Locally unitary groups and regular polytopes *

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Abstract

Complex groups generated by involutory reflexions arise naturally in the modern theory of abstract regular polytopes. The paper investigates this relationship, and explains how the enumeration of certain finite universal regular polytopes can be accomplished through the enumeration of certain types of finite complex reflexion groups. In particular, the paper enumerates all the finite groups and their diagrams which arise in this context, and describes the corresponding regular polytopes.

1 Introduction

In the previous paper [22], we extensively investigated groups which preserve a hermitian form on complex n-space and are generated by n hyperplane reflexions of period 2. In the present paper, we further discuss those reflexion groups which arise in the modern theory of abstract regular polytopes (see [23]).

It is striking that complex hermitian forms occur in the enumeration of certain universal regular polytopes, including several classes of locally toroidal polytopes. It turns out that such a polytope is finite if and only if the corresponding hermitian form is positive definite. Its automorphism group is then a semi-direct product of a finite unitary reflexion group by a small finite group. The link between polytopes and hermitian forms

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generalizes the well-known classical situation, where the structure of a regular tessellation on the sphere or in euclidean or hyperbolic space is determined by a real quadratic form which defines the geometry of the ambient space (see Coxeter [8]); however, in the present context, this link is weaker.

In Section 2, we review some general results obtained in [22] on complex groups generated by involutory reflexions. In the most interesting case, the subgroups generated by all but a few generating reflexions are finite unitary groups; informally, such reflexion groups might be called *locally unitary* (but see Section 2 for the technical definition). The reader is also referred to the original work of Shephard & Todd [33], who were the first to classify completely all the finite unitary groups generated by reflexions (of any period), and to the work of Coxeter [6, 7, 9] and Cohen [3].

In Section 3, we investigate in detail the reflexion groups in complex 4-space; these are generated by four involutory reflexions and are described by a tetrahedral diagram with marked edges and marked triangles. Then, in Section 4, we consider abstract n-generator groups with presentations generalizing those discussed in Section 3 for the case $n = 4$, and study their representations as complex reflexion groups. The results of these two sections are of considerable interest, independent of their applicability in the enumeration of regular polytopes.

In Section 5, we review basic notions about regular polytopes, and then explain in general terms how the results of the previous sections can be applied to enumerate certain types of finite universal regular polytopes. The actual applications of the technique are described in Sections 6 and 7.

2 Complex reflexion groups

We shall require some results obtained in [22] about complex groups G generated by involutory hyperplane reflexions.

Let G be of the form $G = \langle S_1, \ldots, S_n \rangle$, with involutory generating reflexions S_j given by

$$xS_j = x - 2(\langle x, v_j \rangle - \beta_j)v_j$$

for $j = 1, \ldots, n$. In this context, $\langle \cdot, \cdot \rangle$ is an hermitian form on $\mathbb{C}^n$, not necessarily non-singular, β_j is a scalar, and $\langle v_j, v_j \rangle = 1$ for each j; the determinant Δ of the matrix $(\langle v_i, v_j \rangle)_{ij}$ is called the *Schläfli determinant* (see [8, §7.7]). If $\Delta \neq 0$, the vectors $v_1, \ldots, v_n$ are necessarily linearly independent, and we may always assume that $\beta_j = 0$ for each j. In the case when G is finite we will always have a positive definite form (and thus $\Delta > 0$). We allow the possibility that the unit vectors v_j are linearly dependent, to account for the infinite discrete groups in unitary space; in this *degenerate* case $\Delta = 0$. In the examples with $\Delta = 0$, any $n - 1$ of the vectors v_j will be independent and will span $\mathbb{C}^{n-1}$, and we need take only one β_j non-zero. The group G is called *locally unitary* (*locally finite*) if each subgroup generated by $n - 1$ of the reflexions S_j is actually a finite reflexion group, acting on an $(n - 1)$-space on which the form is positive definite.

We now associate with a set $\mathcal{S}$ of generators S_j of G a (basic geometric) *diagram* $\mathcal{D}$. It has n *nodes* labelled $1, \ldots, n$, with j corresponding to S_j. For each $j \neq k$, if the product S_jS_k has finite period, there is a rational number $p_{jk} = p_{kj} \geqslant 2$, not necessarily an integer, such that $|\langle v_j, v_k\rangle| = \cos\frac{\pi}{p_{jk}}$; if the period is infinite (and v_j, v_k are independent), we take $p_{jk} = \infty$. The *branch* joining j and k is then labelled p_{jk}; we follow the standard conventions in excising a branch which would be labelled 2, and omitting the label 3 on branches because of its frequency. If $\Delta \neq 0$, then G is irreducible if and only if the diagram of G associated with $S_1, \ldots, S_n$ is connected. We also define $\Delta(\mathcal{D}) := \Delta$.

A *basic operation* is a change of generators of the following kind: with $j \neq k$, we replace S_j by $S_j' := S_k S_j S_k$, and leave the other $n-1$ generators unchanged. Two sets of generators of G which can be obtained from each other by a sequence of basic operations are called *basically equivalent*. The diagrams corresponding to basically equivalent sets of generators are called *basically equivalent diagrams*. We now have

Theorem 2.1 *Let G be a finite group acting on $\mathbb{C}^n$, which preserves a positive definite hermitian form, and is generated by n involutory hyperplane reflexions with linearly independent normals. Then the set $\mathcal{S}$ of generators of G can be chosen so that the branches of any diagram basically equivalent to that of $\mathcal{S}$ bear only integer marks.*

Theorem 2.2 *Let $G = \langle S_0, \ldots, S_n\rangle$ be an infinite discrete group, which preserves a positive definite hermitian form on $\mathbb{C}^n$, and is generated by involutory reflexions S_j in hyperplanes whose normals v_j span $\mathbb{C}^n$, with any n of them linearly independent, and with a corresponding connected diagram. Then the marks on the branches of any diagram of G can only be $2, 3, 4, 6$ or ∞.*

Let $C = (j(1), \ldots, j(m))$ be a *cycle* of distinct numbers in $\{1, \ldots, n\}$, where we do not regard as distinct from C the same cycle beginning at a different point. We then define

$$\gamma(C) := \prod_{i=1}^{m} \langle v_{j(i)}, v_{j(i+1)}\rangle, \tag{2.1}$$

$$\omega(C) := \begin{cases} 1 & \text{if } m = 1, \\ -\gamma(C) & \text{if } m = 2, \\ 2(-1)^{m-1}\Re\gamma(C) & \text{if } m \geqslant 3, \end{cases} \tag{2.2}$$

$$\vartheta(C) := \arg(-1)^m \gamma(C), \tag{2.3}$$

as long as $\gamma(C) \neq 0$, where $\Re z$ and $\arg z$ denote the real part and argument of a complex number z, with $-\pi < \arg z \leqslant \pi$ (if $z \neq 0$). If C represents a circuit $\mathcal{C}$ in the diagram of G, we define $\omega(\mathcal{C}) := \omega(C)$. Then, if $m \geqslant 3$ and $p_{j(i),j(i+1)} > 2$ for each i, we have

$$\omega(C) = -2\left(\prod_{i=1}^{m} \cos\frac{\pi}{p_{j(i),j(i+1)}}\right)\cos\vartheta(C). \tag{2.4}$$

The angle $\vartheta(C)$ is invariant under cyclic permutation, but changes sign on reversing the cycle. If C represents a circuit $\mathcal{C}$, then the absolute value of $\vartheta(C)$ (and when convenient, $\vartheta(C)$ itself) is called the *turn* of $\mathcal{C}$ and is denoted by $\vartheta(\mathcal{C})$.

A (*complete*) *circuit matching* $\mathcal{M}$ of a diagram $\mathcal{D}$ is a collection of node-disjoint circuits of $\mathcal{D}$ such that each node of $\mathcal{D}$ occurs in exactly one circuit (we allow single nodes and branches traversed in both directions). We denote by $\mathcal{M}(\mathcal{D})$ the family of all complete circuit matchings $\mathcal{M}$ of $\mathcal{D}$.

Theorem 2.3 *Let $\mathcal{D}$ be a diagram of a group G acting on $\mathbb{C}^n$, which preserves a hermitian form, and is generated by n involutory hyperplane reflexions. Then*

$$\varDelta(\mathcal{D}) = \sum_{\mathcal{M}\in\mathcal{M}(\mathcal{D})} \prod_{\mathcal{C}\in\mathcal{M}} \omega(\mathcal{C}). \tag{2.5}$$

We also need the following recursive calculation for certain Schläfli determinants.

Lemma 2.4 *Let G be as in the previous theorem, and let the node 1 of $\mathcal{D}$ belong to a single branch $\{1,2\}$ marked t. If $\varDelta$ is the Schläfli determinant of $\mathcal{D}$, and if $\varDelta_1$ and $\varDelta_{12}$ are those of the subdiagrams obtained by deleting node 1 or nodes 1 and 2, respectively, then*

$$\varDelta = \varDelta_1 - \cos^2\frac{\pi}{t}\,\varDelta_{12}.$$

Let $j(1),\ldots,j(m)$ be distinct numbers. If the product $S_{j(1)}S_{j(2)}\cdots S_{j(m)}S_{j(m-1)}\cdots S_{j(2)}$ has finite period, then it is a complex rotation through an angle $2\pi/q$, where $q\geqslant 2$ is a rational number; in this case we define $p_{j(1),\ldots,j(m)} := q$.

Lemma 2.5 *Let $C=(j(1),\ldots,j(m))$ be a cycle which induces a diagonal-free circuit $\mathcal{C}$ in the diagram $\mathcal{D}$ of G. If the product $S_{j(1)}S_{j(2)}\cdots S_{j(m)}S_{j(m-1)}\cdots S_{j(2)}$ is a genuine rotation, then its rotation angle $2\pi/q$ depends only on the numbers $p_{j(i),j(i+1)}$ and the turn $\vartheta(\mathcal{C})$, the absolute value of $\vartheta(C)$.*

We mainly apply the lemma with $m=3$ or 4. In general, the actual equation for the number $q=p_{j(1),\ldots,j(m)}$ of a diagonal-free circuit is rather complicated. If $p_{k,k+1}=3$ for all but one k, then the numbers $p_{j(1),\ldots,j(m)}$ are invariant under cyclic permutation of the indices (and of course, reversal of the indices); if all $p_{k,k+1}=3$, then $\vartheta(C)=\pm\frac{2\pi}{q}$. If $q=2$, then the circuit itself represents a real group.

For reflexion groups $G=\langle S_1,S_2,S_3\rangle$ in $\mathbb{C}^3$, the basic diagram $\mathcal{D}$ is a triangle with (rational) marks $p:=p_{12}$, $q:=p_{13}$, $r:=p_{23}\,(\geqslant 2)$ on the branches $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, respectively. If the period of $S_1S_3S_2S_3$ in G is finite, then $S_1S_3S_2S_3$ is a complex rotation through an angle $2\pi/s$, with $s:=p_{132}\,(\geqslant 2)$ rational (see Lemma 2.5); its conjugate $S_2S_3S_1S_3$ is a rotation through the same angle. The Schläfli determinant $\varDelta$ now satisfies the fundamental equation

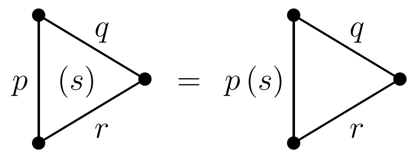
$$4\varDelta = 2\cos\frac{2\pi}{q}\cos\frac{2\pi}{r} - \cos\frac{2\pi}{p} - \cos\frac{2\pi}{s}, \tag{2.6}$$

which only depends on p,q,r and s, and is symmetric between p and s. Similarly, if the periods are finite, we also have rotation angles $2\pi/t$ for the conjugate pair $S_1S_2S_3S_2$ and $S_3S_2S_1S_2$, and $2\pi/t'$ for the pair $S_2S_1S_3S_1$ and $S_3S_1S_2S_1$; as before, $t:=p_{123}$ and $t':=p_{213}$ are rational. The numbers t and t' can be calculated from the data we already have. For example,

$$\cos\frac{2\pi}{t} - \cos\frac{2\pi}{s} = \left(2\cos\frac{2\pi}{r}+1\right)\left(\cos\frac{2\pi}{p} - \cos\frac{2\pi}{q}\right), \tag{2.7}$$

and similarly for t'.

