

# Locally Toroidal Polytopes and Modular Linear Groups

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May 22, 2008

## Abstract

When the standard representation of a crystallographic Coxeter group  $G$  (with string diagram) is reduced modulo the integer  $d \geq 2$ , one obtains a finite group  $G^d$  which is often the automorphism group of an abstract regular polytope. Building on earlier work in the case that  $d$  is an odd prime, we here develop methods to handle composite moduli and completely describe the corresponding modular polytopes when  $G$  is of spherical or Euclidean type. Using a modular variant of the quotient criterion, we then describe the locally toroidal polytopes provided by our construction, most of which are new.

Key Words: locally toroidal polytopes, abstract regular polytopes

AMS Subject Classification (2000): Primary: 51M20. Secondary: 20F55.

## 1 Introduction

Our fascination with the regular polytopes is due not only to their visual appeal and charm, but also to the fact that their symmetry groups appear in such varied and unexpected places. In a recent series of papers, for example, the authors established the basic machinery needed to describe a large class of polytopes whose automorphism groups typically have small index in some finite orthogonal group (see [8, 9, 10]). Indeed, in our analysis there we often had to exploit quite subtle properties of the orthogonal group  $O(n, p, \epsilon)$  on an  $n$ -dimensional vector space over  $\mathbb{Z}_p$ , where  $p$  is an odd prime. Here we take a bit of a detour and consider instead

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\*Supported by NSERC of Canada Grant # 4818

†Supported by NSA-grants H98230-05-1-0027 and H98230-07-1-0005

the possibilities released by more generally working over the ring  $\mathbb{Z}_d$ , with any modulus  $d \geq 2$ . (The rank 4 polytopes described in [11, 12] involve an analogous excursion into the domains of Gaussian and Eisenstein integers; and, of course, the related idea of constructing the automorphism group of a regular map by modular reduction is natural and well established; see [13], for example.)

Our main goal is to extend previous results on *locally toroidal polytopes*, as provided by our construction [10, §4]. To that end, in Sections 2 and 3 we describe the modular reduction of a crystallographic Coxeter group  $G$  with string diagram. In Sections 4 and 5 we completely describe what happens when  $G$  is of spherical or Euclidean type. Finally, after proving a useful quotient criterion (Theorem 6.1), we discuss in Section 7 various new families of locally toroidal polytopes, mainly in ranks 5 and 6.

## 2 Abstract regular polytopes and Coxeter groups

Let us begin with a brief review of some key properties of abstract regular polytopes, referring to [6] for details. An (*abstract*)  $n$ -polytope  $\mathcal{P}$  is a partially ordered set with a strictly monotone rank function having range  $\{-1, 0, \dots, n\}$ . An element  $F \in \mathcal{P}$  with  $\text{rank}(F) = j$  is called a  $j$ -face; typically  $F_j$  will indicate a  $j$ -face;  $\mathcal{P}$  has a unique least face  $F_{-1}$  and unique greatest face  $F_n$ . Each maximal chain or *flag* in  $\mathcal{P}$  must contain  $n + 2$  faces. Next,  $\mathcal{P}$  must satisfy a *homogeneity property*: whenever  $F < G$  with  $\text{rank}(F) = j - 1$  and  $\text{rank}(G) = j + 1$ , there are exactly two  $j$ -faces  $H$  with  $F < H < G$ , just as happens for convex  $n$ -polytopes. It follows that for  $0 \leq j \leq n - 1$  and any flag  $\Phi$ , there exists a unique *adjacent* flag  ${}^j\Phi$ , differing from  $\Phi$  in just the rank  $j$  face. With this notion of adjacency the flags of  $\mathcal{P}$  form a *flag graph*. The final defining property of  $\mathcal{P}$  is that the flag graph for each section must be connected, so that  $\mathcal{P}$  is *strongly flag-connected*. Recall here that whenever  $F \leq G$  are faces of ranks  $j \leq k$  in  $\mathcal{P}$ , then the *section* of  $\mathcal{P}$  determined by  $F$  and  $G$  is given by  $G/F := \{H \in \mathcal{P} \mid F \leq H \leq G\}$ . In fact, this is a  $(k - j - 1)$ -polytope in its own right.

Naturally, the symmetry of  $\mathcal{P}$  is exhibited by its *automorphism group*  $\Gamma(\mathcal{P})$ , containing all order preserving bijections on  $\mathcal{P}$ . Henceforth, we shall consider only *regular polytopes*  $\mathcal{P}$ , for which  $\Gamma(\mathcal{P})$  is, by definition, transitive on flags. Clearly a regular  $n$ -polytope  $\mathcal{P}$  must have all sorts of local combinatorial symmetry. In particular,  $\mathcal{P}$  will be *equivelar* of some type  $\{p_1, \dots, p_{n-1}\}$ , where  $2 \leq p_j \leq \infty$ ; this means that for each fixed  $j \in \{1, \dots, n - 1\}$  and each pair of incident faces  $F$  and  $G$  in  $\mathcal{P}$ , with  $\text{rank}(F) = j - 2$  and  $\text{rank}(G) = j + 1$ , the rank 2 section  $G/F$  has the structure of a  $p_j$ -gon (independent of choice of  $F < G$ ). Thus, each 2-face (polygon) of  $\mathcal{P}$  is isomorphic to a  $p_1$ -gon, and in every 3-face of  $\mathcal{P}$ , each 0-face is surrounded by an alternating cycle of  $p_2$  edges and  $p_2$  polygons, etc.

To further understand the structure of  $\Gamma(\mathcal{P})$  when  $\mathcal{P}$  is regular, we fix a *base flag*  $\Phi = \{F_{-1}, F_0, \dots, F_{n-1}, F_n\}$ , with  $\text{rank}(F_j) = j$ . For  $0 \leq j \leq n - 1$ , let  $\rho_j$  be the (unique) automorphism with  $\rho_j(\Phi) = {}^j\Phi$ . If  $\mathcal{P}$  is regular, then  $\Gamma(\mathcal{P})$  is generated by  $\rho_0, \rho_1, \dots, \rho_{n-1}$ , which are involutions satisfying at least the relations

$$\rho_j^2 = (\rho_{j-1}\rho_j)^{p_j} = (\rho_i\rho_j)^2 = 1, \quad 0 \leq i, j \leq n - 1, |j - i| \geq 2 \quad (1)$$

(with  $j \geq 1$  for  $\rho_{j-1}\rho_j$ ). Also, an *intersection condition* on standard subgroups holds:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \quad (2)$$

for all  $I, J \subseteq \{0, \dots, n-1\}$ . In short,  $\Gamma(\mathcal{P})$  is a very particular quotient of the Coxeter group  $G = [p_1, \dots, p_{n-1}]$ , whose diagram is a string with branches labelled  $p_1, \dots, p_{n-1}$ . (We allow  $p_j = 2$ , in which case the ‘string’ is disconnected.) Conversely, given any group  $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$  generated by involutions and satisfying (1) and (2), one may construct a polytope  $\mathcal{P}$  with  $\Gamma(\mathcal{P}) = \Gamma$  (see [6, Theorem 2E11]). We then say that  $\Gamma(\mathcal{P})$  is a *string C-group*. Since  $\mathcal{P}$  can be uniquely reconstructed from  $\Gamma(\mathcal{P})$ , we may therefore shift our focus to an appropriate class of groups of particular interest.

Recall that if  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are regular  $n$ -polytopes with  $n \geq 2$ , then  $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$  denotes the class of all regular  $(n+1)$ -polytopes whose facets are isomorphic to  $\mathcal{P}_1$  and whose vertex-figures are isomorphic to  $\mathcal{P}_2$ . If this class is non-empty, then it contains a *universal* regular  $(n+1)$ -polytope, denoted  $\{\mathcal{P}_1, \mathcal{P}_2\}$ , which covers any other polytope in the class [6, Th. 4A2].

Let us look more closely at the abstract Coxeter group  $G = [p_1, \dots, p_{n-1}]$ , which is itself a string C-group with respect to the usual generators and which may well be infinite. The corresponding polytope  $\{p_1, \dots, p_{n-1}\} := \mathcal{P}(G)$  is universal in a more local sense, as described in [6, Th. 3D5].

Now, like any finitely generated Coxeter group,  $G$  can be identified with its image under the standard faithful representation in real  $n$ -space  $V$  [4, Cor. 5.4]. Consequently, we may suppose  $G = \langle r_0, \dots, r_{n-1} \rangle$  to be the *linear Coxeter group* generated by certain reflections  $r_j$  on  $V$ . In fact, these reflections leave invariant a symmetric bilinear form  $x \cdot y$  on  $V$ , so that  $G$  is a subgroup of the corresponding orthogonal group  $O(V) \subset GL(V)$ . (Note, however, that  $x \cdot y$  is positive definite if and only if  $G$  is finite [4, Th. 6.4].) We shall let  $e$  denote the identity in the group  $GL(V)$ .

Recalling our earlier description of the regular  $n$ -polytope  $\mathcal{P}$ , we now have an epimorphism

$$\begin{aligned} G &\rightarrow \Gamma(\mathcal{P}) \\ r_j &\mapsto \rho_j. \end{aligned}$$

Intuitively then, we may think of regular polytopes as having maximal reflection symmetry.

