$  for  $X_{\Sigma_P}$ . It turns out that these varieties have a natural structure as an identification space.

Let  $T^n = \mathbb{R}^n/\mathbb{Z}^n$  be the compact  $n$ -torus and  $P$  a rational convex  $n$ -polytope. For each point  $q \in P$ , let  $G(q)$  be the smallest face of  $P$  containing  $q$  in its relative interior, and let  $N(q)$  be the normal subspace to  $G(q)$ . Then  $N(q)$  is rational, so the canonical projection from  $\mathbb{R}^n$  to  $\mathbb{R}^n/\mathbb{Z}^n = T^n$  sends  $N(q)$  to some subtorus  $T(q)$  of  $T^n$ . Then with a little more work, we see that  $X_P = T^n \times P / \sim$ , where  $(t, p) \sim (u, q)$  whenever  $p = q$  and  $tu^{-1} \in T(q)$ . Furthermore,  $T^n$  acts on  $T^n \times P$  by multiplication in the first factor and trivially on the second, and this action induces an action on  $X_P$ , the quotient. So not only do we get the variety  $X_P$ , but we also know the action of  $T^n$ .

**Example:** Let  $P = I$ , the unit interval. Then  $X_P = S^1 \times I / \sim$ , so it is the quotient of a cylinder. The only identifications correspond to the endpoints of  $I$ , causing each of the two boundary circles of the cylinder to shrink down to a point. Then we get that  $X_P$  is a

2-sphere; in fact, it is the complex projective line  $\mathbb{C}P^1$ .

Here are a couple of properties of the action of  $T^n$  on  $X_P$ :

1. The subtori  $T(q)$  are the isotropy subgroups (i.e. stabilizers) for the  $T^n$ -action on  $X_P$ .
2. If  $G(q)$  has dimension  $k$ , then  $T(q)$  has a  $k$ -dimensional orbit. In particular, the vertices of  $P$  correspond to the fixed points of the  $T^n$ -action on  $X_P$ .

### 3 Quasitoric manifolds

Let  $P$  be an  $n$ -dimensional simple polytope with  $m$  facets (faces of codimension 1). By *simple* we mean that each vertex is on  $n$  facets, from which it follows that every face of  $P$  of codimension  $k$  is the intersection of exactly  $k$  facets. Let  $F_1, \dots, F_m$  be the facets of  $P$ . Then we say that an  $n \times m$  matrix  $\chi$  with integer entries is *characteristic* for  $P$  if for each vertex  $v = F_{i_1} \cap \dots \cap F_{i_n}$ , the minor of  $\chi$  given by the columns  $i_1, \dots, i_n$  has determinant  $\pm 1$ . We can consider  $\chi$  as a map from  $T^m$  to  $T^n$ , where the  $m$  circles of  $T^m$  are indexed according to the  $m$  facets of  $P$ . If  $\chi$  is characteristic, then the restriction of  $\chi$  to  $T_{i_1} \times \dots \times T_{i_n}$  is injective whenever  $F_{i_1} \cap \dots \cap F_{i_n}$  is a vertex of  $P$ .

Let  $G(q)$  be, as before, the smallest face of  $P$  having  $q$  in its relative interior. Any face  $G$  of codimension  $k$  can be written  $G = F_{j_1} \cap \dots \cap F_{j_k}$ , and we can define  $T_G = T_{j_1} \times \dots \times T_{j_k}$ . Now we can define the *quasitoric manifold*  $M_P(\chi) = T^n \times P / \sim$ , where  $(t, p) \sim (u, q)$  if and only if  $p = q$  and  $tu^{-1} \in \chi(T_{G(q)})$ . As before, the action of  $T^n$  on  $T^n \times P$  induces an action of  $M_P(\chi)$ , with the following properties:

1. Faces of dimension  $k$  of  $P$  correspond to orbits of dimension  $k$ .
2. In particular, vertices of  $P$  correspond to fixed points.

$M_P(\chi)$  is a smooth, simply connected  $2n$ -manifold. Let us look at a couple of examples of this construction.

**Example:** Let  $P = I$ , the unit interval. Then  $n = 1$  and  $m = 2$  (the two facets are the endpoints). Then if  $\chi$  is characteristic, it is of the form  $(\pm 1, \pm 1)$ ; no matter how we choose the signs,  $M_I(\chi) = \mathbb{C}P^1 \simeq S^2$ .

**Example:** Let  $P$  be a triangle. Then  $n = 2$  and  $m = 3$ . Then there are only two essentially different choices for  $\chi$ ; one yields  $\mathbb{C}P^2$  while the other yields  $\overline{\mathbb{C}P^2}$ , the complex projective plane with the opposite orientation.

We have the following important fact:

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**Proposition:** Every smooth projective toric variety is a quasitoric manifold.

The converse is not true. For example, the connected sum  $\mathbb{C}P^2 \sharp \mathbb{C}P^2$  arises as one of the quasitoric manifolds of a quadrilateral.

## 4 Small covers

The construction of quasitoric manifolds can also be done using the 2-torus  $\mathbb{Z}_2^n$  instead. This yields an identification space  $N_P(\chi)$  (called a small cover) that is  $n$ -dimensional instead of  $2n$  dimensional. This sits naturally inside of  $M_P(\chi)$ ; in particular, if  $\tau$  acts on  $T^n$  by complex conjugation, then this induces an involution  $\tau$  on  $M_P(\chi)$ , and the set of fixed points of  $\tau$  is exactly  $N_P(\chi)$ . Whenever  $M_P(\chi)$  is a toric variety  $X_P$ , then  $N_P(\chi)$  is identified with the real points of  $X_P$ .

## 5 Moment-angle manifolds

Because of the conditions we put on  $\chi$ , we have that  $T^m \cong \ker \chi \times T^n$ . Using this, we can build a larger quasitoric manifold  $Z_P = T^m \times P / \sim$ , where  $(t, p) \sim (u, q)$  if and only if  $p = q$  and  $tu^{-1} \in T_{G(q)}$ . The difference here is that we are using  $T^m$  instead of  $T^n$ . We call  $Z_P$  the *moment-angle manifold* of  $P$ . Note that we took  $T_{G(q)}$  instead of its image  $\chi(T_{G(q)})$ , so that  $Z_P$  doesn't depend at all on  $\chi$ . For any characteristic  $\chi$ , we have that  $\ker \chi$  acts freely on  $Z_P$ , and so  $Z_P / \ker \chi \cong M_P(\chi)$ . So  $Z_P$  acts as a universal cover for all  $M_P(\chi)$  for any fixed polytope  $P$ .