We denote the group G by $G^3(p,s;q,r)$. There is a corresponding diagram $\mathcal{T}^3(p,s;q,r)$ for the group, namely



(2.8)

where the latter alternative form emphasizes that $s\,(=p_{132})$ is *attached* to p, meaning that, in the product with rotation angle $2\pi/s$, the generator represented by the node opposite to the branch marked p is the only one which occurs twice. The latter form of the diagram enables us to attach (if we wish) the corresponding label $t\,(=p_{123})$ to q, and similarly $t'\,(=p_{213})$ to r. The former form now has an interior mark on the triangle, placed in parentheses to indicate that it is attached to p (and is generally not the same as t or t'). Note that we have $G^3(p,s;r,q) = G^3(p,s;q,r) = G^3(s,p;q,r)$.

If at least two of the marks p, q, r are 3, the interior mark s is the same, whichever pair of the other marks we care to call p and q; it is usual now to set $q=r=3$. We then use the first form of the diagram and simplify notation by omitting the parentheses around the interior mark. Similar remarks apply when $p=q=r$.

If p, q, r, s are integers (or ∞), the *generalized triangle group* $\Gamma^3(p,s;q,r)$ is the abstract group with presentation

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_1\sigma_2)^p = (\sigma_1\sigma_3)^q = (\sigma_2\sigma_3)^r = (\sigma_1\sigma_3\sigma_2\sigma_3)^s = \varepsilon. \tag{2.9}$$

Then the geometric group $G^3(p,s;q,r)$ is a quotient of $\Gamma^3(p,s;q,r)$. For the finite geometric groups, we know from Theorem 2.1 that we can indeed assume that each of p,q,r,s (and t,t') is an integer. Following [6, 7], we introduce the alternative notation $[1\ 1\ 1^p]^s := G^3(p,s;3,3)$. Moreover, recall that $[p_1,\ldots,p_{n-1}]$ denotes the Coxeter group with a string diagram on n nodes and with branches labelled $p_1,\ldots,p_{n-1}$; if finite, this is the symmetry group of a regular convex polytope $\{p_1,\ldots,p_{n-1}\}$ (see [8]). Then we have

Theorem 2.6 *The finite irreducible reflexion groups in unitary* 3*-space* $\mathbb{C}^3$ *generated by* 3 *planar reflexions are* $[p,3]$ *with* $p=3,4,5$, *and* $[1\ 1\ 1^p]^s$ *with* $\{p,s\}=\{3,m\}$ *for any* $m \geqslant 2$, $\{4,4\}$ *or* $\{4,5\}$. *In each case, there is only one geometric group (up to conjugacy), and it is isomorphic to the corresponding abstract group.*

p	s	ϑ
p	3	$\pi - \pi/p$
3	s	$2\pi/s$
4	4	$\arccos(-1/2\sqrt{2})$
4	5	$\arccos(-\tau^{-2}/2\sqrt{2})$
5	4	$2\pi/3$

Table 2.1: The turns for the groups $[1\ 1\ 1^p]^s$.

We briefly discuss the degenerate case $\Delta = 0$, when G acts on the unitary plane $\mathbb{C}^2$ (see also [27]). Call a group G *strongly locally finite* if all six parameters p,q,r,s,t and t' are finite. Then we have

Theorem 2.7 *The infinite discrete unitary groups in* $\mathbb{C}^2$*, which are generated by* 3 *involutory reflexions, and are irreducible, non-real, and strongly locally finite, are the three groups* $G^3(4,4;4,4)$*,* $G^3(6,3;4,4)$ *and* $G^3(6,4;3,3)$*. In each case, there is only one geometric group (up to conjugacy).*

We also need the existence of geometric groups $G^3(p,s;q,r)$ for more general parameters than those of the finite or infinite discrete groups. If $q = r = 3$, the corresponding equations for the unit vectors $\{v_1, v_2, v_3\}$ take the form

$$\langle v_1, v_2\rangle = -\cos\frac{\pi}{p}e^{i\vartheta}, \quad \langle v_1, v_3\rangle = -\cos\frac{\pi}{q} = -\frac{1}{2}, \quad \langle v_2, v_3\rangle = -\cos\frac{\pi}{r} = -\frac{1}{2}, \qquad (2.10)$$

where the turn ϑ (loaded on the branch $\{1,2\}$) is related to p, s by the equation

$$\cos\frac{\pi}{p}\cos\vartheta \;=\; \cos^2\frac{\pi}{s} - \cos^2\frac{\pi}{p} - \frac{1}{4}. \qquad (2.11)$$

For the finite groups, this yields the turns listed in Table 2.1. Now, rather than finding the unit vectors which satisfy the equations for a given hermitian form (in the above, usually the standard positive definite form), we pick any basis $\{v_1, v_2, v_3\}$ of $\mathbb{C}^3$, and define a hermitian form by specifying its Gram matrix on this basis as in (2.10). Using this approach we obtain

Theorem 2.8 *Geometric groups* $G^3(p,s;3,3)$ *exist for all integers* $p, s \geqslant 3$ *and for* $\{p,s\} = \{3,2\}$*. Both* $\Gamma^3(p,s;3,3)$ *and* $G^3(p,s;3,3)$ *are infinite unless* $\{p,s\} = \{3,m\}$ *for* $m \geqslant 2$*,* $\{4,4\}$ *or* $\{4,5\}$*.*

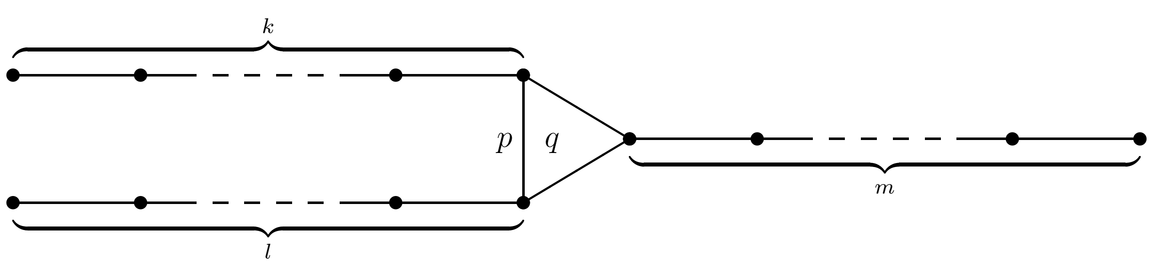


Figure 2.1: The group $[k\,l\,m^p]^q$

The finite unitary reflexion groups were first enumerated in [33]. All the finite non-real irreducible unitary groups generated by n involutory reflexions in $\mathbb{C}^n$ are instances of groups $[k\,l\,m^p]^q$ in unitary $(k+l+m)$-space, whose diagram on $n := k+l+m$ nodes is shown in Figure 2.1. A presentation for the group is obtained by adding to the standard Coxeter-type relations $(S_iS_j)^{p_{ij}} = I$ $(1 \leqslant i < j \leqslant n)$ of the underlying Coxeter diagram, the one extra relation

$$(S_aS_bS_cS_b)^q = I \qquad (2.12)$$

for the triangular circuit with nodes a, b, c and interior mark q; here I denotes the identity transformation. The only groups that actually occur are those listed in Table 2.2.

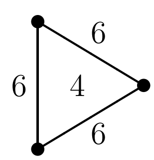
Dimension	Group	Order	Centre
$n(\geqslant 3)$	$[1\,1\,(n-2)^p]^3$	$p^{n-1}n!$	(p,n)
3	$[1\,1\,1^4]^4$	336	2
	$[1\,1\,1^5]^4 = [1\,1\,1^4]^5$	2160	6
4	$[2\,1\,1^4]^3 = [2\,1\,1]^4 = [1\,1\,2]^4$	$64 \cdot 5!$	4
5	$[2\,1\,2]^3$	$72 \cdot 6!$	2
6	$[2\,1\,3]^3$	$108 \cdot 9!$	6

Table 2.2: The finite non-real irreducible unitary reflexion groups generated by n involutions

3 Tetrahedral diagrams

The methods developed in [22] do not easily lead to a solution of the problem of classifying the finite unitary groups in $\mathbb{C}^n$ generated by n involutory hyperplane reflexions when $n \geqslant 4$. To a certain extent, inductive techniques will work. For instance, each subgroup generated by $n-1$ of the reflexions can be transformed (in practice, if perhaps not in theory) into a standard form by basic operations. This then severely restricts the forms which the remaining subgroups of this kind can take. However, there is no way of ensuring that *all* these subgroups can be nicely presented simultaneously. Nevertheless, in certain cases, these methods do lead to significant insights.

One small problem which we encounter is the following. If we decrease the (integer) mark on any branch of the diagram of a finite Coxeter group, then we obtain the diagram of another finite Coxeter group. Actually, a bit more is true. If the group is locally finite (that is, any subgroup generated by $n-1$ of the generating reflexions is finite), but is infinite and acts discretely on $\mathbb{E}^m$ for some m, then decreasing any mark again yields the diagram of a finite group. Unfortunately, for our unitary groups, the analogous results are false. For example, the discrete infinite group $[1\,1\,1^4]^6 = G^3(6,4;3,3)$ can be represented by a diagram with all branches marked 6, and the triangle marked 4, namely the following:



However, if we lower the triangle mark to 3, while the new Schläfli determinant is now positive (namely, 1/8), we do not obtain a finite group. Indeed, if we write the group in the form $G^3(3,6;6,6)$, the calculation for the corresponding value of t from (2.7) gives

$$\cos\frac{2\pi}{t} = \frac{1}{2} + \left(2\cdot\frac{1}{2}+1\right)\left(-\frac{1}{2}-\frac{1}{2}\right) = -\frac{3}{2},$$

which is nonsensical! So, while as a rule of thumb similar considerations to those for Coxeter groups do apply, care must be taken to check each instance.

Our main purpose in this section is to investigate certain groups $\Gamma(\mathcal{D})$ defined by tetrahedral diagrams $\mathcal{D}$. As we shall see, these groups are of importance for the enumeration of corresponding locally toroidal regular polytopes; the relationship is that such a group $\Gamma(\mathcal{D})$ can be "twisted" to yield the appropriate C-group (see Section 5). With these applications in mind, by and large we shall not attempt to classify groups which do not admit suitable twists; in other words, our diagrams will have certain symmetries.

However, we begin the discussion with those diagrams consisting of a triangle with a tail (these are degenerate tetrahedral diagrams). Many of these do permit a twist, and are important to the investigations of Sections 6 and 7 below. Nevertheless, since we know that all the finite reflexion groups have diagrams of this kind, they are clearly of great importance, and so naturally get referred to frequently.