### 3 Crystallographic Coxeter groups and their modular reductions

Now let us specialize. We say that the linear Coxeter group  $G$  is *crystallographic* (with respect to the standard representation) if it leaves invariant some *lattice*  $\sum_{j=0}^{n-1} \mathbb{Z}b_j$  generated by a basis  $\beta = \{b_j\}$  for  $V$ . As described in [5] or [8, Prop. 4.1], there is no loss of generality in assuming that  $\beta$  is a *basic system* for  $G$ , meaning that each  $b_j$  is a *root* for the corresponding reflection  $r_j$ . Thus,

$$r_i(b_j) = b_j + m_{ij}b_i \quad (3)$$

for certain *Cartan integers*  $m_{ij}$ ,  $0 \leq i, j \leq n - 1$ , with all  $m_{ii} = -2$  and  $m_{ij} = 0$  for  $|i - j| \geq 2$ .

Now recall that the string Coxeter group  $G = [p_1, \dots, p_{n-1}]$  is crystallographic if and only if all  $p_j \in \{2, 3, 4, 6, \infty\}$  [8, Prop. 4.1(c)]. If the corresponding *Coxeter diagram*  $\Delta_c(G)$  is connected, then  $G$  admits only a finite number of essentially distinct basic systems  $\beta$ . As we observed in [8, §4], each basic system and corresponding lattice can be encoded in a new diagram  $\Delta(G)$ , a variant of  $\Delta_c(G)$ . Briefly, the branches of  $\Delta(G)$  are no longer labelled; instead, each node  $j$  of  $\Delta(G)$  is labelled by the real number  $b_j^2 = b_j \cdot b_j$ . Each subdiagram on two nodes  $i$  and  $j$  must then be one of those appearing in Table 1 below.

Period of $r_i r_j$	Subdiagram on nodes $i$ (left), $j$ (right)	Cartan integers $m_{ij}, m_{ji}$
2	$\begin{array}{cc} a & c \\ \bullet & \bullet \end{array}$	0, 0
3	$\begin{array}{cc} a & a \\ \bullet & \text{---} \bullet \end{array}$	1, 1
4	$\begin{array}{cc} a & 2a \\ \bullet & \text{---} \bullet \end{array}$	2, 1
6	$\begin{array}{cc} a & 3a \\ \bullet & \text{---} \bullet \end{array}$	3, 1
$\infty$	$\begin{array}{cc} a & 4a \\ \bullet & \text{---} \bullet \end{array}$	4, 1
$\infty$	$\begin{array}{cc} a & a \\ \bullet & \text{===} \bullet \end{array}$	2, 2

Table 1. Possible diagrams for dihedral subgroups  $\langle r_i, r_j \rangle$  of  $G$

For each  $i \neq j$ , we have  $m_{ij}m_{ji} = 4 \cos^2(\pi/p_{ij})$ , where  $p_{ij}$  is the period of the *rotation*  $r_i r_j$ . (In particular,  $p_{j-1, j} = p_j$ .) Note that nodes  $i$  and  $j$  must be clearly distinguished, say as left and right in the Table 1, whenever  $m_{ij} \neq m_{ji}$ . By suitably rescaling the node labels on each connected component of  $\Delta(G)$ , we can assume that these labels are a set of relatively prime positive integers. As a familiar example, consider the usual tessellation  $\mathcal{P}$  of the Euclidean plane by congruent squares. Then  $\mathcal{P}$  is an infinite regular 3-polytope, and  $G = [4, 4] \simeq \Gamma(\mathcal{P})$  admits the diagrams

$$\begin{array}{c} 1 \\ \bullet \end{array} \text{---} \begin{array}{c} 2 \\ \bullet \end{array} \text{---} \begin{array}{c} 1 \\ \bullet \end{array}, \begin{array}{c} 1 \\ \bullet \end{array} \text{---} \begin{array}{c} 2 \\ \bullet \end{array} \text{---} \begin{array}{c} 4 \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} 2 \\ \bullet \end{array} \text{---} \begin{array}{c} 1 \\ \bullet \end{array} \text{---} \begin{array}{c} 2 \\ \bullet \end{array}. \quad (4)$$

Having fixed such a basic system for a crystallographic Coxeter group  $G = [p_1, \dots, p_{n-1}]$ , we can reduce  $G$  modulo any integer  $s \geq 2$ : the natural epimorphism  $\mathbb{Z} \rightarrow \mathbb{Z}_s$  induces a homomorphism of  $G$  onto a subgroup  $G^s$  of  $GL_n(\mathbb{Z}_s)$ , the group of  $n \times n$  invertible matrices over  $\mathbb{Z}_s$ . Our hope, of course, is that the finite group  $G^s$  will be the automorphism group of a regular  $n$ -polytope. (In [8, 9, 10] we examined such groups in the case that  $s$  is an odd prime, so as to exploit the structure of orthogonal groups over finite fields.)

We shall often abuse notation by referring to the modular images of objects by the same name (such as  $r_i$ ,  $e$ ,  $b_i$ ,  $V$ , etc.). In particular,  $\{b_i\}$  will denote the standard basis for  $V = \mathbb{Z}_s^n$ , which in general we must now view as a free module over the ring  $\mathbb{Z}_s$ . We shall see in Lemma 3.1 that  $r_i$  usually continues to act as a reflection after reduction; in any case, we can compute it using (3). However, the situation for metrical quantities such as  $b_i \cdot b_j$ , a rational number which occasionally has denominator 2, is more intricate [8, Eq.

10]. Nevertheless, at least when  $\gcd(6, s) = 1$ , we can interpret  $G^s$  as a subgroup of the orthogonal group  $O(\mathbb{Z}_s^n)$  for the symmetric bilinear form defined on  $\mathbb{Z}_s^n$  by means of the Gram matrix  $[b_i \cdot b_j]$ . Moreover, we can then write

$$r_i(x) = x - 2 \frac{x \cdot b_i}{b_i \cdot b_i} b_i$$

since  $b_i^2$  will be invertible  $\pmod{s}$ . In our earlier work with prime moduli, these issues were a concern only for ‘non-generic’ groups, where  $s = 3$  and  $G$  has some period  $p_j = 6$ . Here, with more general moduli, the analysis is more complicated. It often happens, for instance, that the group  $G^s$  depends essentially on the choice of basic system and the corresponding diagram  $\Delta(G)$ . For example, for the modulus  $s = 4$ , the group  $G^4$  corresponding to the three diagrams in (4) has order 32, 128 and 64, respectively. These are, in fact, the automorphism groups of the regular toroidal maps  $\{4, 4\}_{(2,0)}$ ,  $\{4, 4\}_{(4,0)}$  and  $\{4, 4\}_{(2,2)}$  (see Table 4 below).

Clearly, we must now confront a crucial question: when is  $G^s = \langle r_0, \dots, r_{n-1} \rangle^s$  a string  $C$ -group (i.e. the automorphism group of a finite, abstract regular  $n$ -polytope  $\mathcal{P} = \mathcal{P}(G^s)$ )? Unfortunately, we cannot provide anything like a comprehensive answer here. Instead, for classes of groups  $G$  of particular interest, we shall have to rely more on ad hoc techniques than we did for prime moduli, without trying to exploit in any deep way the structure of orthogonal groups over general rings. Occasionally, we employ GAP [2] to settle ‘small’ cases.

Certainly, the generators  $r_j$  of  $G^s$  satisfy the Coxeter-type relations inherited from  $G$  (see (1), with  $\rho_j$  replaced by  $r_j$ ). However, before confronting the intersection condition (2) for  $G^s$ , we must take a closer look. For example, it might happen that  $r_j = e \pmod{s}$ .

**Notation.** We say that node  $i$  of  $\Delta(G)$  is *e-e* if both Cartan integers  $m_{i,i-1}$  and  $m_{i,i+1}$  are even; *o-e* if just one of the integers is even; and *o-o* if both are odd. For the terminal nodes 0 and  $n - 1$  on the string we shall agree that  $m_{0,-1} = m_{n-1,n} = 0$ .

Note that end nodes can never be o-o. Likewise, a node is e-e if it is labelled  $a$ , while any adjacent nodes are labelled  $4a$ ,  $2a$  or  $a$  (after a double branch), as in

$$\dots - \overset{2a}{\bullet} - \overset{a}{\bullet} - \overset{2a}{\bullet} - \dots, \quad \overset{a}{\bullet} \equiv \overset{a}{\bullet} - \dots, \quad \dots - \overset{2a}{\bullet} - \overset{a}{\bullet}, \quad \text{etc.}$$

Typical o-e nodes are the middle nodes in the subdiagrams

$$\dots - \overset{3a}{\bullet} - \overset{a}{\bullet} - \overset{2a}{\bullet} - \dots \quad \text{or} \quad \dots - \overset{a}{\bullet} \equiv \overset{a}{\bullet} - \overset{c}{\bullet} - \dots$$

(where the integer label  $c$  divides  $a$ ). Let us now summarize basic properties of the generators  $r_i$  for  $G^s$ . Using (3), the calculations are straightforward, if a bit involved.

**Lemma 3.1** *Let  $G = \langle r_0, \dots, r_{n-1} \rangle \simeq [p_1, \dots, p_{n-1}]$  be any crystallographic linear Coxeter group with string diagram. Suppose  $s \geq 2$ , and reduce  $G$  modulo  $s$ . Then*

- (a) *Each  $r_i \in G^s$  has period 2, except that  $r_i = e$  when  $s = 2$  and node  $i$  of  $\Delta(G)$  is e-e.*
- (b)  *$r_i$  and  $r_j$  commute in  $G^s$  when  $i < j - 1$ .*
- (c) *Suppose  $p_i = 2, 3, 4$  or  $6$ . If  $s > 2$ , then  $r_{i-1}r_i$  has period  $p_i$  in  $G^s$  (unchanged from characteristic 0).*

Now let  $s = 2$ . If  $p_i = 3$  or  $6$ , the period of  $r_{i-1}r_i$  is always  $3$ . If  $p_i = 4$ , the period collapses to  $2$  if and only if one of nodes  $i - 1$  or  $i$  is  $e-e$ . For  $p_i = 2$ , the period collapses to  $1$  if and only if both nodes are  $e-e$  (so that  $r_{i-1} = r_i = e$ ).

