Combinatorial Aperiodicity of Polyhedral Prototiles*

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Abstract

The paper studies combinatorial prototiles of locally finite face-to-face tilings of euclidean d-space $\mathbb{E}^d$ by convex d-polytopes. A finite set $\mathcal{P}$ of prototiles is called combinatorially aperiodic if $\mathcal{P}$ admits a locally finite face-to-face tiling by combinatorially equivalent copies of the prototiles in $\mathcal{P}$, but no such tiling with a combinatorial automorphism of infinite order. The paper describes some properties of combinatorially aperiodic protosets and their tilings, and also discusses some open problems and conjectures.

1 Introduction

Aperiodicity is a fascinating phenomenon. Traditionally, an aperiodic protoset is a set of tiles in euclidean d-space $\mathbb{E}^d$ that admit a tiling of $\mathbb{E}^d$ by congruent copies, but no such tiling with a non-trivial translational symmetry. In the light of the discovery of the Schmitt-Conway-Danzer tile and its tiling properties ([4]), the notion of aperiodicity has been revised to require the stronger condition that no tiling by the protoset have a euclidean symmetry of infinite order. The purpose of this short note is to introduce a combinatorial analogue of this stronger notion, called *combinatorial aperiodicity*. Congruence of the tiles is here replaced by combinatorial equivalence of the tiles, and the interest is in locally finite face-to-face tilings of $\mathbb{E}^d$ by convex polytopes each combinatorially equivalent to a polytope from a finite protoset of convex polytopes. At this point it is still open whether or not combinatorially aperiodic protosets actually exist in any dimension. We describe some results about combinatorially aperiodic protosets and their tilings, and also discuss some open problems and conjectures.

2 Background

A *tiling* $\mathcal{T}$ of euclidean d-space $\mathbb{E}^d$ is a countable family of closed subsets of $\mathbb{E}^d$, the *tiles* of $\mathcal{T}$, which cover $\mathbb{E}^d$ without gaps and overlaps ([8]). We shall assume that the tiles are

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convex d-polytopes. A tiling $\mathcal{T}$ by convex d-polytopes is said to be *face-to-face* if the intersection of any two tiles is a face of each tile, possibly the empty face. All tilings are taken to be *locally finite*, meaning that each point of $\mathbb{E}^d$ has a neighborhood that meets only finitely many tiles.

A *protoset* of a tiling $\mathcal{T}$ of $\mathbb{E}^d$ is a minimal subset of tiles of $\mathcal{T}$ such that each tile of $\mathcal{T}$ is congruent to one of those in the subset. The tiles in the subset are the *prototiles* of $\mathcal{T}$, and the protoset is said to *admit* the tiling $\mathcal{T}$. By abuse of notation, we also use this terminology for shapes and sets of shapes that are under consideration for being prototiles or protosets of tilings, respectively.

A tiling $\mathcal{T}$ in $\mathbb{E}^d$ is called *periodic* if its symmetry group is a crystallographic group and thus contains translations in d linearly independent directions. A *non-periodic* tiling has no (non-trivial) translational symmetry at all. A protoset is called *weakly aperiodic* if it admits a tiling of $\mathbb{E}^d$ but if all such tilings are non-periodic (see, for example, [7]). Moreover, a protoset is (*isometrically*) *aperiodic* if it admits a tiling of $\mathbb{E}^d$ but if no such tiling has a symmetry of infinite order. In the euclidean plane, each symmetry of infinite order is a translation or a glide reflection; thus aperiodicity and weak aperiodicity are equivalent concepts for plane tilings.