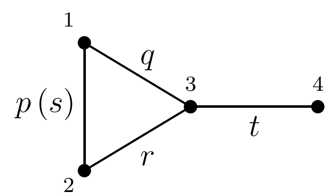


Figure 3.1: The diagram $\mathcal{T}_4(p,s;q,r;t)$

The general triangle with tail is as in Figure 3.1, with integers $p,q,r,s,t \geqslant 2$. As usual, the (involutory) generator S_j of the corresponding geometric group $G := G^4(p,s;q,r;t)$ is associated with the node labelled j. The group now acts on $\mathbb{C}^4$, with generators given by

$$xS_j = x - 2(\langle x, v_j\rangle - \beta_j)v_j,$$

where, as before, $\langle\cdot,\cdot\rangle$ is a hermitian form, v_j a unit vector, and β_j a scalar. In our applications, we usually take the standard positive definite form on $\mathbb{C}^4$, so that $\Delta > 0$ gives the finite groups in $\mathbb{C}^4$, and $\Delta = 0$ the infinite discrete groups in $\mathbb{C}^3$. Again we may take $\beta_j = 0$ for either all j if $\Delta > 0$ (or $\Delta \neq 0$), or all save one j if $\Delta = 0$. Recall that, for a finite group G, the generators $S_1,\ldots,S_4$ can be chosen so that the branches of any diagram basically equivalent to that of $S_1,\ldots,S_4$ bear only integer marks (see Theorem 2.1). For infinite discrete groups G, the branches on any diagram of G can only be $2,3,4,6$ or ∞ (see Theorem 2.2).

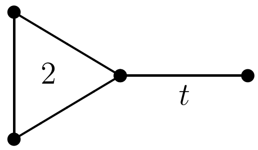
Now, after a little simplification, from (2.6) and Lemma 2.4, or directly from (2.5), we find for the Schläfli determinant $\Delta = \Delta(\mathcal{T}_4(p,s;q,r;t))$ the expression

$$4\Delta = 2\cos\frac{2\pi}{q}\cos\frac{2\pi}{r} - \cos\frac{2\pi}{s} - \left(1-\cos\frac{2\pi}{p}\right)\cos\frac{2\pi}{t} - 1, \tag{3.1}$$

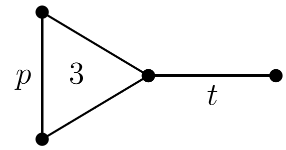
which only depends on p,q,r,s,t.

In fact, we can make some initial observations about strong local finiteness. In this context we call G *strongly locally finite* if G is locally finite with respect to not only the original generators $\{S_1,\ldots,S_4\}$ but also those obtained by applying a single basic change of generators $S_j \mapsto S_kS_jS_k$ to $\{S_1,\ldots,S_4\}$. (The corresponding definition for triangle groups is equivalent to the one given in the previous section.) If the branch $\{3,4\}$ carries a mark $t > 3$, then $\{1,3\}$ and $\{2,3\}$ cannot, nor can such a mark be brought on to $\{1,3\}$ by the basic operation $S_1 \mapsto S_2S_1S_2$ (the new mark would be q if $r = 2$, or s if $q = r = 3$).

Thus any such non-linear diagram (with $t > 3$) can only be of the form



or



The first is the diagram for $[3, 3, t]$, as the above basic operation shows, and so yields a finite group only for $t = 3, 4, 5$ (we have restored $t = 3$ to this list); indeed, since formula (3.1) can be simplified to

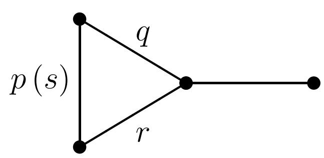
$$8\mathit{\Delta} = 1 - 3\cos\frac{2\pi}{t},$$

the condition $\mathit{\Delta} > 0$ gives $t < 2\pi/\arccos\frac{1}{3}$. For the second, (3.1) gives

$$4\mathit{\Delta} = -\left(1 - \cos\frac{2\pi}{p}\right)\cos\frac{2\pi}{t}.$$

Clearly, for any p, this is non-positive whenever $t > 3$ (but positive when $t = 3$). When $t = 4$, so that $\mathit{\Delta} = 0$, the crystallographic restriction of Theorem 2.2 allows only $p = 2, 3, 4$ or 6, the first case being real; one can show that the corresponding group genuinely is discrete (see [6]).

When $t = 3$, the original diagram is



(3.2)

and the Schläfli determinant is given by

$$8\mathit{\Delta} = 4\cos\frac{2\pi}{q}\cos\frac{2\pi}{r} - \cos\frac{2\pi}{p} - 2\cos\frac{2\pi}{s} - 1. \qquad (3.3)$$

In practice, it is often easier to use Lemma 2.4 directly, and write

$$\mathit{\Delta} = \mathit{\Delta}_4 - \frac{1}{4}\sin^2\frac{\pi}{p},$$

where $\mathit{\Delta}_4$ is the Schläfli determinant of the triangle group. This has the advantage that the basically equivalent diagrams obtained by the change of generators $S_j \mapsto S_k S_j S_k$, with $\{j, k\} = \{1, 2\}$, are treated together.

In effect, we already noted above that $G^4(p, 3; 3, 3; 3) = [1\ 1\ 2^p]^3$ is always finite; the Schläfli determinant is

$$\mathit{\Delta} = \frac{1}{8}\left(1 - \cos\frac{2\pi}{p}\right) = \frac{1}{4}\sin^2\frac{\pi}{p} > 0.$$

The case $p = 2$ is the real (Coxeter) group D_4. Further, with $p = q = r = 3$, we have

$$8\Delta = \frac{1}{2} - 2\cos\frac{2\pi}{s};$$

if $\Delta > 0$ (or even if $\Delta \geqslant 0$), we must have $s \leqslant 4$. This yields the finite groups $[1\ 1\ 2]^s$ with $s = 2, 3, 4$.

For the remaining finite groups with $t = 3$, we have $p \leqslant 5$ and $s \leqslant 5$, and the triangle is obtainable from $[3, 4]$, $[3, 5]$, $[1\ 1\ 1^4]^4$ or $[1\ 1\ 1^4]^5$ by basic changes of generators. Indeed, since the triangle group is finite and necessarily $q, r \leqslant 5$, we first eliminate choices for p, q, r, s by appealing to arguments similar to those used in the proof of Theorem 2.6 (see [22, Theorem 5.3]), and then employ the recursive formula for the determinant to reduce the list further. Excluding the real groups, which we can handle directly, the details for the first step are as follows.

If, say, $r = 4$, then $\Delta_4 > 0$ implies that $\{p, s\} = \{q, t_q\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$, where t_q is the mark on the triangle attached to q (we cannot have $q = 2$ or $t_q = 2$ for a non-real group).

If, say, $r = 3$, then $\Delta_4 > 0$ yields a condition symmetric in p, q, s, which implies that at least one of p, q, s is also 3 and that the pair of the other two is $\{3, m\}$ ($m \geqslant 3$), $\{4, 4\}$ or $\{4, 5\}$ (again, p, q or s cannot be 2 for a non-real group).

Finally, if $q = r = 5$, then $\Delta_4 > 0$ implies that $p < 5$ if $p \leqslant s$ (and $q < 5$ if $q < t_q$); now relabelling leads us back to the two previous cases.

For the remaining possibilities, a basic change of generators will take the triangle into a finite group $[1\ 1\ 1^l]^m$ (with the standard diagram), and so Δ_4 must be the (standard) Schläfli determinant of $[1\ 1\ 1^l]^m$. With the recursive formula for the determinant, we can now further reduce the list as follows.

For $p = 5$, we can eliminate $[1\ 1\ 1^5]^4$ with $\Delta_4 = \frac{1}{8}\tau^{-2}$ as a possibility. However, the real group $[5, 3]$, appearing as $G^3(5, 5; 3, 2)$, $G^3(5, 2; 5, 3)$ or $G^3(5, 3; 5, 5)$, is permitted; the resulting group is $[5, 3, 3]$.

With $p = 4$, we can also eliminate $[1\ 1\ 1^4]^s$ with $s = 4$ or 5; for the former $\Delta = 0$, and we obtain a discrete infinite group (for example, in the form $[1\ 1\ 2^4]^4$). But the real group $[4, 3]$, appearing as $G^3(4, 2; 4, 3)$, is permitted, yielding the group $[4, 3, 3]$.

Finally, with $p = 3$, the only possibilities for the triangle which have not been mentioned so far are $[1\ 1\ 1]^4$ and $[3, 4]$, which can appear additionally as $G^3(3, 3; 3, 4)$ and $G^3(3, 3; 4, 4)$, respectively, resulting in the groups $[1\ 1\ 2^4]^3$ and $[3, 4, 3]$. This now exhausts the list of possible groups.

Let us summarize this discussion. We write $\Gamma^4(p, s; q, r; t)$ for the abstract group determined by the diagram relations; this has generators $\sigma_1, \ldots, \sigma_4$ (say), and is defined by the standard Coxeter type relations (for the marked branches), and the single extra relation $(\sigma_1\sigma_3\sigma_2\sigma_3)^s = \varepsilon$ (represented by the interior mark of the triangle). We now have

Theorem 3.1 *A finite irreducible reflexion group $G^4(p, s; q, r; t)$ in unitary 4-space $\mathbb{C}^4$ whose diagram $\mathcal{T}_4(p, s; q, r; t)$ is a triangle with a tail is one of the following (some groups are listed more than once):*

a) $G^4(3, 2; 3, 3; t) \cong [3, 3, t]$ *for* $t = 3, 4, 5$;

b) $G^4(p, 3; 3, 3; 3) = [1\ 1\ 2^p]^3$ *for* $p \geqslant 2$;

c) $G^4(3, s; 3, 3; 3) = [1\ 1\ 2^3]^s = [1\ 1\ 2]^s$ *for* $s = 2, 3, 4$, *and* $G^4(3, 3; 3, 4; 3) = [1\ 1\ 2]^4$*;*
d) $[3, 3, 4]$ *and* $[3, 3, 5]$, *but not as in the first part, or* $[3, 4, 3]$.
In the first three parts, the groups are isomorphic to the abstract groups determined by the diagram relations.

In later sections, we also require the existence of certain reflexion groups $G^4(p, s; q, r; t)$ which are infinite. We concentrate again on the case $q = r = 3$, which is the most interesting for us. Similar arguments to those for the triangle group $G^3(p, s; 3, 3)$ apply in this case, and prove that a group $G^4(p, s; 3, 3; t)$ exists whenever $G^3(p, s; 3, 3)$ exists and $t \geqslant 2$. In other words, we pick a basis $\{v_1, \ldots, v_4\}$ of $\mathbb{C}^4$ and specify the Gram matrix of a hermitian form on this basis as dictated by the diagram. For $G^4(p, s; 3, 3; t)$, the defining equations comprise those of (2.10) (with $q = r = 3$) and, in addition,

$$\langle v_1, v_4 \rangle = \langle v_2, v_4 \rangle = 0, \quad \langle v_3, v_4 \rangle = -\cos\frac{\pi}{t}.$$

If the triangle represents a finite group (this is the only case we really need), then we choose the turn ϑ in (2.10) as in Table 2.1. Then three cases can occur. If the resulting form on $\mathbb{C}^4$ is positive definite, we are back in the above enumeration. If it is positive semi-definite, the radical must be 1-dimensional, and the group can be viewed as acting on $\mathbb{C}^3$, with a positive definite form; this corresponds to the degenerate case $\Delta = 0$. Finally, if the form on $\mathbb{C}^4$ is indefinite, then the group must necessarily be infinite.