(d) Suppose  $p_i = \infty$ . Then  $r_{i-1}r_i$  has period  $s$  in  $G^s$ , except in the following cases, each when  $s$  is even: for the subdiagram  $\overset{a}{\bullet} \equiv \overset{a}{\bullet}$ , the period becomes  $\frac{s}{2}$  when both nodes are  $e-e$ ; for the subdiagram  $\overset{a}{\bullet} - \overset{4a}{\bullet}$  the period becomes  $2s$  when the node labelled  $a$  is  $o-e$ .

**Remarks.** In the typical case, when all  $r_i$  have period  $2$ , we say that  $G^s$  is a *string group generated by involutions*. Even for modulus  $s = 2$ , it is quite possible that all  $r_i$  be involutions (though not geometrical reflections), so long as  $\Delta(G)$  has special features, as explained later. Assuming now that all  $r_i$  are involutions, we conclude that  $G^s$  is a string C-group if and only if it satisfies the intersection condition (2), with  $r_i = \rho_i$ . Our main problem is therefore to determine when  $G^s$  satisfies (2).

We hinted earlier at the definite advantages of working with prime moduli. For a composite modulus  $s$ , we would at least hope that  $G^s$  somehow splits according to the prime decomposition of  $s$ . However, our hopes for a simple approach are dashed by examples such as the following. Let  $G \simeq [4, 6, 4]$  be the group with diagram

$$\overset{2}{\bullet} \text{ --- } \overset{1}{\bullet} \text{ --- } \overset{3}{\bullet} \text{ --- } \overset{6}{\bullet} .$$

First of all, we find for  $p = 2$  that  $G^2$  is a string C-group of order  $96$ . The middle rotation order collapses and we actually obtain the group for the universal locally projective polytope  $\{\{4, 3\}_3, \{3, 4\}_3\}$ . For  $p = 3$  we get a group  $G^3$  of order  $5184$  for a self-dual polytope of type  $\{4, 6, 4\}$  (see [9, Eq. (33)]).

Now for modulus  $s = 6$  we find that  $G^6$  has order  $248832 = \frac{1}{2}(96 \times 5184)$ . But the intersection condition fails, since  $\langle r_1, r_2 \rangle^6$  has index  $3$  in  $\langle r_0, r_1, r_2 \rangle^6 \cap \langle r_1, r_2, r_3 \rangle^6$ . In other words, the polytopality of  $G^s$  is not determined through the prime factorization of  $s$ . Since, in the end, we are more concerned with locally toroidal groups  $G$ , which do fall to a more direct attack, we shall mainly ignore the prime factorization of  $s$ . (We note, however, that precisely that approach worked in [11, 12]. But for the 4-polytopes considered there, the rotation groups were covered by special linear groups over certain rings of algebraic integers; and the resulting modular groups do split according to the prime factorization.)

Before proceeding, let us set down some useful notation. For any  $J \subseteq \{0, \dots, n - 1\}$ , we let  $G_J^s := \langle r_j \mid j \notin J \rangle$ ; in particular, for  $k, l \in \{0, \dots, n - 1\}$  we let  $G_k^s := \langle r_j \mid j \neq k \rangle$  and  $G_{k,l}^s := \langle r_j \mid j \neq k, l \rangle$ . We also let  $V_J$  be the submodule of  $V = \mathbb{Z}_s^n$  spanned by  $\{b_j \mid j \notin J\}$ , and similarly for  $V_k, V_{k,l}$ . Note that  $V_J$  is  $G_J^s$ -invariant. In particular,  $G_J^s$  acts on  $V_j$ , for  $j = 0$  or  $n - 1$ ; however, this action need not be faithful (see [9, Lemma 3.1]).

## 4 Modular polytopes of spherical type

When  $G = [p_1, \dots, p_{n-1}]$  is finite, the invariant form  $x \cdot y$  on real  $n$ -space  $V$  is positive definite, so that  $G$  acts in a natural way on any sphere  $\mathbb{S}^{n-1}$  with centre  $o \in V$ . Accordingly, we also say that  $G$  is of *spherical type*. If the spherical group  $G$  has a connected diagram, then

up to isomorphism  $\mathcal{P}(G)$  is one of the familiar convex regular  $n$ -polytopes [8, §5-6]. After central projection, such polytopes can usefully be viewed as regular spherical tessellations of the circumsphere  $\mathbb{S}^{n-1}$ .

In [8, §5-6] we showed that  $G \simeq G^p$ , for any odd prime modulus  $p$  and crystallographic string Coxeter group  $G$  of spherical type and in any rank  $n \geq 1$ . When  $s$  is divisible by an odd prime  $p$ , the natural epimorphisms

$$G \rightarrow G^s \rightarrow G^p$$

immediately give  $G \simeq G^s$ . Here we take a different approach, working explicitly with the underlying representation of the spherical group  $G$  in  $GL_n(\mathbb{Z})$ . We confirm that  $G^s \simeq G$  for *any* modulus  $s \geq 3$ , and sometimes even for  $s = 2$ . (For  $n = 1, 2$  such isomorphisms follow at once from Lemma 3.1.) However, since the actual calculations for general rank  $n$  are quite tiresome, we shall simply summarize the results, with brief comments, for each of the relevant families of spherical groups. In fact, to serve later applications, we must generalize a little and consider how a spherical group can embed as a string subgroup of some group  $G$  of higher rank. In other words, we consider certain *spherical subdiagrams* of  $\Delta(G)$ .

Of course, when  $G^s \simeq G$  we also know the structure of the modular polytope  $\mathcal{P}(G^s)$ , which is merely a copy of  $\mathcal{P}(G)$ .

**(a) The group of the  $m$ -simplex:  $A_m \simeq S_{m+1}$ , for  $m \geq 1$ .**

Here, for some label  $a \geq 1$ ,  $\Delta(G)$  has the subdiagram

$$\dots - \overset{a}{\bullet} - \overset{a}{\bullet} - \dots - \overset{a}{\bullet} - \overset{a}{\bullet} - \dots, \quad (5)$$

on  $m$  consecutive nodes  $j, \dots, j + m - 1$ . For all  $m \geq 2$  and all  $s \geq 2$ , we then have

$$\langle r_j, \dots, r_{j+m-1} \rangle^s \simeq A_m .$$

Part (c) of Lemma 3.1 provides the base step of an induction on  $m \geq 2$ . As in [8, §6.1], we then exploit the contragredient representation of  $A_m$ . (Alternatively, we could use the fact that the even subgroup of  $\langle r_j, \dots, r_{j+m-1} \rangle$  is the alternating group of degree  $m + 1$ , which is simple if  $m \geq 4$ ; the cases  $m = 2, 3$  are straightforward.) For  $m = 1$ , we note that  $A_1^2 = \{e\}$ ; otherwise, for  $s \geq 3$ ,  $A_1^s \simeq A_1 \simeq C_2$ .

**(b) The group of the  $m$ -cube:  $B_m$ , for  $m \geq 2$ .**

We must accommodate two distinct basic systems for  $B_m$ . Consider first the subdiagram

$$\dots - \overset{a}{\bullet} - \overset{2a}{\bullet} - \overset{2a}{\bullet} - \dots - \overset{2a}{\bullet} - \overset{2a}{\bullet} - \dots, \quad (6)$$

on nodes  $j, \dots, j + m - 1$  of  $\Delta(G)$ . Then

$$\langle r_j, \dots, r_{j+m-1} \rangle^s \simeq B_m ,$$

for all  $s \geq 3$ , *and* for  $s = 2$  so long as node  $j$  (labelled  $a$ ) is o-e . If, however,  $s = 2$  and node  $j$  is e-e , then  $r_j = e$  and the given generators do not give a string  $C$ -group. Instead, from (a) we see that the subgroup collapses in rank to a copy of  $A_{m-1}^2$ .

Here, and in similar situations below, we obtain dual versions of these results by flipping the diagram end-for-end. Consequently, we may suppose that  $m \geq 3$  for the alternative basic system

$$\dots - \overset{2a}{\bullet} - \overset{a}{\bullet} - \overset{a}{\bullet} - \dots - \overset{a}{\bullet} - \overset{a}{\bullet} - \dots, \quad (7)$$

on nodes  $j, \dots, j + m - 1$ . Again we have

$$\langle r_j, \dots, r_{j+m-1} \rangle^s \simeq B_m,$$

whenever the modulus  $s \geq 3$ . For  $s = 2$ , the subgroup  $\langle r_j, \dots, r_{j+m-1} \rangle^s$  is isomorphic either to  $B_m$  (the group of the cube), or to  $B_m/\{\pm e\}$  (the group of the *hemi-cube*), as detailed in Table 2.

node		$m \geq 3$	
$j$	$j + m - 1$	even	odd
o-o	o-o	$B_m$	$B_m$
o-o	o-e	$B_m/\{\pm e\}$	$B_m$
o-e	o-o	$B_m$	$B_m$
o-e	o-e	$B_m/\{\pm e\}$	$B_m/\{\pm e\}$

Table 2. The group  $B_m^2$  for the diagram (7)

(Since  $m \geq 3$ , node  $j+m-1$  cannot be e-e.) Note that the bottom row covers the case that  $G$  actually equals  $B_m$ , for which there is inevitably a collapse when  $s = 2$ . A crucial step in the verification employs a small observation concerning  $B_m \simeq [4, 3, \dots, 3] = \langle r_0, r_1, \dots, r_{m-1} \rangle$ : if  $\varphi : B_m \rightarrow H$  is a homomorphism which is 1-1 on the subgroups  $\langle r_0, r_1 \rangle$  and  $\langle r_1, \dots, r_{m-1} \rangle$ , then  $\ker \varphi \subseteq \{\pm e\}$ . The proof follows from explicit calculation in  $B_m$ , taken as the semidirect product  $C_2^m \rtimes S_m$ . Note here that  $H$  is isomorphic to  $B_m$  if and only if  $(r_0 r_1 \dots r_{m-1})^m \neq e$ . (To argue from a topological perspective, the regular  $m$ -polytope associated with the group  $\langle r_j, \dots, r_{j+m-1} \rangle^s$  must be a regular tessellation on an  $(m-1)$ -dimensional spherical space-form and hence necessarily be isomorphic to a regular tessellation on the  $(m-1)$ -sphere or real projective  $(m-1)$ -space (see [6, 6C2]). This observation also applies to the next group.)