There exists a considerable body of literature about aperiodicity of protosets, especially planar protosets. The first appearance of an aperiodic protoset was Berger's (1966) discovery of a set of 20426 square tiles with colored edges, so-called Wang-tiles; in particular, this proved that the so-called "Tiling Problem" is undecidable (that is, no algorithm exists that, upon being fed a protoset, decides whether it admits a tiling or not). Many (colored or otherwise) decorated or non-decorated aperiodic protosets have been discovered since then, but all have at least two prototiles. Some famous examples found by Robinson (1971), Penrose (1974, 1978) and Ammann (1977) are surveyed in Grünbaum & Shephard [8] and Senechal [17]; others occur in, for example, [2, 3, 10]. There are aperiodic protosets with only two prototiles in any dimension $d \geq 2$ (Goodman-Strauss [6]).

The tile of [4], known as the Schmitt-Conway-Danzer tile, comes remarkably close to being a single aperiodic prototile in dimension 3. A tiling by *directly* congruent copies of it (that is, mirror images are not allowed) cannot have translational symmetry but must have screw rotational symmetry of infinite order in one direction; if mirror images are allowed, then periodic tilings do exist. The existence of a single aperiodic prototile, an *aperiodic monotile*, is an important open problem about tilings in euclidean spaces.

Aperiodic protosets and their non-periodic tilings have truly amazing properties ([8, 17]). Three basic construction techniques for the protosets and their tilings have emerged over time.

One type of construction involves decorated tiles and matching rules. For example, the two famous Penrose rhombs have their vertices colored black or white, and have orientations given to some of their edges; considered are only those tilings for which the colors at the vertices and the orientations of the sides of adjacent tiles match.

In another type of construction, the aperiodicity is based on the existence of a hierarchical structure on the tilings. These tilings are defined by a substitution rule and an inflation process. The substitution rule tells us how each prototile is decomposed into tiles at a smaller scale, each a homothetic copy of some prototile by a fixed factor λ^{-1}; the tilings are then produced by an iterative process of subdividing tiles and expanding by the "inflation factor" λ.

Finally, the projection method produces the tiles of a non-periodic tiling of $\mathbb{E}^d$ by projecting onto $\mathbb{E}^d$ certain d-faces of the Delone tiling (dual of Voronoi tiling) that is associated with a lattice in a higher-dimensional space (superspace) $\mathbb{E}^n$; the d-faces that are projected down are those whose projection onto the orthogonal complement of $\mathbb{E}^d$ in $\mathbb{E}^n$ lies within a certain "acceptance region" (window).

There are variants and generalizations of these methods, and there is a wealth of interesting examples. However, only a few examples can be obtained by all three processes.

3 Combinatorial aperiodicity

We now investigate a combinatorial version of aperiodicity in tilings by convex polytopes. In this, congruence of the tiles is replaced by combinatorial equivalence of the tiles. We are interested in locally finite tilings $\mathcal{T}$ of $\mathbb{E}^d$ by (non-decorated) *convex* polytopes each combinatorially equivalent to one polytope from a given finite set $\mathcal{P}$ of convex polytopes, the *combinatorial protoset* of *combinatorial prototiles* of $\mathcal{T}$. Again, $\mathcal{P}$ is said to *admit* the tiling $\mathcal{T}$. As before, we use similar terminology also for polytopes and sets of polytopes that are under consideration for being combinatorial prototiles or combinatorial protosets. The tilings $\mathcal{T}$ with a single combinatorial prototile are called *monotypic*. Throughout we insist on convexity of the tiles.

Combinatorial prototiles have been studied in detail in [14, 15]. For each $d \geq 3$ there are polytopes, called *nontiles*, which are not (single) combinatorial prototiles of monotypic tilings of $\mathbb{E}^d$ that are face-to-face. On the other hand, each convex 3-polytope does admit a monotypic tiling of 3-space; however, this tiling will not be face-to-face in general.

A finite protoset $\mathcal{P}$ of convex d-polytopes is called *combinatorially aperiodic* if $\mathcal{P}$ admits a locally finite face-to-face tiling $\mathcal{T}$ by combinatorially equivalent convex copies of the prototiles, but if no such tiling has a combinatorial automorphism of infinite order. Thus an automorphism of a locally finite face-to-face tiling with protoset $\mathcal{P}$ must have finite order.

