Unbounded Solutions of the Modified Korteweg-De Vries Equation

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Focusing mKdV Equation on $\mathbb{R} \times \mathbb{R}$

$w = w(x,t) \qquad\qquad (x,t) \in \mathbb{R} \times \mathbb{R}$

$$\begin{aligned} w_t + w^2 w_x + w_{xxx} &= 0 \\ w(x,0) &= w_0(x) \in C^\infty(\mathbb{R}) \end{aligned}$$

$w_o(x) \sim \sum_{k=0}^{\infty} a_k^{\pm} \cdot (\pm x)^{\beta_k} \qquad$ as $x \to \pm\infty$

- $a_k^+, a_k^- \in \mathbb{R}$
- $\beta_k \in \mathbb{R}$
- $\beta_k > \beta_{k+1}$
- $\beta_k \to -\infty$ as $k \to \infty$
- $\beta_0 \le \frac{1}{2}$

The Spaces $S^\beta(\mathbb{R} \times I)$ and $S^\beta(\mathbb{R})$

$\beta \in \mathbb{R}$, $\qquad I \subset \mathbb{R}$ interval, $\qquad f \in C^\infty(\mathbb{R} \times I)$.

$f(x,t) \in S^\beta(\mathbb{R} \times I)$ if there exists a pair of formal power series where $\beta_0 = \beta$

$$\sum_{k=0}^{\infty} a_k^{\pm}(t)\,(\pm x)^{\beta_k}$$

asymptotic to $f(x,t)$ as $x \to \pm\infty$.

i.e. for every $J \subset I$ compact, $N, i, j \geq 0$ there exists $C_{J,N,i,j} > 0$ such that

$$\left| \partial_t^i \partial_x^j \left(f(x,t) - \sum_{k=0}^{N} a_k^{\pm}(t)\,(\pm x)^{\beta_k} \right) \right| \leq C_{J,N,i,j}\, |x|^{\beta_{N+1}-j}$$

for any $\pm x \geq 1$ and $t \in J$.

- $f(x) \in S^\beta(\mathbb{R})$ if the a_k are independent of t.

Main Theorem

- For any $\beta \leq \frac{1}{2}$ and for any $w_0 \in S^{\beta}(\mathbb{R})$ there exists $T > 0$ and a unique solution $w(x,t) \in S^{\beta}(\mathbb{R} \times [0,T])$ of the focusing mKdV equation.

$$w_t + w^2 w_x + w_{xxx} = 0$$

$$w(x,0) = w_0(x)$$

- If j is the smallest index such that $a_j^{+} \neq 0$ (resp. $a_j^{-} \neq 0$) then the coefficient $a_j^{+}(t)$ (resp. $a_j^{-}(t)$) in the asymptotic expansion of the solution is a nonvanishing continuous function of t and all preceeding coefficients are identically zero.

$$w(x,t) \sim a_j^{\pm}(t) \cdot (\pm x)^{\beta_j} + \sum_{k=j+1}^{\infty} a_k^{\pm}(t) \cdot (\pm x)^{\beta_k}$$

as $x \longrightarrow \pm\infty$

The Spaces $F^{\beta}(\mathbb{R}_{\pm} \times I)$ and $F^{\beta}(\mathbb{R}_{\pm})$:

$\beta \in \mathbb{R}$, $I \subset \mathbb{R}$ interval.

$$F^{\beta}(\mathbb{R}_{\pm}\times I) = \left\{ \sum_{k=0}^{\infty} a_k^{\pm}(t)\,(\pm x)^{\beta_k} \mid a_k,\ \beta_k \text{ satisfy} \right.$$

- $a_k^{\pm}(t) \in C^{\infty}(I)$
- $\beta_0 = \beta$
- $\beta_k > \beta_{k+1}$
- $\beta_k \to -\infty$ as $k \to \infty$

$f(x) \in F^{\beta}(\mathbb{R}_{\pm})$ if the a_k are constants.

Formal Solutions

A pair of series $f_\pm \in F^\beta(\mathbb{R}_\pm \times I)$ with $f_\pm(\cdot, 0) \in F^\beta(\mathbb{R}_\pm)$ that are solutions is called a *formal solution*.