**(c) The group of the 24-cell:  $F_4$ .**

We must consider a subdiagram such as

$$\dots - \overset{a}{\bullet} - \overset{a}{\bullet} - \overset{2a}{\bullet} - \overset{2a}{\bullet} - \dots \quad (8)$$

on nodes  $j, \dots, j + 3$  in  $\Delta(G)$ . By part (b), the natural mapping

$$\varphi : F_4 \rightarrow \langle r_j, r_{j+1}, r_{j+2}, r_{j+3} \rangle$$

is 1-1 on subgroups  $\langle r_j, r_{j+1}, r_{j+2} \rangle$  and  $\langle r_{j+1}, r_{j+2}, r_{j+3} \rangle$ . A similar small observation now gives  $\ker \varphi \subseteq \{\pm e\}$ . No matter how the subdiagram is embedded in  $\Delta(G)$  we find that

$$\langle r_j, r_{j+1}, r_{j+2}, r_{j+3} \rangle^s \simeq \begin{cases} F_4, & \text{if } s \geq 3; \\ F_4/\{\pm e\}, & \text{if } s = 2. \end{cases} \quad (9)$$

## 5 Modular polytopes of Euclidean type

Suppose now that  $G = [p_1, \dots, p_{n-1}]$  is a string Coxeter group of *Euclidean* (or *affine*) type, with connected diagram (no  $p_j = 2$ ). Then  $G$  acts as the full symmetry group of a certain regular tessellation  $\mathcal{T} \simeq \mathcal{P}(G)$  of Euclidean space  $\mathbb{A}^{n-1}$ . Indeed,  $G$  must be one of the Coxeter groups displayed in the left column of Table 3, though perhaps with generators specified in dual order. Note that each of these groups is crystallographic.

A *regular  $n$ -toroid*  $\mathcal{P}$  is the quotient of such a tessellation  $\mathcal{T}$  by a non-trivial normal subgroup  $L$  of translations in  $G$ . Thus every toroid can be viewed as a finite, regular tessellation of the  $(n-1)$ -torus. We refer to [6, 1D and 6D-E] for a complete classification; briefly, for each group  $G$  the distinct toroids are indexed by a *type vector*  $\mathbf{q} := (q^k, 0^{n-1-k}) = (q, \dots, q, 0, \dots, 0)$ , where  $q \geq 2$  and  $k = 1, 2$  or  $n-1$ . (For  $G = [3, 3, 4, 3]$ , the case  $k = 4$  is subsumed by the case  $k = 1$ .) Anyway,  $L$  is generated (as a normal subgroup of  $G$ ) by the translation

$$\bar{t} := t_1^q \cdots t_k^q,$$

where  $\{t_1, \dots, t_{n-1}\}$  is a standard set of generators for the full group  $T$  of translations in  $G$ . The modular toroids  $\mathcal{P}(G^p)$  described in [8, §6B] are special instances; with one exception, we had there  $\mathbf{q} = (p, 0, \dots, 0)$ .

For completeness we also list in Table 3 the infinite dihedral group  $[\infty]$ , which of course has rank 2 and acts on the Euclidean line  $\mathbb{A}^1$ . The corresponding 2-toroids are then regular polygons inscribed in a ‘1-torus’, namely, in an ordinary circle.

Before proceeding to a classification of the groups  $G^s$ , we take a closer look at the geometric action of groups of affine Euclidean isometries. Suppose then that  $G = \langle r_0, \dots, r_{n-1} \rangle$  is of Euclidean type (here always with connected diagram). From [4, §6.5] we recall that the invariant quadratic form  $x \cdot y$  on real  $n$ -space  $V$  must be positive semidefinite, so that the *radical subspace*  $\text{rad}(V) = \langle c \rangle$  is 1-dimensional. Since  $r_j(c) = c$ , for  $0 \leq j \leq n-1$ ,  $G$  is in fact a subgroup of  $\widehat{O}(V)$ , the pointwise stabilizer of  $\text{rad}(V)$  in  $O(V)$ .

To actually exploit the structure of  $G$  as a group of (affine) isometries on Euclidean  $(n-1)$ -space, we pass to the contragredient representation of  $G$  in the dual space  $\check{V}$  (as in [4, 5.13]). Since  $c$  is fixed by  $G$ , we see that  $G$  leaves invariant any translate of the  $(n-1)$ -space

$$U = \{\mu \in \check{V} : \mu(c) = 0\}.$$

Next, for each  $w \in V$  define  $\mu_w \in \check{V}$  by  $\mu_w(x) := w \cdot x$ . The mapping  $w \mapsto \mu_w$  factors to a linear isomorphism between  $V/\text{rad}(V)$  and  $U$ , and so we transfer to  $U$  the *positive definite* form induced by  $V$  on  $V/\text{rad}(V)$ . Now choose any  $\alpha \in \check{V}$  such that  $\alpha(c) = 1$ , and let  $\mathbb{A}^{n-1} := U + \alpha$ . Putting all this together we may now think of  $\mathbb{A}^{n-1}$  as *Euclidean  $(n-1)$ -space*, with  $U$  as its *space of translations*. Indeed, each fixed  $\tau \in U$  defines an isometric translation on  $\mathbb{A}^{n-1}$ :

$$\mu \mapsto \mu + \tau, \quad \forall \mu \in \mathbb{A}^{n-1}.$$

It is easy to check that this mapping on  $\mathbb{A}^{n-1}$  is induced by a unique isometry  $t \in \widehat{O}(V)$ , namely the *transvection*

$$\begin{aligned} t(x) &= x - \tau(x)c, \\ &= x - (x \cdot a)c, \end{aligned}$$

where  $\tau = \mu_a$  for suitable  $a \in V$ . (Remember here that we employ the contragredient representation of  $\widehat{O}(V)$  on  $\check{V}$ , not just that of  $G$ .) In summary, we can therefore safely think of translations as transvections.

In the following table we list those Euclidean Coxeter groups which are relevant to our analysis (see [8, §6B]). Concerning the group  $G = [4, 3^{n-3}, 4]$  (for the familiar cubical tessellation of  $\mathbb{A}^{n-1}$ ), we recall our convention that  $3^{n-3}$  indicates a string of  $n-3 \geq 0$  consecutive 3's.

The group $G$	$\dim(\mathbb{A}^{n-1})$	One possible diagram $\Delta(G)$	The corresponding vector $c \in \text{rad}(V)$
$[4, 3^{n-3}, 4]$	$n - 1 \geq 2$	$\overset{2}{\bullet} - \overset{1}{\bullet} - \overset{1}{\bullet} \dots \overset{1}{\bullet} - \overset{1}{\bullet} - \overset{2}{\bullet}$	$c = b_0 + 2(b_1 + \dots + b_{n-2}) + b_{n-1}$
$[3, 3, 4, 3]$	4	$\overset{1}{\bullet} - \overset{1}{\bullet} - \overset{1}{\bullet} - \overset{2}{\bullet} - \overset{2}{\bullet}$	$c = b_0 + 2b_1 + 3b_2 + 2b_3 + b_4$
$[3, 6]$	2	$\overset{1}{\bullet} - \overset{1}{\bullet} - \overset{3}{\bullet}$	$c = b_0 + 2b_1 + b_2$
$[\infty]$	1	$\overset{1}{\bullet} = \overset{1}{\bullet}$	$c = b_0 + b_1$

Table 3. Euclidean Coxeter Groups

An investigation of the action of these discrete reflection groups on the Euclidean space  $\mathbb{A}^{n-1}$  shows, in each case, that  $G \simeq T \rtimes H$  splits as the semidirect product of the (normal) subgroup  $T$  of translations with a certain (finite) *point group*  $H$  (see [4, Prop. 4.2]). We can and do display each group in the table so that  $H = G_0 = \langle r_1, \dots, r_{n-1} \rangle$ .

Now we are in a position to survey the modular reduction of the Euclidean groups in Table 3. Again we more generally consider Euclidean subgroups

$$E = \langle r_j, \dots, r_{j+m} \rangle \simeq T \rtimes \langle r_{j+1}, \dots, r_{j+m} \rangle \quad (10)$$

of our usual group  $G$ ; and once more we allow various possible basic systems. Notice that we specifically assume that  $E$  is embedded in  $G$  so that the point subgroup (of spherical type) is  $\langle r_{j+1}, \dots, r_{j+m} \rangle$ . Because of this, we can use the splitting in (10) to actually perform explicit calculations, although the details are quite involved. We begin with

**Lemma 5.1** *Let  $G$  be a crystallographic linear Coxeter group with string diagram. Suppose that  $E = \langle r_j, \dots, r_{j+m} \rangle$  is the (Euclidean) subgroup of  $G$  corresponding to one of the subdiagrams displayed in Table 4 or Table 5, so that  $E = T \rtimes H$ , with translation group  $T$  and (spherical) point group  $H = \langle r_{j+1}, \dots, r_{j+m} \rangle$ . Also suppose that  $s, m$ , and the nodes  $j, j+m$  are restricted in one of the various ways indicated in the Tables, so in particular  $H \simeq H^s$ . Let  $\varphi : E \rightarrow E^s \subseteq G^s$  be the natural epimorphism for modulus  $s \geq 2$ .*

- (a) Then  $\ker(\varphi) \subset T$ .