Note that we are not requiring the stronger condition that each tiling $\mathcal{T}$ must have a finite combinatorial automorphism group $\Gamma(\mathcal{T})$. This condition would lead to a stronger notion of combinatorial aperiodicity than is intended here. Moreover, we are also not defining here what it could mean that a tiling $\mathcal{T}$ is "combinatorially periodic"; it seems that a definition would have to require at a minimum that $\Gamma(\mathcal{T})$ has only finitely many orbits on the flags of $\mathcal{T}$.

Very little seems to be known about general properties of combinatorial automorphisms of face-to-face tilings $\mathcal{T}$ that are not induced by isometries of the underlying space. For example, it does not seem to be obvious that an automorphism of $\mathcal{T}$ of finite order must leave one face of $\mathcal{T}$ invariant.

Our first observation is straightforward. If a combinatorially aperiodic protoset admits a face-to-face tiling by *congruent* copies of the prototiles, then such a tiling cannot have a symmetry of infinite order (because this would yield a combinatorial automorphism of infinite order); that is, in the context of face-to-face tilings, a combinatorially aperiodic protoset is also isometrically aperiodic provided it does admit a tiling by congruent copies.

Combinatorial aperiodicity requires a property to hold for a larger class of tilings (and in this sense it is stronger than isometrical aperiodicity), but it does not require the existence of a tiling by congruent copies (and in this sense it is weaker).

Every tiling of the real line by closed intervals has a combinatorial automorphism of infinite order, namely the "shift" in one direction. Thus there is no combinatorially aperiodic protoset on the line. The same is also true for the plane but is not quite as obvious.

Theorem 3.1 *There is no combinatorially aperiodic protoset in the plane.*

Proof: Let p, q be positive integers such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, and let $\{p, q\}$ denote the regular tiling of the hyperbolic plane by regular p-gons, q meeting at each vertex. Then there exists a tiling $\mathcal{T}'$ of the euclidean plane which is combinatorially equivalent to $\{p, q\}$ ([9, 15]). To construct $\mathcal{T}'$ we begin with a convex p-gon which contains the origin o as an interior point. We now add new convex p-gons such that, after each step, the resulting finite patch of tiles is star-shaped with respect to o (every ray emanating from o intersects the boundary of the patch in exactly one point), and every vertex on the boundary is in at most two tiles. More precisely, if x is a vertex on the boundary of an existing patch which is contained in two tiles, we add $q - 2$ new p-gonal tiles with vertex x, which, together with the two existing tiles, fully surround x; with some care, this new patch is again star-shaped, and each new vertex is in at most two tiles and sufficiently far away from the origin. The latter condition guarantees that the iteration of the process will yield a tiling $\mathcal{T}'$ of the entire plane. The tiling is indeed combinatorially equivalent to the hyperbolic tiling $\{p, q\}$ because it is "freely generated" from p-gons where q meet at each vertex. Note that the combinatorial automorphism group of $\mathcal{T}'$ is isomorphic to the symmetry group G of $\{p, q\}$ and thus contains elements of infinite order.

Now suppose that $\mathcal{P} := \{P_1, \ldots, P_n\}$ is a combinatorially aperiodic protoset in the euclidean plane, with P_i a p_i-gon for each i. The above arguments show that we cannot have $n = 1$; in fact, if $p := p_1$ and q is chosen properly, we obtain a tiling $\mathcal{T} := \mathcal{T}'$ by p_1-gons which has an automorphism of infinite order.