If $f_\pm = \sum_{k=0}^\infty a_k^\pm(t)\,(\pm x)^{\beta_k}$ is a formal solution then

$$f_t + f^2 f_x + f_{xxx} = 0$$

$\Longrightarrow$

$$\sum_{k=0}^\infty \dot{a}_k(t)\,x^{\beta_k} + \Big(\sum_{k=0}^\infty a_k(t)\,x^{\beta_k}\Big)^2\Big(\sum_{k=0}^\infty a_k(t)\,\beta_k\,x^{\beta_k-1}\Big)$$
$$+\Big(\sum_{k=0}^\infty a_k(t)\,\beta_k\,(\beta_k-1)\,(\beta_k-2)\,x^{\beta_k-3}\Big) = 0$$

By comparing coefficients of x^{β_k}

$$\dot{a}_k = P_k(a_0, a_1, \ldots, a_k)$$

where P_k is a polynomial.

Asymptotic Solutions

- $S^{-\infty}(\mathbb{R} \times I)$ is the space of functions asymptotic to 0.

- $S^{-\infty}(\mathbb{R})$ is the space of Schwarz functions.

- $f(x,t) \in S^{\beta}(\mathbb{R} \times I)$ is called an *asymptotic solution* if

$$\begin{aligned} f_t + f^2 f_x + f_{xxx} &= g \in S^{-\infty}(\mathbb{R} \times I) \\ f(x,0) - w_0(x) &= u_0(x) \in S^{-\infty}(\mathbb{R}) \end{aligned}$$

Asymptotic solutions are constructed from formal solutions by a standard procedure found in ΨDO theory,

$$F^{\beta} \longrightarrow S^{\beta}$$

$$\sum_{k=0}^{\infty} a_k(t)\, x^{\beta_k} \longmapsto f(x,t)$$

Genuine Solutions

- If

$$f(x,t) \sim \sum_{k=0}^{\infty} a_k^{\pm}(t)\, x^{\beta_k}$$

and $u(x,t) \in S^{-\infty}(\mathbb{R} \times I)$ then

$$u + f \sim \sum_{k=0}^{\infty} a_k^{\pm}(t)\, x^{\beta_k}$$

- If $f(x,t) \in S^{\beta}(\mathbb{R} \times I)$ is an asymptotic solution and $w(x,t) := u(x,t) + f(x,t)$ is a genuine solution then

$$w_t + w^2 w_x + w_{xxx} = 0$$

$\Longrightarrow$

$$\begin{aligned} u_t + u^2 u_x + u_{xxx} + \left(u^2 f\right)_x + \left(f^2 u\right)_x + g &= 0 \\ u(x,0) = u_0(x) \end{aligned}$$

- $g(x,t) \in S^{-\infty}(\mathbb{R} \times \mathbb{R})$

- $u_0(x) \in S^{-\infty}(\mathbb{R})$

Obtaining a Solution by Discretization

Take $h, k \in (0,1)$ and let $x_n := nh$, $t_j := jk$ for $n, j \in \mathbb{Z}$, and $u(x_n, t_j) := u_{n,j}$.
Define discrete derivative operators,

$$D_+ u(x) := \frac{u(x+h) - u(x)}{h}$$

$$D_- u(x) := \frac{u(x) - u(x-h)}{h}$$

$$D_0 u(x) := \frac{u(x+h) - u(x-h)}{2h}$$

Discretize the new equation

$$D_{t,+}u_{n,j} + u_{n,j}^2 D_0 u_{n,j+1} + D_+^2 D_- u_{n,j+1} +$$

$$+ 2f_{n,j} u_{n,j} D_0 u_{n,j+1} + 2f_{n,j}(f_x)_{n,j} u_{n,j+1} +$$

$$(f_x)_{n,j} u_{n,j} u_{n,j+1} + f_{n,j}^2 D_0 u_{n,j+1} + g_{n,j} = 0$$

- Define

  $Q_j u \ := \ u_{n,j}^2 D_0 u + D_+^2 D_- u + 2 f_{n,j} u_{n,j} D_0 u +$

  $2 f_{n,j} (f_x)_{n,j} u \ + \ (f_x)_{n,j} u_{n,j} u \ + \ f_{n,j}^2 D_0 u$

- $P_j \, u := \Big(I + k Q_j \Big) \, u.$

- Discrete mKdV becomes

$$P_j \, u_{j+1} = u_j - k g_j$$

- After several estimates

$$\Big(P_j u, u \Big)_{S_h} \ \geq \ \frac{1}{2} \, \| u \|_{S_h}^2$$

  which implies P_j is invertible at least for a finite amount of time T.