- (b)  $E^s$  is a string  $C$ -group, namely the automorphism group of a regular  $m$ -toroid.  
(c) If  $T^s$  acts faithfully on the  $\mathbb{Z}_s$ -submodule spanned by  $b_j, \dots, b_{j+m}$ , then

$$T^s \cap \langle r_{j+1}, \dots, r_{j+m}, \dots, r_{j+l} \rangle^s = \{e\},$$

for any  $l \geq m$ .

**Proof.** As always, our calculations may well depend on the underlying choice of basic system  $\{b_i\}$  for  $G$ , as encoded in the diagram  $\Delta(G)$ . By inspection of the various diagrams in Tables 4 and 5, we confirm in each case that  $E = T \rtimes H$ , with  $H = \langle r_{j+1}, \dots, r_{j+m} \rangle$ . Furthermore, we also observe that the radical of  $\sum_{k=j}^{j+m} \mathbb{R}b_k$  is spanned by an *integral* vector  $c = \sum_{k=j}^{j+m} x_k b_k$ , in which the coefficient of  $b_j$  is  $x_j = 1$ .

Now for part (a) let  $g = th \in \ker(\varphi)$ , with  $t \in T, h \in H$ , so that  $t \equiv h^{-1} \pmod{s}$ . For  $j \leq i \leq j+m$ , we have  $t(b_i) = b_i + z_i c$ , with  $z_i \in \mathbb{Z}$  (the coefficient of  $b_j$  in  $c$  is 1), since  $t$  is a translation and the lattice  $\sum_{k=j}^{j+m} \mathbb{Z}b_k$  is invariant under  $E$ ; likewise  $h^{-1}(b_i) = b_i + v_i$ , with  $v_i \in \sum_{k=j+1}^{j+m} \mathbb{Z}b_k$ , since  $h \in \langle r_{j+1}, \dots, r_{j+m} \rangle$ . Thus  $z_i \equiv 0 \pmod{s}$ , so that  $h^{-1} \equiv e \pmod{s}$ . Since reduction modulo  $s$  is faithful on  $H$ , we have  $h = e$  (in characteristic 0), and  $g = t \in T$ .

For part (b) we first of all note that the subgroups  $H = \langle r_{j+1}, \dots, r_{j+m} \rangle$  and  $A := \langle r_j, \dots, r_{j+m-1} \rangle$  are spherical, since the various constraints on  $s, m_{j,j-1}, m_{j+m,j+m+1}$  in Tables 4 and 5 guarantee that both subgroups are faithfully represented mod  $s$ ; see Section 4. Now (b) follows at once from (a), since  $\ker(\varphi)$  is a normal subgroup of translations; see [6, 6D-E]. Here we also need to make a forward appeal to the computation of the type vector  $\mathbf{q}$  of Tables 4 and 5, eliminating the possibility that the index of  $\ker(\varphi)$  in  $T$  is too small for  $E^s$  to be polytopal. (We can also give a direct proof of the intersection property of  $E^s$  using [6, Prop. 2E16(a)]. Since the subgroups  $A, H$  are both (spherical) string  $C$ -groups, we need only show that  $A^s \cap H^s \subseteq \langle r_{j+1}, \dots, r_{j+m-1} \rangle^s$ . So suppose  $g \in A$  and  $h \in H$  (both in characteristic 0) such that  $g \equiv h \pmod{s}$ . Then  $h^{-1}g =: t \in \ker(\varphi) \subseteq T$ . Now let  $\mathcal{T}$  be the regular tessellation in Euclidean  $m$ -space associated with  $E$ , let  $o$  be the base vertex of  $\mathcal{T}$ , and let  $z$  be the center of the base facet (tile)  $F$  of  $\mathcal{T}$ . Then  $t^{-1}(h^{-1}(z)) = g^{-1}(z) = z$ , so  $t$  must be the translation by the vector  $h^{-1}(z) - z$ . Since  $h^{-1}(z)$  is the center of the facet  $h^{-1}(F)$  of  $\mathcal{T}$  and  $o$  is a vertex of  $h^{-1}(F)$ , the two vertices  $h^{-1}(z)$  and  $z$  of the dual of the vertex-figure of  $\mathcal{T}$  at  $o$  are equivalent under  $t$  and thus under  $\ker(\varphi)$ . Hence, if  $t$  is non-trivial, then reduction modulo  $s$  collapses the vertex-figure of  $\mathcal{T}$  at  $o$ , contrary to the fact that  $H^s$  is isomorphic to  $H$ . Therefore,  $t$  must be trivial and  $g = h \in A \cap H = \langle r_{j+1}, \dots, r_{j+m-1} \rangle$ . It follows that the modular images of  $g$  and  $h$  are in  $\langle r_{j+1}, \dots, r_{j+m-1} \rangle^s$ , as required. Alternatively we can argue here as follows. The translation vectors of the conjugates of  $t$  under  $H$  generate a sublattice of  $\ker(\varphi)$  with very small index in  $T$ ; however, our computation of the type vectors  $\mathbf{q}$  has shown that this cannot occur.)

For part (c) we let  $\varphi(t) = \varphi(h) \in T^s \cap \langle r_{j+1}, \dots, r_{j+l} \rangle^s$ . Again  $t(b_i) \equiv b_i \pmod{s}$  for  $j \leq i \leq j+m$ , so that by hypothesis we have  $t \equiv e \pmod{s}$ .  $\square$

**Remarks.** We have seen that  $H \simeq H^s$  always holds when  $s \geq 3$  and occasionally when  $s = 2$ ; under the constraints on  $m$  indicated in Tables 4 and 5, it also holds for  $s = 2$ . A consequence of our calculations is that, for all the cases detailed in Tables 4 and 5, the semidirect splitting (10) of  $E = \langle r_j, \dots, r_{j+m} \rangle$  (in characteristic 0) survives reduction modulo  $s$ . Thus,  $E^s \simeq T^s \rtimes H^s$ , although it is not necessarily the case that  $T^s \simeq \mathbb{Z}_s^m$ .

Of course, taking the  $r_i$ 's in reverse order, we obtain a dual version of Lemma 5.1. In applications, we must then take care that the subdiagrams in Tables 4 and 5, along with the attached constraints, really have been flipped end-for-end.

Next we must deal with the specific features of each group  $G$ . Guided by [6, 6D-E], we can, with some effort, write out explicit matrices for standard generators  $t_1, \dots, t_m$  of the translation subgroup  $T \subset \langle r_j, \dots, r_{j+m} \rangle$ . Such matrices incorporate the unspecified, but crucial, Cartan integers  $m_{j,j-1}$  and  $m_{j+m,j+m+1}$  and furthermore vary a little with the choice of the underlying basic system. But from Lemma 5.1(b) we know that  $E^s$  is a string  $C$ -group. To finish off its description, we identify the type vector  $\mathbf{q}$  by calculating the periods of the key translations  $t_1, t_1 t_2$  and  $t_1 t_2 \dots t_m$ . It is convenient now to separate our results into two lots:

**(a) The groups  $[4, 3^{m-2}, 4]$  ( $m \geq 2$ ).**

When  $\langle r_j, \dots, r_{j+m} \rangle \simeq [4, 3^{m-2}, 4]$ , we must contend with the three distinct basic systems shown in Table 4. For any  $s \geq 3$ , we observe that  $\langle r_j, \dots, r_{j+m} \rangle^s$  is the group of a suitable cubic toroid  $\{4, 3^{m-2}, 4\}_{\mathbf{q}}$  of rank  $m + 1$  (on the  $m$ -torus), whose type vector  $\mathbf{q}$  is also displayed in the Table. The same holds for  $s = 2$ , so long as terminal nodes  $j$  and  $j + m$  are constrained as indicated. This restriction guarantees that the facet and vertex-figure subgroups are spherical, with the correct rank  $m$  (see Section 4 above). For any other terminal node types when  $s = 2$ , one finds that  $\langle r_j, \dots, r_{j+m} \rangle^2$  either fails to have involutory generators (so is not a string  $C$ -group) or is *locally projective* rather than toroidal (see [6, 14A] and [3]).