We can also rule out the case $n \geq 2$. It is not difficult to see that, if q is even, each p-gonal 2-face of the hyperbolic tiling $\{p, q\}$ is a fundamental region for the subgroup H of G that is generated by the reflections in the sides of the 2-face (see, for example, [12, p.269,275]). In fact, G is the semi-direct product of H and the symmetry group of the 2-face. Clearly, this subgroup H contains elements of infinite order. Now, if we dissect some 2-face P in an edge-to-edge manner into smaller polygons, then the images under H yield an H-invariant tiling in which every old 2-face is dissected in exactly the same way. If each smaller polygon is a p_i-gon for some i, with each i actually occurring, we then have an edge-to-edge tiling of the hyperbolic plane, in which each tile is a p_i-gon for some i, again with each i occurring, and which admits H as a group of symmetries. We now apply this observation with $p := (p_1 + \ldots + p_n) - 2n$, using the following dissection of the p-gon: split the boundary of the p-gon into n edge-disjoint paths, with the i^{th} path consisting of $p_i - 2$ edges, and then join their endpoints to the center of the p-gon. In our final step we employ the isomorphism between $\{p, q\}$ and $\mathcal{T}'$, and pass from the resulting tiling of the hyperbolic plane to a corresponding tiling $\mathcal{T}$ of the euclidean plane by dissecting each tile of $\mathcal{T}'$ into convex p_i-gons as dictated by the isomorphism. Then $\mathcal{T}$ is a locally finite face-to-face (edge-to-edge) tiling of $\mathbb{E}^2$ with protoset $\mathcal{P}$. Moreover, by construction, $\mathcal{T}$ has combinatorial automorphisms of infinite order, including those that correspond to elements of infinite order in H. □

We now discuss higher dimensions. Here it is not known whether or not combinatorially aperiodic protosets exist.

Problem 3.2 *Are there combinatorially aperiodic protosets in dimensions $d \geq 3$?*

Call a set $\{P_1, \ldots, P_n\}$ of convex d-polytopes *facet-forming* if there exists a convex $(d+1)$-polytope Q such that each facet of Q is combinatorially equivalent to some P_i, and each P_i is combinatorially equivalent to some facet of Q. A vertex of a convex d-polytope is *simple* if it is contained in exactly d edges.

Theorem 3.3 *Let $\mathcal{P} := \{P_1, \ldots, P_n\}$ be a facet-forming set of convex d-polytopes. If $P_1, \ldots, P_n$ can be realized as the types of facets of a $(d+1)$-polytope Q which has a simple vertex, then $\{P_1, \ldots, P_n\}$ admits a locally finite face-to-face tiling of $\mathbb{E}^d$ which is periodic. Thus $\mathcal{P}$ is not combinatorially aperiodic.*

Proof: We use the same idea as in [14, Thm.3]. Let x be a simple vertex of Q, and let T be the d-simplex spanned by the $d+1$ neighbors of x in the boundary complex of Q; then T is the vertex-figure of Q at x. We now project the facets of Q which do not contain x, centrally from x into T. This yields a face-to-face "tiling" of T into polytopes each combinatorially equivalent to some P_i. We then map T affinely onto the standard fundamental region (d-simplex) for the symmetry group of the cubical tessellation in $\mathbb{E}^d$; this results in a corresponding tiling of the fundamental d-simplex. In the final step we apply all the symmetries of the cubical tessellation to generate a tiling $\mathcal{T}$ of the entire space. Clearly, $\mathcal{T}$ is periodic.

Each tile in $\mathcal{T}$ is combinatorially equivalent to a facet of Q which does not contain x. If each prototile P_i is represented by one of these facets, then $\mathcal{T}$ is indeed a locally finite face-to-face tiling with protoset $\mathcal{P}$. However, if there exists a prototile P_i which only is represented by a facet which contains x, then P_i will not occur as a tile of $\mathcal{T}$. In this case we replace Q by a new polytope Q', for which each P_i is represented by a facet not containing the simple vertex, and then apply the above construction to Q' instead of Q. To obtain Q' we adjoin a small projective copy of Q along a facet of Q which does not contain x (as described in, for example, [13, p.121]), and repeat this construction once more if necessary until the desired property holds. □

The basic idea of the proof works in more general circumstances. If the $(d+1)$-polytope $\mathcal{Q}$ has a vertex-figure T that admits a suitable tiling $\mathcal{T}'$ of $\mathbb{E}^d$, then we can map the "tiling" in T obtained by projection, into the tiles of $\mathcal{T}'$ to create a new tiling $\mathcal{T}$ with tiles combinatorially equivalent to $P_1, \ldots, P_n$.