The following theorem summarizes the results for those diagrams which occur in the enumeration of locally toroidal regular polytopes.

Theorem 3.2 *Let* $(p, s) = (p, 3), (3, s), (4, 4), (4, 5)$ *or* $(5, 4)$, *and let* $t = 3, 4$ *or* 5. *Then there exists an infinite geometric group* $G^4(p, s; 3, 3; t)$, *and so the abstract group* $\Gamma^4(p, s; 3, 3; t)$ *is also infinite, unless* p, s *and* t *are as in the first three parts of Theorem 3.1.*

We need one further comment, which applies in the context of a general hermitian form. Recall that a group generated by involutions $\sigma_1, \ldots, \sigma_k$ is called a *C-group* if it satisfies the *intersection property*

$$\langle \sigma_i \mid i \in I \rangle \cap \langle \sigma_i \mid i \in J \rangle = \langle \sigma_i \mid i \in I \cap J \rangle \tag{3.4}$$

for all subsets I, J of $\{1, \ldots, k\}$ (see [23, Section 2E]; this occurs in another context in Section 5 below). The geometric groups, corresponding to the diagrams, are C-groups (with generators $S_1, \ldots, S_4$), as we can see from the geometry; for example, to prove that

$$\langle S_i, S_j, S_k \rangle \cap \langle S_i, S_j, S_l \rangle = \langle S_i, S_j \rangle,$$

with i, j, k, l distinct, observe that an element in the intersection must fix both the k-th and l-th coordinate (with respect to the basis $v_1, \ldots, v_4$) of each vector, and must therefore belong to the subgroup $\langle S_i, S_j \rangle$ of the unitary group $\langle S_i, S_j, S_k \rangle$. The corresponding abstract group $\Gamma^4(p, s; q, r; t) = \langle \sigma_1, \ldots, \sigma_4 \rangle$ is then also a C-group; this follows from a variant of the quotient criterion of [23, Theorem 2E17], since the natural homomorphism which identifies the generators σ_i of the abstract group with the generating reflexions S_i of the geometric group is one-to-one on at least one 3-generator subgroup. This establishes

Theorem 3.3 *For the diagram $\mathcal{T}_4(p,s;q,r;t)$, if the geometric group $G^4(p,s;q,r;t)$ exists, then both $G^4(p,s;q,r;t)$ and $\Gamma^4(p,s;q,r;t)$ are C-groups.*

We now come to general tetrahedral diagrams, which play an important rôle in Section 6. When $n = 4$, we have the following obvious remark; here and elsewhere, we employ the notation and terminology of Section 2. In particular, if $C := (j(1),\ldots,j(m))$ is a cycle in a diagram, we set $\vartheta_{j(1)\ldots j(m)} := \vartheta(C)$.

Lemma 3.4 *If all the branch marks on a tetrahedral diagram with nodes* $1,2,3,4$ *are at least* 3*, then the turns of the triangles satisfy*

$$\vartheta_{234} - \vartheta_{134} + \vartheta_{124} - \vartheta_{123} \equiv 0 \pmod{2\pi}. \tag{3.5}$$

Proof The 4-cycle $C := (1,\ldots,4)$ in the diagram can be viewed in two ways as a concatenation of 3-cycles, each determined by a diagonal of C. From [22, Lemma 4.9] we then have

$$\vartheta_{123} + \vartheta_{134} \equiv \vartheta_{1234} \equiv \vartheta_{234} + \vartheta_{124} \pmod{2\pi},$$

and hence the result follows at once. □

We can now put our finger on one of the main problems. When we specify a triangle group by means of a diagram, say on the nodes 1, 2 and 3, we may give the marks p_{12}, p_{13}, p_{23} and (let us suppose) p_{132}. If all these marks are at least 3, then this only yields the group and its generators geometrically up to a choice of sign of the turn ϑ_{132} (see the proof of [22, Lemma 5.5]). It follows that, when we attempt to construct a tetrahedral diagram out of compatible triangular diagrams, then this ambiguity of sign may possibly yield different groups. (In fact, we shall give a specific example of this phenomenon below.)

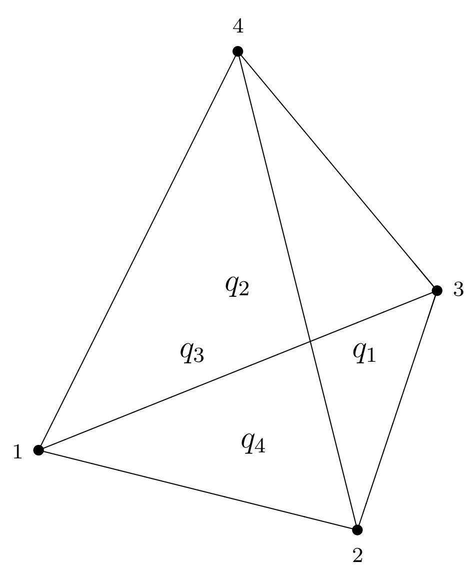


Figure 3.2: The diagram $\mathcal{T}_4(q_1,\ldots,q_4)$

Consider first the tetrahedral diagram $\mathcal{D} = \mathcal{T}_4(q_1,\ldots,q_4)$ of Figure 3.2, whose branch marks are all 3 (and are therefore omitted), and whose triangle marks (which are now unambiguous) are integers q_1, q_2, q_3, q_4 ($\geqslant 2$), with q_i the mark on the face opposite to the node i. Let $\Gamma(\mathcal{D}) = \langle \sigma_1, \ldots, \sigma_4 \rangle$ be the abstract group corresponding to $\mathcal{D}$, which is the Coxeter group determined by the unmarked tetrahedron, factored by the extra relations

$$(\sigma_i \sigma_j \sigma_k \sigma_j)^{q_m} = \varepsilon, \quad \text{with } \{i, j, k, m\} = \{1, 2, 3, 4\}.$$

Any attempt at constructing a (locally unitary) representation

$$r \colon \Gamma(\mathcal{D}) \to G = \langle S_1, \ldots, S_4 \rangle \subseteq \mathrm{GL}_4(\mathbb{C}),$$

with G preserving a hermitian form, must bear in mind (3.5) when it applies, which it certainly does here. As we saw in Section 2, the turn on the triangle opposite node i is $\pm 2\pi/q_i$; we conclude that the q_i must satisfy the relation

$$\frac{1}{q_1} \pm \frac{1}{q_2} \pm \frac{1}{q_3} \pm \frac{1}{q_4} \equiv 0 \pmod 1, \tag{3.6}$$

for suitable choices of signs.

Now there are just two (essentially) distinct cases where the diagram may permit a twist, to yield a string C-group. First, the q_i are equal in pairs, say $q_1 = q_2 = s$ and $q_3 = q_4 = q$ (the notation is chosen to conform with a more general diagram below). It is trivial to choose signs so as to satisfy (3.6). Indeed, unless $s = q = 2$ or 4, or (say) $s = 3$ and $q = 6$, the only solution (up to permutation if $s = q$) is

$$\pm(\frac{1}{s} - \frac{1}{s}) \pm (\frac{1}{q} - \frac{1}{q}) = 0.$$

(Note that, if $s = 2$, then we could change a sign in the first term and sum to ± 1; however, this yields nothing new.) The case $s = q = 2$ conceals the group $[3, 3, 3]$ of the 4-simplex, as we shall see. For $s = q = 4$, we also have

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1, \tag{3.7}$$

and for $s = 3$ and $q = 6$ we have

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = 1; \tag{3.8}$$

we shall comment further on these below.

In the general case, given a geometric group G, we can load the turns $\pm 2\pi/s$ on the branch $\{3, 4\}$, and the turns $\pm 2\pi/q$ into $\{1, 2\}$. Indeed, by rescaling the normal vectors (if need be), we can ensure that $\langle v_i, v_j \rangle = -\frac{1}{2}\,(= -\cos\frac{\pi}{3})$ for all $\{i, j\}$, except $\{1, 2\}$ and $\{3, 4\}$; the latter determine the signs of the turns. (There are four possibilities for choices of sign each leading to a group G. Any two such groups are conjugate in the group of unitary transformations of $\mathbb{C}^4$, by arguments similar to those used in the proof of [22, Lemma 5.5].) We evaluate the Schläfli determinant Δ, using (2.5). The only extra information to be noted is that the three circuits of four branches have turns 0 and $\pm(2\pi/q \pm 2\pi/s)$. Thus we have

$$\Delta = 1 - 6 \cdot \tfrac{1}{4} + 3 \cdot \tfrac{1}{16} - 2 \cdot \tfrac{1}{4} \cos \tfrac{2\pi}{q} - 2 \cdot \tfrac{1}{4} \cos \tfrac{2\pi}{s} - \tfrac{1}{8} - \tfrac{1}{8} \cos(\tfrac{2\pi}{q} + \tfrac{2\pi}{s}) - \tfrac{1}{8} \cos(\tfrac{2\pi}{q} + \tfrac{2\pi}{s}),$$

from which we easily deduce that

$$16\Delta = 9 - 4\left(\cos\frac{2\pi}{s} + 2\right)\left(\cos\frac{2\pi}{q} + 2\right). \tag{3.9}$$

For $\Delta > 0$ (that is, for the finite groups), we see easily that the only solutions of (3.9) are $\{s, q\} = \{2, 2\}$, $\{2, 3\}$ and $\{2, 4\}$. Here, we have actually just found an alternative diagram for $[1\ 1\ 2]^q$, as we can see using the basic change of generators $S_4 \mapsto S_3S_4S_3$. Further, of course, the case $q = 2$ as well gives $[3, 3, 3]$; the generator S_i can be represented by the transposition $(i\ 5)$, for $i = 1, \ldots, 4$. Observe that $s = q = 3$ gives $\Delta = 0$, a specific example of the same more general case which we shall meet later.

The other case is, say, $q_1 = p$, $q_2 = q_3 = q_4 = q$. Excluding the case $p = q$ which was covered above (but we shall need to bear it in mind for applications), we see that the only choice of signs to satisfy (3.6) is effectively

$$\frac{1}{p} - \frac{1}{q} - \frac{1}{q} - \frac{1}{q} = 0,$$

which results in $q = 3p$. (Changing the signs of the fractions $1/q$ and summing to 1 actually yields nothing new, because of the restriction that p and q be integers, with the exception of the excluded case $p = q = 4$. Note that the case $(p, q) = (2, 6)$ can occur here in two ways, because the terms can sum to 0 or 1.) By rescaling the normals of G (if need be), we can now load the turns $2\pi/q$ on the branches $\{2, 3\}$, $\{3, 4\}$ and $\{4, 2\}$ (we write the last branch this way to hint at the corresponding orientation). We again use (2.5) to calculate the Schläfli determinant. The three circuits of four branches each have turn $\pm 4\pi/3p$, and so Δ is now given by

$$\Delta = 1 - 6\cdot\tfrac{1}{4} + 3\cdot\tfrac{1}{16} - 3\cdot\tfrac{1}{4}\cos\tfrac{2\pi}{3p} - \tfrac{1}{4}\cos\tfrac{2\pi}{p} - 3\cdot\tfrac{1}{8}\cos\tfrac{4\pi}{3p},$$

which after simplification yields

$$16\Delta = 1 - 12\cos^2\frac{2\pi}{3p} - 16\cos^3\frac{2\pi}{3p}. \tag{3.10}$$

Bearing in mind that $p \geqslant 2$ is an integer, we easily see that (3.10) has no solutions with $\Delta > 0$ (or even $\Delta = 0$).