Subdiagram of $\Delta(G)$ on nodes $j, \dots, j + m$	Modulus $s$	Affine dim. $m \geq 2$	Constraints on nodes $j, j + m$	Type vector $\mathbf{q}$
$-\overset{2a}{\bullet} - \overset{a}{\bullet} \dots \overset{a}{\bullet} - \overset{2a}{\bullet} -$	odd $s \geq 3$ even $s \geq 4$ even $s \geq 4$ even $s \geq 4$ $s = 2$	any $m$ odd $m$ odd $m$ even $m$ odd	— at least one o-o both o-e — both o-o	$(s, 0, \dots, 0)$ $(s, 0, \dots, 0)$ $(\frac{s}{2}, \frac{s}{2}, \dots, \frac{s}{2})$ $(\frac{s}{2}, \frac{s}{2}, \dots, \frac{s}{2})$ $(2, 0, \dots, 0)$
$-\overset{a}{\bullet} - \overset{2a}{\bullet} \dots \overset{2a}{\bullet} - \overset{a}{\bullet} -$	odd $s \geq 3$ even $s \geq 4$ even $s \geq 4$ $s = 2$	any any any any	— at least one o-e both e-e both o-e	$(s, 0, \dots, 0)$ $(s, 0, \dots, 0)$ $(\frac{s}{2}, 0, \dots, 0)$ $(2, 0, \dots, 0)$
$-\overset{4a}{\bullet} - \overset{2a}{\bullet} \dots \overset{2a}{\bullet} - \overset{a}{\bullet} -$	odd $s \geq 3$ even $s \geq 4$ even $s \geq 2$	any any any	— $j + m$ is e-e $j + m$ is o-e	$(s, 0, \dots, 0)$ $(s, 0, \dots, 0)$ $(s, s, 0, \dots, 0)$

Table 4. Groups for the cubic toroids

**(b) The special groups  $[3, 3, 4, 3]$  ( $m = 4$ ),  $[3, 6]$  ( $m = 2$ ) and  $[\infty]$  ( $m = 1$ ).**

Similar remarks apply to the remaining Euclidean groups  $\langle r_j, r_{j+1}, r_{j+2}, r_{j+3}, r_{j+4} \rangle \simeq [3, 3, 4, 3]$ ,  $\langle r_j, r_{j+1}, r_{j+2} \rangle \simeq [3, 6]$  or  $\langle r_j, r_{j+1} \rangle \simeq [\infty]$  (and their duals). For the first two groups we may exclude the modulus  $s = 2$ , for which there is a collapse in either the facet or vertex-figure. Our calculations are summarized in Table 5. The resulting polytopes are regular toroids  $\{3, 3, 4, 3\}_{\mathbf{q}}$  of rank 5 (on the 4-torus),  $\{3, 6\}_{\mathbf{q}}$  of rank 3 (on the 2-torus), and regular polygons  $\{q\}$  (on the 1-torus), when  $\mathbf{q} = (q)$  in the latter case. Note for the group  $[3, 6]$  that the residue of the Cartan integer  $m_{j,j-1} \pmod{3}$  is a consideration (see [8, 5.6]).

**Remark.** We have surveyed here the Euclidean subgroups  $E$  of  $G$ . We emphasize that any reduced subgroup  $E^s$  not explicitly covered (up to duality) by an entry in Table 4 or Table 5 will fail in some way to be the group of a regular toroid.

Subdiagram of $\Delta(G)$ on nodes $j, \dots, j+m$	Modulus $s$	Affine dim. $m$	Constraints on nodes $j, j+m$	Type vector $\mathbf{q}$
$-\overset{a}{\bullet}-\overset{a}{\bullet}-\overset{a}{\bullet}-\overset{2a}{\bullet}-\overset{2a}{\bullet}-$	odd $s \geq 3$ even $s \geq 4$ even $s \geq 4$	4 4 4	— node $j$ is o-o node $j$ is o-e	$(s, 0, 0, 0)$ $(s, 0, 0, 0)$ $(\frac{s}{2}, \frac{s}{2}, 0, 0)$
$-\overset{2a}{\bullet}-\overset{2a}{\bullet}-\overset{2a}{\bullet}-\overset{a}{\bullet}-\overset{a}{\bullet}-$	any $s \geq 3$	4	—	$(s, 0, 0, 0)$
$-\overset{a}{\bullet}-\overset{a}{\bullet}-\overset{3a}{\bullet}-$	$s \equiv \pm 1 \pmod{3}$ ( $s > 2$ ) $s \equiv 0 \pmod{3}$ $s \equiv 0 \pmod{3}$	2 2 2	— $m_{j,j-1} \equiv \pm 1 \pmod{3}$ $m_{j,j-1} \equiv 0 \pmod{3}$	$(s, 0)$ $(s, 0)$ $(\frac{s}{3}, \frac{s}{3})$
$-\overset{3a}{\bullet}-\overset{3a}{\bullet}-\overset{a}{\bullet}-$	any $s \geq 3$	2	—	$(s, 0)$
$-\overset{a}{\bullet}=\overset{a}{\bullet}-$	odd $s \geq 3$ even $s \geq 4$ even $s \geq 4$ $s = 2$	1 1 1 1	— some node o-e both nodes e-e both nodes o-e	$(s)$ $(s)$ $(\frac{s}{2})$ $(2)$
$-\overset{4a}{\bullet}-\overset{a}{\bullet}-$	odd $s \geq 3$ even $s \geq 4$ even $s \geq 2$	1 1 1	— node $j+1$ is e-e node $j+1$ is o-e	$(s)$ $(s)$ $(2s)$

Table 5. Groups for the special toroids

## 6 The Quotient Criterion

The following result is a modular variant of the *quotient criterion* in [6, 2E17]. As usual there is a dual version with subgroups  $G_{n-1}$  and  $G_0$  interchanged.

**Theorem 6.1** *Let  $G = \langle r_0, \dots, r_{n-1} \rangle$  be a crystallographic linear Coxeter group with string diagram, and suppose  $G^s$  is a string C-group for modulus  $s \geq 2$ . Suppose also that  $s|d$  and that either*

(a)  $G_{n-1}$  is of spherical type and that  $G_{n-1} \simeq G_{n-1}^s$  (so that the underlying basic system of  $G$  is restricted as explained in §4 when  $s = 2$ ); or

(b)  $G_{n-1} = T \rtimes G_{0,n-1}$  is of Euclidean type, with translation group  $T$  and (faithfully represented) spherical point group  $G_{0,n-1} \simeq G_{0,n-1}^s$  (so that  $n \geq 3$  and the underlying basic system of  $G$  is restricted as explained in §5). Also assume in this case that

$$T^d \cap \langle r_1, \dots, r_{n-1} \rangle^d = \{e\}. \quad (11)$$

Then  $G^d$  is a string C-group.

**Proof.** We adapt the proof of [6, 2E17]. Since  $s|d$  we have natural epimorphisms  $\eta : G \rightarrow G^d$  and  $\varphi : G^d \rightarrow G^s$ . For clarity we avoid our customary abuse of notation and take care to distinguish the standard generators  $q_j := \eta(r_j)$  of  $G^d$  and  $s_j := \varphi(q_j)$  of  $G^s$ . Since  $G^s$  is a string C-group, each  $s_j$  and hence each  $q_j$  is an involution. By [6, 2E16(b)], we need only show that  $G_{n-1}^d$  is a string C-group and, for  $1 \leq k \leq n-1$ , that  $G_{n-1}^d \cap \langle q_k, \dots, q_{n-1} \rangle \subseteq \langle q_k, \dots, q_{n-2} \rangle$ . So, beginning with the latter, let  $g \in G_{n-1}^d \cap \langle q_k, \dots, q_{n-1} \rangle$ ; then  $\varphi(g) \in \langle s_k, \dots, s_{n-2} \rangle \subseteq G_{n-1}^s$ , since  $G^s$  is a string C-group.

In the spherical case (a),  $\varphi$  is 1-1 on  $G_{n-1}^d$ , since  $G_{n-1} \simeq G_{n-1}^s (\simeq G_{n-1}^d)$ . Thus  $g \in \langle q_k, \dots, q_{n-2} \rangle$ .

Consider the Euclidean case (b). There exists (a unique)  $h \in \langle q_k, \dots, q_{n-2} \rangle$  with  $\varphi(h) = \varphi(g)$ . Applying Lemma 5.1 to  $\varphi \circ \eta$  (restricted to  $G_{n-1}$ ), we have  $g = th$  for some translation  $t \in T^d$ . By (11) we get  $t = e$ , so that  $g \in \langle q_k, \dots, q_{n-2} \rangle$ .

Finally,  $G_{n-1}^d$  is a string C-group in each case. This follows from applying our considerations in Sections 4 and 5 to  $G_{n-1}$ , since switching from  $s$  to a multiple  $d$  merely eases any constraints which could prevent  $G_{n-1}^d$  from being a string C-group.  $\square$

**Example and Remarks.** In general, some condition like (11) is necessary. Consider, for instance, the diagrams

$$\bullet \overset{a}{\text{---}} \overset{1}{\bullet} \overset{1}{\text{---}} \bullet \quad \text{and} \quad \bullet \overset{1}{\text{---}} \overset{a}{\bullet} \overset{a}{\text{---}} \bullet. \quad (12)$$

For  $a \in \{1, 2, 3\}$ , the corresponding groups of rank 3 reduce to string C-groups for any modulus  $d > 2$ . In the left diagram we can even take  $a = 4$  and so obtain a polyhedron of type  $\{d, d\}$ , for odd  $d \geq 3$ , or type  $\{d, \frac{d}{2}\}$ , for even  $d \geq 4$ . However, taking  $a = 4$  in the right diagram, we find that the intersection condition fails precisely when the modulus  $d = 2s$ , with  $s$  odd: for then  $t = (r_0 r_1)^s = (r_1 r_2)^s \neq e \pmod{d}$ ; and  $t \in T^d \cap \langle r_1, r_2 \rangle^d$  directly contradicts (11). We shall see that the fault lies in the embedding of the subdiagrams for facet and vertex-figure.