We now describe a construction of tilings which shares some of the features of the construction of non-periodic tilings by inflation. Recall that a *d-diagram* $\mathcal{D} := \{D\} \cup \mathcal{C}$ in $\mathbb{E}^d$ consists of a convex d-polytope D and a finite (face-to-face) d-complex $\mathcal{C}$ of convex polytopes, such that D is the union of the members in $\mathcal{C}$, each proper face of D is a member of $\mathcal{C}$, and each member of $\mathcal{C}$ intersects the boundary of D in a (possibly empty) member of $\mathcal{C}$ ([18]); we call D the *support* of $\mathcal{D}$, and the d-polytopes in $\mathcal{C}$ the *facets* of $\mathcal{D}$. The Schlegel diagrams of convex $(d+1)$-polytopes yield examples of d-diagrams, but not every diagram is a Schlegel diagram.

Lemma 3.4 *Let $\mathcal{D} := \{D_1\} \cup \mathcal{C}_1$ be a d-diagram in $\mathbb{E}^d$ with support D_1, and let D_0 be a facet of $\mathcal{D}$ contained in the interior of D_1. Let $\varphi :\ \mathbb{E}^d \mapsto \mathbb{E}^d$ be a similarity transformation that maps D_0 onto D_1. Define*

$$\mathcal{T} := (\cup_{n\geq 0}\ \varphi^n(\mathcal{C}_1 \setminus \{D_0\})) \cup \{D_0\}, \quad \mathcal{T}' := \cup_{n\in\mathbb{Z}}\ \varphi^n(\mathcal{C}_1 \setminus \{D_0\}).$$

Then,
a) $\mathcal{T}$ is a tiling of $\mathbb{E}^d$, in which every tile is similar to a facet of $\mathcal{C}_1$, with every facet of $\mathcal{C}_1$ actually occurring;
b) $\mathcal{T}'$ is a tiling of $\mathbb{E}^d$ minus a point in the interior of D_0, in which every tile is similar to a facet of $\mathcal{C}_1 \setminus \{D_0\}$, with every facet of $\mathcal{C}_1 \setminus \{D_0\}$ actually occurring.

Proof: For each integer n define $D_n := \varphi^n(D_0)$ and $\mathcal{C}_n := \varphi^{n-1}(\mathcal{C}_1)$; then $\{D_n\} \cup \mathcal{C}_n$ is a d-diagram, the image of $\mathcal{D}$ under φ^{n-1}. Then $\mathcal{T}$ and $\mathcal{T}'$ are obtained as the union of the complexes $\mathcal{C}_n \setminus \{D_{n-1}\}$, in the case of $\mathcal{T}$ with $n \geq 1$ and with D_0 added as the initial tile. It helps to think of the tiles in $\mathcal{C}_n \setminus \{D_{n-1}\}$ as the *tiles of level n* (they lie between the boundaries of D_{n-1} and D_n), and, in the case of $\mathcal{T}$, of the initial tile D_0 as an additional tile of level 1. Note that the polytopes D_n are not tiles, except for D_0 in the case of $\mathcal{T}$. By construction, each tile distinct from D_0 is similar to a d-polytope in $\mathcal{C}_1 \setminus \{D_0\}$; this will generally not be true for the initial tile D_0 of $\mathcal{T}$, so that we must include D_0 and take $\mathcal{C}_1$ instead of $\mathcal{C}_1 \setminus \{D_0\}$.