If we wish to consider diagrams $\mathcal{T}_4(q_1, q_2, q_3, q_4)$ which have less symmetry, then many more possibilities present themselves. We have no interest in fully investigating them here; suffice it to remark that there is evidence that none of them corresponds to a finite group. For instance, a comparatively crude estimate shows that the Schläfli determinant $\Delta < 0$ if all $q_i \geqslant 5$ (whatever signs are carried by the turns), so such a group must necessarily be infinite. If we then set (say) $q_1 = 2$, 3 or 4 in turn, and exclude the trivial cases where $q_i = q_j$ occur with opposite signs, then a little work shows that this effectively restricts the set $\{q_1, q_2, q_3, q_4\}$ to a finite (if rather long) list. It would be tedious to go into more detail, particularly since we do not need such diagrams. However, to illustrate the general principle, take $q_1 = 2$, $q_2 = 3$, and $q_3 = r$, $q_4 = s$ with $r \leqslant s$. The two equations

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{r} \pm \frac{1}{s} = 1,$$

which imply that $1/r \pm 1/s = 1/6$, have integer solutions

$$(r, s) = (7, 42),\ (8, 24),\ (9, 18),\ (10, 15),\ (12, 12),$$

or

$$(r, s) = (2, 3),\ (3, 6),\ (4, 12),\ (5, 30),$$

respectively. In any event,

$$\Delta = -\frac{1}{4}\left(\cos\frac{2\pi}{r} + \cos\frac{2\pi}{s}\right),$$

which is negative unless $r \leqslant 3$ (and is zero when $r = 3$). The only case with $\Delta > 0$ is $r = 2$, which we have already met.

We can summarize this stage of the discussion as follows.

Theorem 3.5 *Geometric groups in* $\mathbb{C}^4$ *exist for all diagrams* $\mathcal{T}_4(s, s, q, q)$ *with* $s, q \geqslant 2$, *and all diagrams* $\mathcal{T}_4(p, 3p, 3p, 3p)$ *with* $p \geqslant 2$. *The only groups which are finite (indeed, finite irrespective of compatible choices of sign of the turns) are those with* $s = 2$ *and* $q = 2, 3, 4$ *(up to permutation of s and q). The corresponding finite group is* $[1\ 1\ 2]^q = [1\ 1\ 2^3]^q$, *and is isomorphic to the abstract group defined by the diagram relations. Furthermore, whether finite or not, the geometric and abstract groups with diagrams* $\mathcal{T}_4(s, s, q, q)$ *and* $\mathcal{T}_4(p, 3p, 3p, 3p)$ *are C-groups.*

Proof The parts of the theorem which have not been mentioned hitherto are the first and the last two. For the last two, note that the basic change of generators $S_4 \leftrightarrow S_3S_4S_3$ transforms the diagrams $\mathcal{T}_4(2, 2, q, q)$ into those of $[1\ 1\ 2]^q$. The abstract groups defined by $\mathcal{T}_4(2, 2, q, q)$ permit the same basic change of generators, which again transforms the diagrams into those for $[1\ 1\ 2]^q$; since we know from Coxeter [6, 7] that the abstract and geometric groups are isomorphic in these cases, this completes the proof. Both abstract and geometric groups are C-groups, for the same reason as in Theorem 3.3.

For the first part, we employ the same technique as for triangles with tails (see Theorem 3.2). Having chosen a basis $\{v_1, \ldots, v_4\}$ of $\mathbb{C}^4$, each triangle in the diagram determines a set of three equations for $\langle v_i, v_j\rangle$ as in (2.10) (here applied with $p = q = r = 3$). In the equations for $\mathcal{T}_4(s, s, q, q)$, we load the turns $\vartheta = 2\pi/q$ or $2\pi/s$ on the branches $\{1, 2\}$ or $\{3, 4\}$, respectively; for $\mathcal{T}_4(p, 3p, 3p, 3p)$, the turns $2\pi/3p$ are loaded on $\{2, 3\}$, $\{3, 4\}$ and $\{4, 2\}$. In each case the six equations specify the Gram matrix of a hermitian form on $\mathbb{C}^4$. Now we are in the same situation as before. In particular, the positive definite forms give the finite groups we have already enumerated. The positive semi-definite forms yield infinite groups in $\mathbb{C}^3$, namely those obtained above for $\Delta = 0$; the only possibility is $\mathcal{T}_4(3, 3, 3, 3)$, giving a discrete group. Finally, if the form is indefinite, then the group is infinite. □

Before we move on, let us also consider the anomalous cases of (3.7) and (3.8). The first, in fact, can be treated in the same way as the case $q_1 = p$, $q_2 = q_3 = q_4 = q$, but now with $p = \frac{4}{3}$ and $q = 4$ (note that $\frac{3}{4} \equiv -\frac{1}{4} \pmod 1$); in particular, the assignment of turns to branches is the same. The corresponding Schläfli determinant is just that of

(3.10) with $p = \frac{4}{3}$, and so $\Delta = 1/16$. Though this is far from obvious, what we have here is another concealed form of the group $[1\ 1\ 2^3]^4 = [1\ 1\ 2]^4$, represented as a group with diagram $\mathcal{T}_4(4,4,4,4)$. We now have two geometric groups with diagram $\mathcal{T}_4(4,4,4,4)$; the new group is finite, and the other (obtained from Theorem 3.5 with $s = q = 4$) is infinite.

For the second anomalous case, we rescale the normals (if need be) such that the contributions to turns are loaded on three branches, namely $\pi/3$ on $\{1,3\}$ and $\{2,4\}$ (again, this indicates the orientations), and π on $\{3,4\}$. Then two of the circuits of length 4 have turns π, while the third has turn $2\pi/3$. From (2.5), we obtain $\Delta = 0$, so that even with a non-standard choice of turns we cannot get a finite group.

The *abstract* groups with diagram $\mathcal{T}_4(p,q,q,q)$ are important in applications to regular polytopes of type $\{3,6,3\}$. The corresponding geometric diagrams $\mathcal{T}_4(p,q,q,q)$, and hence locally unitary groups, can only exist if $q = p$ or $3p$. This is unfortunate, because it renders the techniques we shall describe in Section 5 of limited utility. However, since we know that $\mathcal{T}_4(p,p,p,p)$ yields an infinite group whenever $p \geqslant 3$, and that $\mathcal{T}_4(p,3p,3p,3p)$ yields an infinite group whenever $p \geqslant 2$, we can appeal to the obvious quotient relations to assert

Theorem 3.6 *The abstract group which satisfies the relations induced by the (abstract) diagram* $\mathcal{T}_4(q_1,\ldots,q_4)$ *is infinite in at least the following cases:*
a) $p \mid q_1,\ldots,q_4$ *for some* $p \geqslant 3$*;*
b) $p \mid q_1$ *and* $3p \mid q_2,q_3,q_4$ *for some* $p \geqslant 2$.

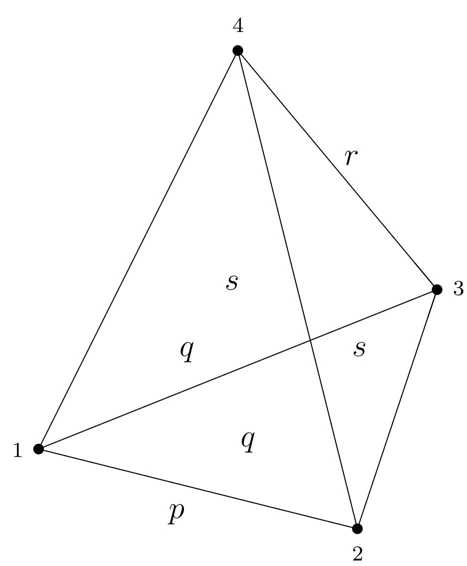


Figure 3.3: The diagram $\mathcal{S}_4(p,q;r,s)$

We now discuss groups G with the more general diagram $\mathcal{S}_4(p,q;r,s)$ illustrated in Figure 3.3. Here all edge marks are 3, except for those on the edges $\{1,2\}$ and $\{3,4\}$, which are p and r, respectively. For obvious reasons, we shall only consider those marks p,q,r,s which correspond to finite 3-generator groups $[1\ 1\ 1^p]^q$ and $[1\ 1\ 1^r]^s$. Thus, by

Theorem 2.6,

$$(p,q) = (p,3),\ (3,q),\ (4,4),\ (4,5) \text{ or } (5,4),$$

with $p, q \geqslant 2$, and similarly for (r,s). We allow $p = 2$ or $r = 2$, in which case we regard the corresponding edge as missing. Note also that, by our diagram conventions, $\mathcal{S}_4(3,q;3,s) = \mathcal{T}_4(s,s,q,q)$.

The calculation for the corresponding Schläfli determinant Δ is very similar to that of (3.9), if rather more elaborate. We write ϑ for the turn of the subdiagram $[1\ 1\ 1^p]^q$ and φ for that of $[1\ 1\ 1^r]^s$; we can appeal to Table 2.1 for the actual values of ϑ and φ. As before, given G we can ensure that the turns are loaded on the branches $\{1,2\}$ and $\{3,4\}$ (again the orientation does not matter). The turns for the two 4-circuits which contain the edges $\{1,2\}$ and $\{3,4\}$ are $\vartheta \pm \varphi$, while that for the third is 0. Thus (2.5) gives

$$\begin{aligned} \Delta \;=\;& 1 - 4\cdot\tfrac14 - \cos^2\tfrac{\pi}{p} - \cos^2\tfrac{\pi}{r} + 2\cdot\tfrac{1}{16} + \cos^2\tfrac{\pi}{p}\cos^2\tfrac{\pi}{r} + 2\cdot\left(-2\cdot\tfrac14\cos\tfrac{\pi}{p}\cos\vartheta\right) \\ &+ 2\cdot\left(-2\cdot\tfrac14\cos\tfrac{\pi}{r}\cos\varphi\right) - \tfrac12\cos\tfrac{\pi}{p}\cos\tfrac{\pi}{r}\cos(\vartheta+\varphi) - \tfrac12\cos\tfrac{\pi}{p}\cos\tfrac{\pi}{r}\cos(\vartheta-\varphi) - \tfrac18. \end{aligned}$$

Using $\cos(\vartheta+\varphi) + \cos(\vartheta-\varphi) = 2\cos\vartheta\,\cos\varphi$, and substituting $\cos\frac{\pi}{p}\cos\vartheta = \cos^2\frac{\pi}{q} - \cos^2\frac{\pi}{p} - \frac14$ from (2.11) with $q = s$ (and similarly for φ), we see that this expression simplifies to

$$\Delta = \sin^2\frac{\pi}{p}\sin^2\frac{\pi}{r}\left[1 - \left(1 + \frac{3 - 4\sin^2(\pi/q)}{4\sin^2(\pi/p)}\right)\left(1 + \frac{3 - 4\sin^2(\pi/s)}{4\sin^2(\pi/r)}\right)\right]. \qquad (3.11)$$

An alternative, but somewhat less useful, way of expressing this determinant is the following.

$$\begin{aligned} 16\Delta =& -5 + 2\cos\frac{2\pi}{p} + 2\cos\frac{2\pi}{r} - 6\cos\frac{2\pi}{q} - 6\cos\frac{2\pi}{s} \\ &+ 4\cos\frac{2\pi}{p}\cos\frac{2\pi}{s} + 4\cos\frac{2\pi}{q}\cos\frac{2\pi}{r} - 4\cos\frac{2\pi}{q}\cos\frac{2\pi}{s}. \end{aligned}$$

Now if one of r or s is 2, then the other must be 3, with a similar restriction on p and q. Let us consider this case first. With $r = 3$ and $s = 2$, the criterion $\Delta > 0$ reduces to

$$2\sin^2\frac{\pi}{p} + 4\sin^2\frac{\pi}{q} > 3.$$

A case by case check (recall that $q = 2$ implies that $p = 3$) shows that the admissible values are

$$(p,q) = (3,2), \quad (p,3) \text{ for any } p \geqslant 2, \quad (3,4).$$

Indeed, the basic change of generators $S_4 \mapsto S_3S_4S_3$ yields a diagram consisting of a triangle with a tail, namely that of the group $[1\ 1\ 2^p]^q$. Note additionally that (again as it must) $(p,q) = (4,4)$ gives $\Delta = 0$; the corresponding infinite group $[1\ 1\ 2^4]^4$ is known to be discrete (see [6]).