To explain what is going on we use Lemma 5.1(c) (with  $s = d, j = 0, m = n - 2, l = m + 1$ ). Thus (11) is fulfilled whenever  $T^d$  acts faithfully on the  $\mathbb{Z}_d$ -submodule  $V_{n-1}$ . This holds, for example, when dropping node  $n - 1$  has no effect on the embedding constraints for  $G_{n-1}$ , as described in the Tables. To see this, note that  $r_i$  induces a mapping  $\tilde{r}_i$  on  $V_{n-1}$ , for  $0 \leq i \leq n - 2$ . Clearly,  $K^d := \langle \tilde{r}_0, \dots, \tilde{r}_{n-2} \rangle$  is just the (toroidal) group corresponding to the the subdiagram of  $\Delta(G)$  obtained by deleting node  $n - 1$ . If, as we suppose, this deletion has no effect on the constraints on node  $n - 2$ , it must be that  $G_{n-1}^d$  and  $K^d$  have the same type vector  $\mathbf{q}$ , as given in the Tables. Since the corresponding spherical point groups are isomorphic, it follows that  $G_{n-1}^d \simeq K^d$  and that  $T^d$  acts faithfully on  $V_{n-1}$ . Thus  $G^d$  is a string  $C$ -group. In particular, we now see that (11) is redundant whenever  $d$  is odd and in several other instances. This leads to an important simplification: for  $d$  odd we need only check that  $G^s$  is a string  $C$ -group for some *odd prime* divisor  $s = p$ . Occasionally, the modulus  $s = 4$  is another keystone.

## 7 Locally toroidal polytopes

In this Section, we consider *locally toroidal* regular polytopes, that is polytopes of rank  $n \geq 4$  whose facets and vertex-figures are globally spherical or toroidal, as described above, with at least one kind toroidal. The  $n$ -polytopes of this kind have not yet been fully classified, although quite a lot is known (see [6, Chs. 10-12]).

As usual, we begin with a crystallographic linear Coxeter group  $G = \langle r_0, \dots, r_{n-1} \rangle$ , but immediately discard degenerate cases in which the underlying diagram  $\Delta(G)$  is disconnected. (In such cases  $G$  is reducible; and  $\mathcal{P}(G)$  and its quotients have the sort of ‘flatness’ described in [6, 4E].)

In [9] we discussed all locally toroidal 4-polytopes  $\mathcal{P}(G^p)$  which arise from our construction with prime modulus  $p$ . Since our methods for general moduli  $s$  add little to the discussion of such polytopes in [6, Chs. 10–11] and [9], we examine here just one group of rank 4, namely  $G = [3, 6, 3]$ , with diagram

$$\bullet \text{---} \overset{3}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{1}{\bullet} .$$

When  $s = 4$  we find that  $G^4$  has order 7680 and is the automorphism group of a locally toroidal 4-polytope in the class  $\langle \{3, 6\}_{(4,0)}, \{6, 3\}_{(4,0)} \rangle$ . Next we note in Table 5 that there are no embedding constraints on node 2. We conclude from Theorem 6.1(b) (and the subsequent remarks) and from [9, p. 345] that  $G^d$  is a string  $C$ -group whenever the modulus  $d$  is divisible by either 4 or an odd prime, that is, whenever  $d \geq 3$ . The polytope  $\mathcal{P}(G^d)$  is in the class  $\langle \{3, 6\}_{\mathbf{q}}, \{6, 3\}_{\mathbf{r}} \rangle$ , where always  $\mathbf{q} = (d, 0)$ , but  $\mathbf{r} = (d, 0)$  when  $3 \nmid d$  and  $\mathbf{r} = (\frac{d}{3}, \frac{d}{3})$  when  $3 \mid d$ . This construction complements the approach in [6, 11E].

Turning to higher rank  $n > 4$ , we observe that any spherical facet, or vertex-figure, must be of type  $\{3^{n-2}\}$ ,  $\{4, 3^{n-3}\}$ ,  $\{3^{n-3}, 4\}$  or  $\{3, 4, 3\}$  ( $n = 5$  only). Likewise, the required Euclidean section must have type  $\{4, 3^{n-4}, 4\}$  or when  $n = 6$ ,  $\{3, 3, 4, 3\}$  or  $\{3, 4, 3, 3\}$ . As described in [6, Lemma 10A1], these constraints severely limit the possibilities: in rank 5, we have just  $G = [4, 3, 4, 3]$  acting on hyperbolic space  $\mathbb{H}^4$ ; and in rank 6 we have  $G = [4, 3, 3, 4, 3]$ ,  $[3, 4, 3, 3, 3]$  or  $[3, 3, 4, 3, 3]$ , all acting on  $\mathbb{H}^5$ . Thus we may complete our discussion by examining the modular polytopes which result from these groups in ranks 5

and 6.

## 7.1 Rank 5: the group $G = [4, 3, 4, 3]$

Here we must contend with the four distinct basic systems encoded in the diagrams

$$\begin{array}{ccc}
 \begin{array}{cccccc}
 1 & \text{---} & 2 & \text{---} & 2 & \text{---} & 4 & \text{---} & 4 \\
 \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
 \end{array} & & \begin{array}{cccccc}
 1 & \text{---} & 2 & \text{---} & 2 & \text{---} & 1 & \text{---} & 1 \\
 \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
 \end{array} \\
 (a) & & (b) \\
 \begin{array}{cccccc}
 2 & \text{---} & 1 & \text{---} & 1 & \text{---} & 2 & \text{---} & 2 \\
 \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
 \end{array} & & \begin{array}{cccccc}
 4 & \text{---} & 2 & \text{---} & 2 & \text{---} & 1 & \text{---} & 1 \\
 \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
 \end{array} \\
 (c) & & (d)
 \end{array} \tag{13}$$

When the modulus is an odd prime  $p$ , the four corresponding finite groups  $G^p$  are isomorphic string  $C$ -groups; and we recall from [10, §4.1] that

$$G^p = \begin{cases} O_1(5, p, 0), & \text{if } p \equiv \pm 1 \pmod{8} \\ O(5, p, 0), & \text{if } p \equiv \pm 3 \pmod{8} \end{cases} \tag{14}$$

Note that  $O_1(5, p, 0)$  has order  $p^4(p^4 - 1)(p^2 - 1)$  and index two in  $O(5, p, 0)$  (see [8, pp. 300-301]). The facets of the corresponding regular 4-polytope  $\mathcal{P}(G^p)$  are toroids  $\{4, 3, 4\}_{(p,0,0)}$ , which one could construct by identifying opposite square faces of a  $p \times p \times p$  cube [8, 6.4]. Of course, the vertex-figures are copies of the 24-cell  $\{3, 4, 3\}$ .

Next, for modulus  $s = 4$ , we may check directly on GAP that  $G^4$  is a string  $C$ -group for each of the basic systems in (13). Diagrams (a), (b), (c) give polytopes of type  $\{ \{4, 3, 4\}_{(4,0,0)}, \{3, 4, 3\} \}$ , whose respective automorphism groups have orders  $g = 2^{16} \cdot 3^2$ ,  $g$ , and  $4g$ . On the other hand, diagram (d) gives a polytope of type  $\{ \{4, 3, 4\}_{(4,4,0)}, \{3, 4, 3\} \}$  whose group has order  $16g$ . By [6, 12B1], none of these polytopes can be universal for their type. However, with different generators, the third group, of order  $4g = 2\,359\,296$ , is the automorphism group for the universal polytope of type  $\{ \{4, 3, 4\}_{(2,2,2)}, \{3, 4, 3\} \}$  and hence is known to be isomorphic to  $(\mathbb{Z}_2^6 \rtimes \mathbb{Z}_2^5) \rtimes F_4$  (see [6, Thm. 8F19 and Table 12B1]).

Now consider *any* modulus  $d > 2$ , which again is divisible either by an odd prime  $s$  or by  $s = 4$ . We immediately conclude from Theorem 6.1(a), in its dual form, that  $G^d$  is a string  $C$ -group for each diagram in (13) and for each modulus  $d > 2$ .

If  $d$  is *odd*, it is easy to check that the four diagrams deliver isomorphic groups. Indeed, a change from any one of the four basic systems to another is accomplished by rescaling various  $b_j$ 's by powers of 2 (see [8, p. 305]). Since 2 is invertible modulo  $d$ , the corresponding linear groups are conjugate in  $GL_5(\mathbb{Z}_d)$ ; and, crucially, such isomorphisms pair off the specified generating reflections. Consulting Table 4 (with  $s$  replaced by  $d$ ), we conclude that the resulting non-universal polytope has type

$$\{ \{4, 3, 4\}_{(d,0,0)}, \{3, 4, 3\} \} . \tag{15}$$

For  $d$  *even*, we have already observed that a change in basic system may well alter the corresponding group and polytope. Referring again to Table 4, we do find that diagrams (a), (b), (c) in (13) provide polytopes of the type displayed in (15), now with  $d$  even. However, diagram (13)(d) gives a polytope of type  $\{ \{4, 3, 4\}_{(d,d,0)}, \{3, 4, 3\} \}$ .



The final subgroup  $K = \langle t_0, \dots, t_5 \rangle$  generated by

$$(t_0, t_1, t_2, t_3, t_4, t_5) := (r_2, r_1, r_0, r_3 r_2 r_1 r_2 r_3, r_4, r_5) \quad (19)$$

has index 10 in  $G$  and is isomorphic to  $[4, 3, 3, 4, 3]$ . Now diagrams (16)(a),(b) lead to diagrams (20)(a),(b) below:

$$\begin{array}{ccc}
 \begin{array}{cccccc}
 2 & 1 & 1 & 1 & 2 & 2 \\
 \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet
 \end{array} & & \begin{array}{cccccc}
 1 & 2 & 2 & 2 & 1 & 1 \\
 \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet
 \end{array} \\
 (a) & & (b) & & & & & & & & & & \\
 \begin{array}{cccccc}
 4 & 2 & 2 & 2 & 1 & 1 \\
 \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet
 \end{array} & & \begin{array}{cccccc}
 1 & 2 & 2 & 2 & 4 & 4 \\
 \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet
 \end{array} \\
 (c) & & (d) & & & & & & & & & & \\
 \end{array} \quad (20)$$

The group  $K$  admits the two other basic systems shown in (20)(c),(d). (See [6, 12A2]. Each group described above acts on  $\mathbb{H}^5$  with a simplicial fundamental domain of finite volume. In [7], these indices were computed by dissecting a simplex for  $H$  (or  $K$ ) into copies of the simplex for  $G$ .)