- $(u, v)_{S_h} \ := \ (\langle x \rangle \, u, \langle x \rangle \, v)_{L_h^2} \ +$
  $+ \ \Big(\langle x \rangle \, D_+^3 u, \langle x \rangle \, D_+^3 v \Big)_{L_h^2} \ + \ \Big(D_+^5 u, D_+^5 v \Big)_{L_h^2}$

- The discrete solution satisfies the estimate

$$\left\| D_+^m \left(\langle x \rangle^N u(\cdot, t_j) \right) \right\|_{L_h^2} \leq C_{N,m}$$

for $t_j \in [0, T]$.

Smoothing Operator: *Stummel* (1967)

$$(I_h u)(x) := \sum_{l=-\infty}^{\infty} u(x_l) \left(\frac{\sin \frac{\pi}{h}(x_l - x)}{\frac{\pi}{h}(x_l - x)} \right)$$

Properties: If $u = u(x_n) \in L_h^2$ and $I_h u := U$ then

- $U \in C^\infty(\mathbb{R})$

- $U(x_n) = u(x_n)$ if $x_n = nh$ for $n \in \mathbb{Z}$.

- $$\left(\frac{2}{\pi}\right)^j \left\| \frac{\partial^j}{\partial x^j} x^N U \right\|_{L^2} \leq \left\| D_+^j x^N u \right\|_{L_h^2} \leq \left\| \frac{\partial^j}{\partial x^j} x^N U \right\|_{L^2}$$

- Apply I to u in x and in t to obtain a family of smooth functions $U^{h,k}$ for $h, k \in (0, 1)$.

- The estimates imply that the family is equicontinuous and uniformly bounded in h, k.

- By Arzela-Ascoli Theorem there is a convergent subsequence and a limit function U.

- By taking limit of discrete equation we see that U satisfies the equation.

- The discrete Schwarz estimates and smoothing operator estimates imply that U lies in $S^{-\infty}(\mathbb{R} \times [0, T])$ Q.E.D.

Related Results

Theorem: *Kappeler, Perry, Shubin, Topalov* (2007)
Global existence and uniqueness is true for the defocusing mKdV equation

$$w_t - w^2 w_x + w_{xxx} = 0$$

with $w(x,t) \in S^\beta(\mathbb{R} \times \mathbb{R})$ for $\beta < \frac{1}{2}$ and $a_0^\pm$ is independent of t.

Theorem: *Bondareva* (1985)
Global existence and uniqueness is true for the KdV equation

$$w_t + w w_x + w_{xxx} = 0$$

with $w(x,t) \in S^\beta(\mathbb{R} \times \mathbb{R})$ for $\beta < 1$ and $a_0^\pm$ is independent of t.

Theorem: Menikoff (1972)
Global existence is true for KdV in a class larger than $S^\beta(\mathbb{R} \times \mathbb{R})$ with $\beta < 1$.

Another Related Result for KdV

Theorem: *Kenig, Ponce, Vega* (1997)
If

$w_0 = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 + f(x)$

k is odd, $a_k > 0$, f - Schwarz function, then the KdV equation has a unique global solution $w(x,t) \in C^\infty(\mathbb{R} \times [0,\infty))$ satisfying the condition:

For any $T > 0$ there exists $C_T > 0$ such that for $(x,t) \in \mathbb{R} \times (0,T)$

1. $|u(x,t)| \le C_T + \frac{|x|}{t}$

2. $|\partial_x u(x,t)| \le C_T + \frac{1}{t}$

3. $$\int_0^T \left(\int_{-\infty}^{\infty} \left| \left(1 + |x|^{j-2}\right) \partial_x^j u(x,t) \right|^2 dx \right)^{\frac{1}{2}} dt \;\le\; C_{T,j}$$
   for $j = 2, 3, \ldots$

Future Work

1. Prove global existence for focusing mKdV, i.e. construct a solution

$$w(x,t) \in S^{\beta}(\mathbb{R} \times \mathbb{R})$$

for $\beta < \frac{1}{2}$.

2. Use the same method to solve other equations, e.g. NLS .