With $s = 3$, then, independently of p and r, the criterion $\Delta > 0$ reduces at once to $\sin^2\frac{\pi}{q} > \frac34$, or $q < 3$; that is, $q = 2$. As we have just pointed out, this then implies that $p = 3$, and so we are reduced to the case immediately above.

We may now suppose that $p, q, r, s \geqslant 3$. With $q = s = 3$, (3.11) implies that $\Delta = 0$, no matter what p or r may be. In fact, the corresponding group is infinite and can be viewed as acting on $\mathbb{C}^3$ (equipped with the standard positive definite form). In particular, we can choose specific generating reflexions S_i with unit normals v_i given by

$$\begin{aligned} v_1 &= (e^{2i\pi/p}/\sqrt{2}, -1/\sqrt{2}, 0), \\ v_2 &= (1/\sqrt{2}, -1/\sqrt{2}, 0), \\ v_3 &= (0, 1/\sqrt{2}, -e^{2i\pi/r}/\sqrt{2}), \\ v_4 &= (0, 1/\sqrt{2}, -1/\sqrt{2}), \end{aligned}$$

ensuring that the reflexion hyperplanes do not contain a common point. However, it is generally non-discrete; appealing to Theorem 2.2 shows that the only permissible values of p and r for discreteness are 2, 3, 4 or 6, and, in fact, 4 and 3 or 6 cannot occur together (if they do, then the special group contains a rotation of period 12, and so the translation subgroup cannot be discrete). The possible pairs $\{p, r\}$ are thus

$$\{p, r\} = \{2,2\}, \quad \{2,3\}, \quad \{2,4\}, \quad \{2,6\}, \quad \{3,3\}, \quad \{3,6\}, \quad \{4,4\}, \quad \{6,6\}.$$

In fact, the group is generated by the conjugates of the involutory reflexions R_1 and R_2 in the complex group $p[4]2[3]2[4]r$; it is just for these values of $\{p, r\}$ that this infinite group is known to be discrete (see [17, §7.2] or [9, §12.7 and §13.2]).

Finally, with $q \geqslant 3$ and $s \geqslant 3$, we see that $\Delta < 0$ if $q > 3$ or $s > 3$, again irrespective of the values of p and r. Nevertheless, the geometric groups indeed exist in these cases, by arguments similar to those for the above diagrams $\mathcal{T}_4(s, s, q, q)$. In the defining equations for the hermitian form, we must again load the turns for $[1\ 1\ 1^p]^q$ and $[1\ 1\ 1^r]^s$ on the branches $\{1, 2\}$ or $\{3, 4\}$, respectively. Then the resulting form is indefinite if $q > 3$ or $s > 3$, yielding an infinite geometric group.

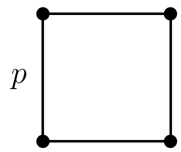
We can summarize the above discussion as follows; once again, the fact that all groups are C-groups follows for the same reason as that of Theorem 3.3.

Theorem 3.7 *For the diagram of Figure 3.3, geometric groups G exist (at least) whenever $(p, q) = (p, 3)$, $(3, q)$, $(4, 4)$, $(4, 5)$ or $(5, 4)$, with $p, q \geqslant 2$, and $(r, s) = (r, 3)$, $(3, s)$, $(4, 4)$, $(4, 5)$ or $(5, 4)$, with $r, s \geqslant 2$. Apart from the groups listed in Theorem 3.5, the only finite examples are those with $(p, q; r, s) = (p, 3; 3, 2)$ for $p \geqslant 2$ (up to interchange of (p, q) and (r, s)); then $G \cong [1\ 1\ 2^p]^3$. Whether finite or not, the geometric and abstract groups are C-groups.*

We shall not consider more general tetrahedral diagrams in which all edges are present (that is, carry a mark other than 2). As we have said, the classification problem using our techniques is complicated. Indeed, for tetrahedral diagrams, listing the different kinds of diagram corresponding to choices of generators of the group is a problem akin to that of listing the Goursat tetrahedra, the reflexions in whose faces generate a finite subgroup of the orthogonal group of Euclidean 4-space (see [8, §14.8]). Indeed, it is little different in the real case—it is merely that our approach does not distinguish between the up to eight spherical tetrahedra bounded by the same planes.

Again because they do not admit twists which relate them to groups which occur later, we shall not look at those tetrahedral diagrams with a single missing edge; the case $p = 2$ of the diagram in Figure 3.3 covers all we need subsequently. However, there is one further class of tetrahedral diagrams which does deserve further investigation, because of the light it sheds on the general classification problem, namely that consisting of the single circuits (with integer marks).

For a single 4-circuit $\mathcal{D}$ to give a locally finite group $G = G(\mathcal{D})$, it must be of the form



(no two marks greater than 3 can occur on adjacent branches). Moreover, to avoid the real cases and obviously infinite subgroups, we must have $3 \leqslant p, q \leqslant 5$. (Actually, for completeness of the discussion, it is convenient to allow $p, q = 2$ here.) Now we have two different numbers for the period p_{1234} of the product $S_1S_2S_3S_4S_3S_2$ of the generators $S_1, \ldots, S_4$ taken in cyclic order around the circuit, depending upon the starting point, except when one of p or q is 3 (see [22, equation 4.9]). In this latter case, when (say) $q = 3$, the group is actually $[1\ 1\ 2^s]^p$, with $s := p_{1234}$ the mark on the circuit, and we have already enumerated the finite groups of this kind in Theorem 3.1. Indeed, if $q = 3$, the change of generators described at the end of [22, §4] replaces the diagram for $[1\ 1\ 2^s]^p$ by the given diagram $\mathcal{D}$ (with interior mark s).

More generally, the periods are different; we are now confined to the cases $p, q = 4$ or 5, although the discussion still covers all possibilities. If the period is s, say, when the branch $\{1, 4\}$ carries the mark p (or q), then, in terms of the turn ϑ of the 4-circuit, we have

$$\cos^2 \frac{\pi}{s} = \cos^2 \frac{\pi}{p} + \cos^2 \frac{\pi}{q} + 2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\vartheta,$$

while if it is t when $\{1, 2\}$ is marked p (or q), then

$$\cos^2 \frac{\pi}{t} = \frac{1}{4} + 4\cos^2\frac{\pi}{p}\cos^2\frac{\pi}{q} + 2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\vartheta.$$

Of course, as we have remarked, if (say) $q = 3$, then $s = t$. In any case, we see that

$$\cos^2\frac{\pi}{t} - \cos^2\frac{\pi}{s} = 4\cos^2\frac{\pi}{p}\cos^2\frac{\pi}{q} - \cos^2\frac{\pi}{p} - \cos^2\frac{\pi}{q} + \frac{1}{4},$$

or

$$\cos\frac{2\pi}{t} - \cos\frac{2\pi}{s} = 2\left(\cos\frac{2\pi}{p} + \frac{1}{2}\right)\left(\cos\frac{2\pi}{q} + \frac{1}{2}\right). \tag{3.12}$$

On the other hand, a straightforward calculation using (2.5) shows that the Schläfli determinant $\varDelta$ is given by

$$16\varDelta = 9 - 16\cos^2\frac{\pi}{p} - 16\cos^2\frac{\pi}{q} + 16\cos^2\frac{\pi}{p}\cos^2\frac{\pi}{q} - 8\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\vartheta.$$

Eliminating the term involving the turn then gives the two parallel formulae

$$16\Delta = \begin{cases} 9 - 12\cos^2\dfrac{\pi}{p} - 12\cos^2\dfrac{\pi}{q} + 16\cos^2\dfrac{\pi}{p}\cos^2\dfrac{\pi}{q} - 4\cos^2\dfrac{\pi}{s}, \\ 10 - 16\cos^2\dfrac{\pi}{p} - 16\cos^2\dfrac{\pi}{q} + 32\cos^2\dfrac{\pi}{p}\cos^2\dfrac{\pi}{q} - 4\cos^2\dfrac{\pi}{t}, \end{cases}$$
$$= \begin{cases} \left(2\cos\dfrac{2\pi}{p} - 1\right)\left(2\cos\dfrac{2\pi}{q} - 1\right) - 2\left(\cos\dfrac{2\pi}{s} + 1\right), \\ 8\cos\dfrac{2\pi}{p}\cos\dfrac{2\pi}{q} - 2\cos\dfrac{2\pi}{t}, \end{cases} \tag{3.13}$$

which express Δ in terms of p, q and r or t, respectively.

We first treat the case $p = q = 5$. We quickly see that, for $\Delta > 0$, we must have $s = 2$. A little work (whose details are not worth including—but consider the effect of the implied basic operations) then shows that we have found $[3, 3, 5]$ in a different guise.

Finally, we have the case $q = 4$, say. This implies that

$$16\Delta = \begin{cases} -2\cos\dfrac{2\pi}{p} - 2\cos\dfrac{2\pi}{s} - 1, \\ -2\cos\dfrac{2\pi}{t}, \end{cases}$$

the latter irrespective of the value of p. We could carry out the analysis from this point (indeed $\Delta > 0$ implies that $t = 2$ or 3, and $\{p, s\} = \{2, 3\}, \{2, 4\}, \{2, 5\}$ or $\{3, 3\}$), but it is easier to appeal to Theorem 3.1. Now the reflexions $S_1, S_2S_3S_2, S_4, S_3$ (in this order) generate a subgroup $G^4(4, s; p, 3; 3)$ of G, which is finite if G is finite. But we know from Theorem 3.1 (and its proof) that the only finite groups of this kind are $G^4(4, 3; 3, 3; 3) \cong [1\ 1\ 2^4]^3$ and $G^4(4, 2; 4, 3; 3) \cong [4, 3, 3]$. Inspection ofthese cases shows that the original first group is $[1\ 1\ 2]^4 = [1\ 1\ 2^3]^4$ (now $p = 3$ implies that $t = s$), while the second is isomorphic to $[3, 4, 3]$ (apply the change of generators $S_4 \mapsto S_2S_3S_4S_3S_2$).