It is immediate that $\mathcal{T}$ and $\mathcal{T}'$ are actually tilings as stated. In fact, φ maps D_0 onto the strictly larger set D_1, and hence φ must be a similarity transformation that expands with a factor $c > 1$. If r_0 and R_0, respectively, are the inradius and circumradius of the polytope D_0, then the inradius and circumradius of D_n are $r_n = c^n r_0$ and $R_n = c^n R_0$ for each n. But $r_n \to \infty$ as $n \to \infty$, and $R_n \to 0$ as $n \to -\infty$, and therefore we have the desired tiling property. The point of $\mathbb{E}^d$ that lies in each set D_n is the singular point of $\mathcal{T}'$ (that is, the point not covered by $\mathcal{T}'$). □

Let $\mathcal{T}$ be a locally finite face-to-face tiling of $\mathbb{E}^d$ (or a subset of $\mathbb{E}^d$) by convex d-polytopes. A *homomorphism* of $\mathcal{T}$ (or of a subcomplex of $\mathcal{T}$, respectively) is a mapping of the set of faces of $\mathcal{T}$ (or of the subcomplex) onto itself that is incidence-preserving. An automorphism of $\mathcal{T}$ then is a bijective mapping φ of $\mathcal{T}$ for which both φ and φ^{-1} are homomorphisms. The set of all homomorphisms of $\mathcal{T}$ forms a semigroup. This is an analogue of the semigroup of self-similarities arising in the context of quasicrystals ([1]). The tiling $\mathcal{T}$ of $\mathbb{E}^d$ produced in Lemma 3.4 will generally not have any automorphisms; however, if we remove the initial (open) tile D_0, then we obtain an injective homomorphism of infinite order, namely the mapping induced by φ. For the tiling $\mathcal{T}'$ of $\mathbb{E}^d$ minus a point, this mapping is a genuine automorphism.

Theorem 3.5 *Let $\mathcal{P} := \{P_1, \ldots, P_n\}$ be a facet-forming set of convex d-polytopes. Then $\mathcal{P}$ admits a locally finite face-to-face tiling of $\mathbb{E}^d$, which has only finitely many similarity* *classes of convex tiles, and which has an injective homomorphism of infinite order if one (open) tile is removed.*

Proof: The result is a generalization and a stronger version of [14, Thm.2]. We apply the construction of Lemma 3.4 with a suitable initial diagram $\mathcal{D} := \{D_1\} \cup \mathcal{C}_1$ and initial facet D_0 of $\mathcal{C}_1$.

We begin by constructing a $(d+1)$-polytope Q with a pair of facets that are translates of each other. Let Q' be any convex $(d+1)$-polytope such that each facet of Q' is combinatorially equivalent to some P_i, and each P_i is combinatorially equivalent to some facet of Q'. We now alter Q' as follows. First, by adjoining small projective copies of Q' along facets of Q' if need be (again see, [13, p.121]), we can assume that Q' has a pair of disjoint facets, and that every P_i (still) occurs as the type of a facet of Q'. Second, by applying a suitable projective transformation which maps the intersection subspace of the corresponding pair of supporting hyperplanes to infinity, we can also assume that these two disjoint facets are parallel. Third, with yet another projective transformation we can move a hyperplane that is parallel and very close to one of these facets, to infinity; the effect is that the resulting polytope Q'' now has a pair of parallel facets, where one facet, F (say), is so huge that the orthogonal projection onto the affine hull of F maps the set $Q'' \setminus F$ into the interior of F. Finally, to obtain the desired polytope Q we reflect Q'' in the affine hull of F, and adjoin Q'' and its mirror image along F. Then Q has a pair of facets (parallel to F) that are translates of each other. Moreover, each facet of Q is combinatorially equivalent to some P_i, and each P_i is combinatorially equivalent to some facet of Q.