In [10, §4.2] we showed that  $G^p, H^p, K^p$  are string  $C$ -groups for any odd prime modulus  $p$ . In fact, all three are isomorphic to

$$\begin{cases} O_1(6, p, +1), & \text{if } p \equiv \pm 1 \pmod{8} \\ O(6, p, +1), & \text{if } p \equiv \pm 3 \pmod{8} \end{cases} \quad (21)$$

Of course, we require different generators in the three cases, as indicated in (17) and (19). Thus, the indices 5 and 10 in characteristic 0 collapse to 1 under reduction mod  $p$ . For any prime  $p \geq 3$ ,  $O_1(6, p, +1)$  has order  $p^6(p^4 - 1)(p^3 - 1)(p^2 - 1)$  and index two in  $O(6, p, +1)$  (see [8, pp. 300-301]).

Now suppose that the modulus is any *odd* integer  $d \geq 3$ . Just as in the previous subsection, the two diagrams in (16) give isomorphic groups, as do the four diagrams in (20). Furthermore, by the remarks following Theorem 6.1 we see that  $G^d, H^d$  and  $K^d$  are then string  $C$ -groups. In each case, the type vector for a toroidal section is  $\mathbf{q} = (d, 0, 0, 0)$ .

The situation for *even* moduli is more complicated. Once more, we may discard the modulus  $d = 2$ , which invariably causes a collapse to the hemi-24-cell in any section of type  $\{3, 4, 3\}$ . Let us consider the three groups in turn.

### The Polytopes $\mathcal{P} = \mathcal{P}(G^d)$ .

Using GAP, we find that  $G^4$  is a string  $C$ -group of order  $2^{26} \cdot 3^2 \cdot 5$  for either diagram in (16). It follows from Theorem 6.1(a) in its dual form that  $G^d$  is a string  $C$ -group for any modulus  $d > 2$ . From either diagram in (16) we obtain a locally toroidal polytope in the class

$$\langle \{3, 4, 3, 3\}_{(d,0,0,0)}, \{4, 3, 3, 3\} \rangle.$$

We note that the toroidal facets of  $\mathcal{P}(G^d)$  each have  $3d^4$  vertices [6, Table 6E1]; and, of course, the vertex-figures are 5-cubes  $\{4, 3, 3, 3\}$ . Although the two admissible diagrams do yield string  $C$ -groups, we have no general proof that these groups are isomorphic when  $d$  is *even*, though this is true for  $d = 4$ .

The following theorem establishes [6, Conjecture 12C2] concerning the existence of locally toroidal regular 6-polytopes of type  $\{3, 4, 3, 3, 3, 3\}$ .

**Theorem 7.1** *The universal regular 6-polytopes  $\{\{3, 4, 3, 3\}_{(d,0,0,0)}, \{4, 3, 3, 3\}\}$  and  $\{\{3, 4, 3, 3\}_{(d,d,0,0)}, \{4, 3, 3, 3\}\}$  exist for all  $d \geq 2$ .*

**Proof.** First note that the case  $d = 2$  was settled in [6, pp.460-461]. So let  $d > 2$ . We now appeal to our earlier remark that a non-empty class of regular polytopes contains a (unique) universal member (see [6, 4A2]). Thus, the existence of a universal polytope of the first kind (type vector  $\mathbf{q} = (d, 0, 0, 0)$ ) follows directly from our construction of a member of its class, namely  $\mathcal{P}(G^d)$ . For the existence of the universal polytopes of the second kind (type vector  $\mathbf{q} = (d, d, 0, 0)$ ) we refer to the discussion in [6, pp.460-462], where it was shown that the existence of the universal polytopes of the second kind is implied by existence of universal polytopes of the first kind. (In fact, some of the arguments provided there can now be simplified using properties of  $G^d$ .)  $\square$

The full classification of the finite universal polytopes of each kind is still open, but three of these are known to be finite, including

$$\{\{3, 4, 3, 3\}_{(3,0,0,0)}, \{4, 3, 3, 3\}\},$$

with automorphism group  $\mathbb{Z}_3 \rtimes O(6, 3, +1)$  ( $= \mathbb{Z}_3 \rtimes G^3$ ). See [10, §4.2].

**The Polytopes  $\mathcal{P} = \mathcal{P}(H^d)$ .**

We have already indicated that for  $d$  odd the polytope  $\mathcal{P}(H^d)$  lies in the class

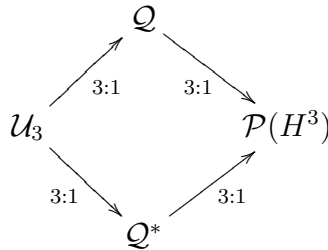
$$\langle \{3, 3, 4, 3\}_{(d,0,0,0)}, \{3, 4, 3, 3\}_{(d,0,0,0)} \rangle.$$

In fact,  $\mathcal{P}(H^d)$  admits an order reversing bijection and so is *self-dual*.

The modulus  $p = 3$  is of particular interest. In [10, §4.2] we gave a new construction for the corresponding (finite!) self-dual universal polytope

$$\mathcal{U}_{H^3} := \{ \{3, 3, 4, 3\}_{(3,0,0,0)}, \{3, 4, 3, 3\}_{(3,0,0,0)} \}.$$

Indeed,  $\Gamma(\mathcal{U}_{H^3}) \simeq (\mathbb{Z}_3 \oplus \mathbb{Z}_3) \rtimes H^3$  under a non-trivial action of  $H^3$  on the abelian factor. Thus  $\mathcal{U}_{H^3}$  is a 9-fold cover of  $\mathcal{P}(H^3)$  ([6, Table 12D1]); and trapped between we find a twin pair  $\mathcal{Q}, \mathcal{Q}^*$  of *non-self-dual* polytopes, with the same toroidal facets and vertex-figures:



Turning to even moduli, we again find that  $H^4$  is a string  $C$ -group (of index 5 in  $G^4$ ); and we note that there are no embedding constraints on node 4 (look at the second diagram in Table 5). Thus, by the discussion following Theorem 6.1, we conclude that  $H^d$  is a string  $C$ -group for all  $d > 2$ . When  $d$  is even, the corresponding polytope is in the class

$$\langle \{3, 3, 4, 3\}_{(d,0,0,0)}, \{3, 4, 3, 3\}_{(\frac{d}{2}, \frac{d}{2}, 0, 0)} \rangle,$$

and hence is certainly not self-dual.

Notice that the type vectors for the facets and vertex-figures of the polytopes  $\mathcal{P}(H^d)$  are related in that they involve the same parameter  $d$ . Thus we cannot expect our methods to completely settle Conjecture 12D3 of [6] concerning the existence of locally toroidal regular 6-polytopes of types  $\{3, 3, 4, 3, 3\}$ , for which the parameters for the facets and vertex-figures may vary independently. The same remark applies to the polytopes  $\mathcal{P}(K^d)$  studied next, and Conjecture 12E3 of [6] for the corresponding type  $\{4, 3, 3, 4, 3\}$ .

**The Polytopes  $\mathcal{P} = \mathcal{P}(K^d)$ .**

For odd  $d \geq 3$  the four diagrams in (20) give isomorphic polytopes in the class

$$\langle \{4, 3, 3, 4\}_{(d,0,0,0)}, \{3, 3, 4, 3\}_{(d,0,0,0)} \rangle .$$

Here the facets are cubical toroids; facets and vertex-figures each have  $d^4$  vertices.

Suppose then that  $d \geq 4$  is even. A calculation with GAP reveals the at first surprising result that the intersection condition (2) fails for diagrams (20)(b)(d), at least when  $d = 4, 6$ . Noting that dropping the last node in each case alters the constraints on node 4, we therefore abandon these diagrams.

For diagram (20)(a) we easily verify that  $K^4$  is a string  $C$ -group (of index 10 in  $G^4$ ). Note that there are no embedding constraints on node 4; see the first diagram in Table 4, with  $m = 4$  and  $s = d$  even. From Theorem 6.1, we thus obtain a polytope in the class

$$\langle \{4, 3, 3, 4\}_{(\frac{d}{2}, \frac{d}{2}, \frac{d}{2}, \frac{d}{2})}, \{3, 3, 4, 3\}_{(\frac{d}{2}, \frac{d}{2}, 0, 0)} \rangle , \quad (\text{even } d \geq 4) .$$

Here the facets have  $d^4/2$  vertices; and each vertex-figure has  $d^4/4$  vertices.

The analysis for diagram (20)(c) is similar, although the particular location of the subgroup  $[3, 4, 3]$  prevents an automatic verification of condition (11). Nevertheless, by brute-force calculation, we find that (11) holds for any modulus  $d \geq 2$ . On the other hand, for  $d = 4$  with this basic system, we can independently check on GAP that  $K^4$  is indeed a string  $C$ -group, with (unexpected) order  $2^{29} \cdot 3^2$ . It follows from Theorem 6.1(b) that  $K^d$  is a string  $C$ -group for any modulus  $d \geq 3$ . In particular, when  $d \geq 4$  is even we obtain a polytope in the class

$$\langle \{4, 3, 3, 4\}_{(d,d,0,0)}, \{3, 3, 4, 3\}_{(d,0,0,0)} \rangle .$$

Here the facets each have  $2d^4$  vertices.

**Acknowledgement.** We wish to thank an anonymous referee for the careful reading of the manuscript and several helpful comments on it.

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