If $q = 4$ and $t = 4$ above, then we have the degenerate case $\Delta = 0$. The expression of Δ in terms of p, s then gives

$$\cos\frac{2\pi}{p} + \cos\frac{2\pi}{s} + \frac{1}{2} = 0,$$

which we can write in the form

$$\cos\frac{2\pi}{p} + \cos\frac{2\pi}{s} + \cos\frac{\pi}{3} = 0.$$

Now we know that all the solutions of *Gordan's equation*

$$\cos x\pi + \cos y\pi + \cos z\pi = 0 \quad (0 \leqslant x, y, z \leqslant 1)$$

in rational numbers are permutations of $(x, \frac{1}{2}, 1-x)$ with $0 \leqslant x \leqslant \frac{1}{2}$, $(x, \frac{2}{3}-x, \frac{2}{3}+x)$ with $0 \leqslant x \leqslant \frac{1}{3}$, $(\frac{1}{5}, \frac{3}{5}, \frac{2}{3})$ or $(\frac{1}{3}, \frac{2}{5}, \frac{4}{5})$ (see [8, p.274]). The second kind will not contribute any solutions to our equation, except possibly when $(p, s) = (2, 6)$ or $(6, 2)$; however, bearing in mind that $p = 2$ would give the *finite* group $[3, 4, 3]$, we can then rule out these possibilities entirely by observing that the group relation for $s = 2$ (that is, $(S_1S_2S_3S_4S_3S_2)^2 = I$) would already force $p = 2$ or 4 (that is, $(S_1S_4)^4 = I$). Finally, eliminating all but the *integer* values of p and s, we find the two solutions

$$(p, s) = (3, 4) \quad \text{or} \quad (4, 3).$$

The case $(p, s) = (3, 4)$ (which implies that $t = s$, and thus $t = 4$ anyway) is familiar; this is just the discrete group $[1\ 1\ 2^4]^4$ in another guise. Strangely enough, this is also true for $(p, s) = (4, 3)$, though the isomorphism is less obvious.

4 Abstract groups and diagrams

The dihedral groups are the basic 2-generator subgroups of Coxeter groups; these have the property that presentations for the groups involve at most pairs of involutory generators. The obvious next step, motivated by what we have done in the previous sections, is to consider groups generated by involutions, where now relators can employ two or three generators. It might initially be thought that the generalized triangle groups of Section 2 provide suitable paradigms for such 3-generator subgroups, giving a class of groups designated by diagrams which are suitably labelled simplicial 2-complexes.

However, the next natural question asks whether the obvious representations of such groups as complex matrices are faithful. We have seen, in [22, §5], that even in the "real" case, the natural homomorphism $\Gamma^3(5, 3; 5, 5) \to G^3(5, 3; 5, 5)$ is not an isomorphism. Moreover, in Section 3, we saw that the 3-generator subgroups need not determine the geometry of the whole group; that is, it can happen that more than one geometric group is associated with the same diagram.

In the context of enumerating the finite locally toroidal regular polytopes (see Section 5), the central criterion seems to be that, if we can find an infinite quotient of a subgroup (of finite index) of a group in which we are interested, then the original group itself must be infinite. Only when the corresponding quotient must necessarily be finite do we then look further into the problem, to determine if the corresponding unitary representation is faithful, and thus to establish that the original group must now itself be finite.

The core of the problem is that a complex representation imposes very strong conditions on the abstract group. First, a single triangle relator (of the form $S_iS_jS_kS_j$) determines the corresponding turn, and hence any other (non-Coxeter-type) relators for that triangle. (We may make exactly the same assumption here as we saw we could make in Section 2, namely that all the marks on diagrams basically equivalent to one implied by the abstract group relations are integers. While there may be representations of an abstract group corresponding to diagrams with fractional marks, to determine its finiteness we need not consider them.) Second, the turns on individual triangles are not independent—they must satisfy the compatibility relations (3.5) for any 4 nodes of the diagram. Third, as we have seen in [22] for circuit diagrams—for example, $\mathcal{J}_6(1, 1; 2, 2; 3)$ in Figure 4.1, which is that of an infinite discrete group—it may not be the case that triangle relators are actually most appropriate; we may need those on longer circuits. (In this diagram, the mark 3 in the square stands for the relation $(S_2S_3S_4S_5S_4S_3)^3 = I$.) The cumulative effect of these objections is that groups generated by involutions with relators involving three as well as two generators are somewhat rarely faithfully represented as unitary groups.

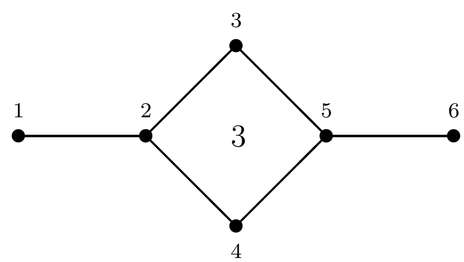


Figure 4.1: The diagram $\mathcal{J}_6(1,1;2,2;3)$.

All that notwithstanding, in the context of regular polytopes, their groups are usually abstract, even though we are often interested in their realizations (or, more generally, models). For that reason, it is appropriate to consider abstract groups with presentations corresponding to those we have discussed earlier, and their representations as complex linear groups.

We begin by discussing abstract diagrams; let us emphasize that we have been following Coxeter [6, 7] in associating diagrams with *geometric* groups. The most general abstract situation is the following. We set $\mathcal{N} := \{1,\ldots,n\}$, the set of *nodes*; each $i \in \mathcal{N}$ will be associated with an involutory generator σ_i of a group $\Gamma := \langle \sigma_1,\ldots,\sigma_n\rangle$. Let $\mathcal{S}$ be the set of finite sequences of elements of $\mathcal{N}$, with successive elements distinct (this eliminates some obvious trivialities below). A *marking* (or *labelling*) on $\mathcal{N}$ is a mapping $p\colon \mathcal{S} \to \{2,3,\ldots,\infty\}$, such that $p_i = 2$ (for a 1-element sequence) and $p_{i(1),\ldots,i(k)} = p_{i(k),\ldots,i(1)}$ for sequences in $\mathcal{S}$ in reverse order; the pair $\mathcal{D} := (\mathcal{N}, p)$ is called an *abstract diagram*. (We write the argument of p as a suffix, to accord with the notation introduced before Lemma 2.5; the next definition mimics what we saw for geometric groups in that section.) The group $\Gamma = \Gamma(\mathcal{N}, p)$ is then defined by

$$\begin{aligned}\Gamma := \langle \sigma_1,\ldots,\sigma_n \mid\ & (\sigma_{i(1)}\sigma_{i(2)}\cdots\sigma_{i(k)}\sigma_{i(k-1)}\cdots\sigma_{i(2)})^{p_{i(1),\ldots,i(k)}} = \varepsilon \\ & \text{for each } (i(1),\ldots,i(k)) \in \mathcal{S}\rangle.\end{aligned} \tag{4.1}$$

In theory, what we are doing here is specifying the periods of all products of pairs of conjugates of the generators. Thus a mark "∞" does not necessarily mean that the corresponding group element has infinite period, merely that it is unspecified. In applications, only a small number of sequences in $\mathcal{S}$ will receive a finite mark, including the 2-element sequences which specify Coxeter-type relations.

We could, in fact, further generalize this definition, by allowing $p_i \geqslant 2$, which would allow generators which are not involutions; however, the way we have specified p would then make less sense.

Since each defining relation in (4.1) involves an even number of generators, the subgroup Γ^+ of Γ consisting of the *even* elements (products of an even number of generators σ_i) has index 2, and hence $\Gamma = \Gamma^+ \ltimes C_2$ (see [11]). As generators of Γ^+ we can take the products $\sigma_i\sigma_j$ (with $i,j \in \mathcal{N}$), but usually a small subset of these will suffice; then

(4.1) translates into a corresponding presentation for Γ^+. Note that, in general, these presentations are not equivalent to those studied in [2, 29, 34, 35].

It is immediately obvious that we may impose certain compatibility conditions on the marking p. For instance, suppose that $p_{jk} = 3$; then $\sigma_i\sigma_j\sigma_k\sigma_j = \sigma_i\sigma_k\sigma_j\sigma_k$ for each i, so that the actual period of $\sigma_i\sigma_j\sigma_k\sigma_j$ is a common divisor of p_{ijk} and p_{ikj}.

With certain of these groups $\Gamma(\mathcal{N}, p)$, we can associate a diagram $\mathcal{D}$ in a more concrete sense. First, we join each pair of distinct nodes $i, j \in \mathcal{N}$ by a *branch* marked p_{ij} $(= p_{ji})$; we employ our standard conventions, in omitting a branch labelled 2, and omitting a mark 3 on any remaining branch. If $i(1), \ldots, i(k)$ are the nodes in cyclic order of a diagonal-free circuit $\mathcal{C}$ in $\mathcal{D}$, and all but at most one of the branches in $\mathcal{C}$ are unlabelled, then the periods of the elements $\sigma_{i(1)}\sigma_{i(2)}\cdots\sigma_{i(k)}\sigma_{i(k-1)}\cdots\sigma_{i(2)}$ are independent of the starting point $i(1)$ (because all these elements are conjugate), and so we may take the corresponding marks $p_{i(1),\ldots,i(k)}$ to be the same. In such a case, we may unambiguously give $\mathcal{C}$ the mark $p_{i(1),\ldots,i(k)}$. Thus $\mathcal{D}$ is a diagram with marked or unmarked branches, and a mark for every diagonal-free circuit in which at most one branch is marked.

There are several questions one might wish to ask about such a group $\Gamma := \Gamma(\mathcal{N}, p)$. As far as we are concerned, two are important. First, what conditions guarantee that Γ does not collapse, particularly to a group with generators corresponding to a proper subset of $\mathcal{N}$? As a related problem we also mention that of preassigning the structure of the *basic* 3-generator subgroups $\langle\sigma_i, \sigma_j, \sigma_k\rangle$ for Γ. Second, when is Γ finite, and if so, is it naturally isomorphic to a reflexion group? In certain circumstances, we can answer these questions.

Theorem 4.1 *Let $n \geqslant 3$, and let $2 \leqslant s \leqslant \infty$. Let p be defined by*

$$p_{jk} := 3 \quad (1 \leqslant j < k \leqslant n), \qquad p_{jkl} = s \quad (1 \leqslant j < k < l \leqslant n).$$

Then each basic 3-generator subgroup of $\Gamma = \Gamma(\mathcal{N}, p)$ is isomorphic to the unitary group $[1\,1\,1]^s$. Moreover, Γ is finite in just two cases:
a) $n = 3$ and $s < \infty$, when $\Gamma \cong [1\,1\,1]^s$;
b) $n \geqslant 3$ and $s = 2$, when $\Gamma \cong S_{n+1}$, the symmetric group on $n + 1$ elements.

Proof We construct a representation of Γ as a reflexion group $G = \langle S_1, \ldots, S_n\rangle$. It is easiest here to specify the Gram matrix for the corresponding hermitian form; this has entries α_{jk} $(:= \langle v_j, v_k\rangle)$ given by

$$2\alpha_{jk} := \begin{cases} 2, & \text{if } j = k, \\ -e^{2i\pi/s}, & \text{if } j < k, \\ -e^{-2i\pi/s}, & \text{if } j > k. \end{cases} \tag{4.2}$$

Of course, S_jS_k has period 3 whenever $j \neq k$, which is consistent with $p_{jk} = 3$. Further, if $j < k < l$, then the turn in the cycle (j, k, l) is

$$\frac{2\pi}{s} + \frac{2\pi}{s} - \frac{2\pi}{s} = \frac{2\pi}{s},$$

so that $S_jS_kS_lS_k$ has period s, which again is consistent with $p_{jkl} = s$. Then it follows that $\langle\sigma_j, \sigma_k, \sigma_l\rangle \cong \langle S_j, S_k, S_l\rangle \cong [1\ 1\ 1]^s$, as required.