We now take a suitable Schlegel diagram of Q. If F_0 and F_1 are the two distinguished facets of Q, we project from a suitable point beyond F_1, onto the supporting plane of F_0. The result is a d-diagram $\mathcal{D}$ whose support D_1 is the image of F_1 under the projection; the initial facet is given by $D_0 := F_0$ and is homothetic to D_1. Now Lemma 3.4 applies. □

The last two theorems support the following conjecture.

Conjecture 3.6 *A facet-forming set of convex d-polytopes is not combinatorially aperiodic.*

The second part of Lemma 3.4 can sometimes be used to construct arbitrarily large (in a combinatorial sense) patches of tiles in situations when a global tiling of space is not known to exist. For example, it is not known if there is a monotypic face-to-face tiling of $\mathbb{E}^d$ all of whose tiles are d-cubes with one vertex cut off. On the other hand, there is such a tiling of $\mathbb{E}^d$ minus a point. In fact, if we cut off one pair of antipodal vertices of the $(d+1)$-cube by hyperplanes orthogonal to the corresponding connecting diagonal, we obtain a convex $(d+1)$-polytope $\mathcal{Q}$ which has as facets two simplices and $2d+2$ cubes with one vertex cut off. Then, if we project from a point beyond one of these two simplices, we arrive at a Schlegel diagram $\mathcal{D}$ of $\mathcal{Q}$ to which Lemma 3.4 applies (with simplices D_0 and D_1). Moreover, if the projection point is on the diagonal of the original cube, the facets of $\mathcal{D}$ distinct from D_0 and D_1 form only two congruence classes, and hence the tiles of the corresponding tiling form only two similarity classes.

In dimension 3, there is a lot of freedom to choose the metrical shape of the tiles within a combinatorial equivalence class of tiles. This is generally not true in dimensions $d \geq 4$. The answers to Problem 3.2 may thus be different in the two cases $d = 3$ and $d \geq 4$. We conjecture that, if $d \geq 4$, there are indeed protosets of convex d-polytopes which admit locally finite face-to-face tilings, but only tilings with automorphisms of finite order. In other words, we have

Conjecture 3.7 *There are combinatorially aperiodic protosets in dimensions $d \geq 4$.*

In fact, there might even exist a combinatorially aperiodic protoset for which every tiling has a trivial combinatorial automorphism group.

It would also be interesting to settle the existence of a combinatorially aperiodic *monotile* (single combinatorially aperiodic prototile). As we mentioned before, there are examples of convex polytopes in dimension 3 which are not (single) prototiles of monotypic face-to-face tilings ([14]). On the other hand, every convex 3-polytope that is simplicial (has triangular 2-faces) does admit such a tiling ([9]); hence, for the class of simplical 3-polytopes, no obstruction arises from the condition to admit a tiling. It would already be interesting to decide the question for simplicial 3-polytopes.

Problem 3.8 *Does every simplicial convex 3-polytope admit a locally finite face-to-face tiling of $\mathbb{E}^3$ which has a combinatorial automorphism of infinite order?*

A similar question may of course also be asked for finite protosets consisting of simplicial convex 3-polytopes. Note that there are simplicial 3-polytopes which are not facet-forming ([13]); therefore we cannot expect to obtain a complete answer to Problem 3.8 by solving Conjecture 3.6.

We have not addressed normality in the above. (Recall that a tiling by convex polytopes is *normal* if its tiles are uniformly bounded.) If we require the face-to-face tilings $\mathcal{T}$ to be normal, the picture changes completely and we arrive at a different notion of combinatorial aperiodicity. For example, there are simplicial 3-polytopes that do not tile face-to-face and normally ([5]), and hence these are excluded from the onset because they do not admit a tiling. It is also not known if every facet-forming set of convex d-polytopes admits any normal tiling of $\mathbb{E}^d$ at all, let alone a normal face-to-face tiling.

Another variant of the problem may allow tiles which are topologically more complicated, like solid tori or more general solid handlebodies, knotted or unknotted. Examples of such tilings have been described in, for example, Kuperberg [11].

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