The rest of the proof follows at once. If $n = 3$, then for finiteness of Γ we clearly have $s < \infty$. If $n \geqslant 4$, then Theorem 3.7 says that each 4-generator subgroup is infinite if $s \geqslant 3$; that is, if Γ is finite, then $s = 2$. If $s = 2$, we change the generators of Γ to $\tau_n := \sigma_n$ and $\tau_i := \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i < n$, and obtain the presentation $\tau_i^2 = (\tau_i\tau_{i+1})^3 = (\tau_i\tau_j)^2 = \varepsilon$, with $|i-j| \geqslant 2$, which shows that Γ is a quotient of S_{n+1}. On the other hand, the generating transpositions $(i\ \ n+1)$, with $i = 1, \ldots, n$, for S_{n+1} satisfy the relations for the generators σ_i of Γ, so that indeed $\Gamma \cong S_{n+1}$. □

In a somewhat similar way, we can also deal with the following more general kind of group, which we will need later. If $\mathcal{E}$ and $\mathcal{F}$ are subsets of pairs (2-subsets) and triples (3-subsets) in the node set $\mathcal{N} = \{1, \ldots, n\}$, we say that $\mathcal{E}$ is *compatible* with $\mathcal{F}$ if each 2-element subset of every triple in $\mathcal{F}$ belongs to $\mathcal{E}$. In our applications, $\mathcal{E}$ and $\mathcal{F}$ will consist of edge or face sets of a simplicial 2-complex on n vertices, for instance, a regular map with triangular faces on some surface.

Theorem 4.2 *Let $\mathcal{E}$ and $\mathcal{F}$ be subsets of pairs and triples in $\mathcal{N} = \{1, \ldots, n\}$, such that $\mathcal{E}$ is compatible with $\mathcal{F}$. Let $2 \leqslant s \leqslant \infty$, and define p by*

$$p_{jk} = 3 \quad (\{j,k\} \in \mathcal{E}); \qquad p_{jkl} = s \quad (\{j,k,l\} \in \mathcal{F}).$$

Then the basic 3-generator subgroups of $\Gamma := \Gamma(\mathcal{N}, p)$ which correspond to triples in $\mathcal{F}$ are isomorphic to the unitary group $[1\,1\,1]^s$. If Γ is finite, then $\mathcal{E}$ consists of all pairs in $\mathcal{N}$, and one of the following holds:
a) $n = 3$ *and* $s < \infty$*;*
b) $n \geqslant 3$ *and* $s = 2$.
In case (b), if $\mathcal{F}$ consists of all triples in $\mathcal{N}$, then $\Gamma \cong S_{n+1}$.

Proof We modify the Gram matrix whose entries are those of (4.2), by setting $\alpha_{jk} = -1$ if $j \neq k$ with $\{j,k\} \notin \mathcal{E}$ (otherwise α_{jk} is as before). Thus S_jS_k has infinite period for such pairs $\{j,k\}$, which is consistent with $p_{jk} = \infty$, and so Γ is infinite unless $\mathcal{E}$ contains all pairs in $\mathcal{N}$. Moreover, the group of Theorem 4.1, with all branches marked 3 and all triangles marked s, is clearly a quotient of Γ, and so is finite if Γ is finite. The current assertions now follow at once from Theorem 4.1. □

Notice that the only cases which Theorem 4.2 leaves open are those where $\mathcal{E}$ consists of all pairs in $\mathcal{N}$, some triples in $\mathcal{N}$ are marked 2, while others are unmarked. In our application of Theorem 4.2 in Section 6, when $\mathcal{F}$ will correspond to the set of faces of a regular map with vertex-set $\mathcal{N}$, we shall see that only one case needs further attention.

5 The basic enumeration technique

We begin our discussion with a brief introduction to the underlying general theory of regular polytopes (see [20] or [23, Chapter 2]). An (*abstract*) *polytope of rank* n, or simply an *n-polytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function whose range is $\{-1, 0, \ldots, n\}$. The elements of rank j are called the *j-faces* of $\mathcal{P}$, or *vertices*, *edges* and *facets* of $\mathcal{P}$ if $j = 0$, 1 or $n-1$, respectively. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ each contain exactly $n+2$ faces, including the unique minimal face F_{-1} and unique maximal face F_n of $\mathcal{P}$. Two flags are called *adjacent* if they differ by one element; then $\mathcal{P}$ is *strongly flag-connected*, meaning that, if Φ and Ψ are two flags, then they can be joined by a sequence of pairwise adjacent flags, each of which contains $\Phi \cap \Psi$. Finally, if F and G are an $(j-1)$-face and an $(j+1)$-face with $F < G$, then there are exactly *two* j-faces H such that $F < H < G$.

When F and G are two faces of a polytope $\mathcal{P}$ with $F \leqslant G$, we call $G/F := \{H \mid F \leqslant H \leqslant G\}$ a *section* of $\mathcal{P}$. We may usually safely identify a face F with the section F/F_{-1}. For a face F the section F_n/F is called the *co-face of* $\mathcal{P}$ *at* F, or the *vertex-figure at* F if F is a vertex.

An n-polytope $\mathcal{P}$ is *regular* if its (*automorphism*) *group* $\Gamma(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$ be a fixed or *base* flag of $\mathcal{P}$. The group $\Gamma(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \ldots, \rho_{n-1}$ (*with respect to* Φ), where ρ_j is the unique automorphism which keeps all but the j-face of Φ fixed. These generators satisfy relations

$$(\rho_i\rho_j)^{p_{ij}} = \varepsilon \quad (i, j = 0, \ldots, n-1), \tag{5.1}$$

with

$$p_{ii} = 1, \quad p_{ij} = p_{ji} \geqslant 2 \ (i \neq j), \quad p_{ij} = 2 \ \text{ if } |i-j| \geqslant 2. \tag{5.2}$$

The numbers $p_j := p_{j-1,j}$ ($j = 1, \ldots, n-1$) determine the (*Schläfli*) *type* $\{p_1, \ldots, p_{n-1}\}$ of $\mathcal{P}$. Further, $\Gamma(\mathcal{P})$ has the intersection property (3.4) (with respect to the distinguished generators), namely

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \ \text{ for all } I, J \subset \{0, \ldots, n-1\}.$$

Observe that, in a natural way, the group of F_k is $\langle \rho_0, \ldots, \rho_{k-1} \rangle$, while that of the co-face F_n/F_k is $\langle \rho_{k+1}, \ldots, \rho_{n-1} \rangle$.

By a *string C-group*, we mean a group which is generated by involutions such that (5.1), (5.2) and (3.4) hold. A string C-group is a C-group whose underlying diagram is a string. The group of a regular polytope is a string C-group. Conversely, given a string C-group, there is an associated regular polytope of which it is the automorphism group (see [20, 23]).

Given regular n-polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the vertex-figures of $\mathcal{P}_1$ are isomorphic to the facets of $\mathcal{P}_2$, we denote by $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ the *class* of all regular $(n+1)$-polytopes $\mathcal{P}$ with facets isomorphic to $\mathcal{P}_1$ and vertex-figures isomorphic to $\mathcal{P}_2$. If $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle \neq \emptyset$, then any such $\mathcal{P}$ is a quotient of a universal member of $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$; this *universal polytope* is denoted by $\{\mathcal{P}_1, \mathcal{P}_2\}$ (again, see [20, 23]).

We now move on to locally toroidal regular polytopes of rank 4. Recall that a regular polytope is *locally toroidal* if its sections which are not spherical are regular toroids (see [20]); in other words, if the rank is 4, the facets and vertex-figures which are not isomorphic to Platonic solids are regular maps on the 2-torus.

We begin the discussion by describing the basic technique which is applied to enumerate those finite universal locally toroidal regular 4-polytopes which are of some type $\{6, 3, p\}$ or $\{p, 3, 6\}$ with $p = 3, 4, 5, 6$, or $\{3, 6, 3\}$. We shall largely concentrate on the types $\{6,3,p\}$, but our method will also work for certain, but not all, polytopes of type $\{3,6,3\}$ (see Section 7).

In particular, we shall investigate the universal regular polytopes

$$ {}_p\mathcal{T}^4_{\mathbf{s}} := \{\{6,3\}_{\mathbf{s}}, \{3,p\}\} \qquad (p = 3,4,5), $$

with $\mathbf{s} = (s^k, 0^{2-k})$ and $k = 1,2$, as well as

$$ {}_6\mathcal{T}^4_{\mathbf{s},\mathbf{t}} := \{\{6,3\}_{\mathbf{s}}, \{3,6\}_{\mathbf{t}}\}, $$

with $\mathbf{s} = (s^k, 0^{2-k})$, $\mathbf{t} = (t^l, 0^{2-l})$ and $k, l = 1,2$. (The formal definitions are given in (5.3) below.) Our assumptions on the parameters are as follows. For all the types, $s \geqslant 2$ if $k = 1$, and $s \geqslant 1$ if $k = 2$, and, similarly, for the last type, $t \geqslant 2$ if $l = 1$, and $t \geqslant 1$ if $l = 2$. Here the superscript denotes the rank, which is 4, and the subscript p to the left is the last entry in the Schläfli symbol. (In [15], the polytopes with $p = 3$ were denoted by $\mathcal{H}_{s,0}$ or $\mathcal{H}_{s,s}$, respectively.)

The basic construction tool for these polytopes is *twisting*, which here means the extending of a group by suitable group automorphisms. The twisting operations are performed on abstract groups W of the type discussed in the previous sections (where they were denoted by Γ) and in [22], and are determined by diagrams which contain labelled circuits (usually triangles). In particular, we shall require the results of Sections 3 and 4 about the enumeration of the finite groups which belong to certain types of diagrams.

We also need some basic facts about the toroidal polyhedron $\{6,3\}_{\mathbf{s}}$, which occurs as the facet of ${}_p\mathcal{T}^4_{\mathbf{s}}$ and ${}_6\mathcal{T}^4_{\mathbf{s},\mathbf{t}}$; its group is denoted by $[6,3]_{\mathbf{s}}$. In particular, $[6,3]_{(s^k,0^{2-k})}$ is $[6,3] = \langle \rho_0, \rho_1, \rho_2 \rangle$, factored out by the single extra relation

$$ \begin{cases} (\rho_0\rho_1\rho_2)^{2s} & = \varepsilon \quad \text{if } k = 1, \\ (\rho_2(\rho_1\rho_0)^2)^{2s} & = \varepsilon \quad \text{if } k = 2 \end{cases} \tag{5.3} $$

(see [11, §8.4]). Note that $\{3,6\}_{(s,0)} = \{3,6\}_{2s}$. (Recall that $\{p,q\}_r$ denotes the regular map obtained from $\{p,q\}$ by identifying r steps along a Petrie polygon; again, see [11].)

For the polytopes ${}_p\mathcal{T}^4_{\mathbf{s}}$, we now consider the corresponding abstract group ${}_p\Gamma^4_{\mathbf{s}}$, which is defined as the Coxeter group $[6,3,p] = \langle \rho_0, \ldots, \rho_3 \rangle$, factored out by the extra relation in (5.3). For either choice of k, this extra relation involves only the first three generators ρ_0, ρ_1, ρ_2, and turns the facet $\{6,3\}$ of the hyperbolic honeycomb $\{6,3,p\}$ (see [5]) into its finite quotient $\{6,3\}_{\mathbf{s}}$. In particular, ${}_p\mathcal{T}^4_{\mathbf{s}}$ exists if and only if ${}_p\Gamma^4_{\mathbf{s}}$ is a C-group, whose subgroups $\langle \rho_0, \rho_1, \rho_2 \rangle$ and $\langle \rho_1, \rho_2, \rho_3 \rangle$ are isomorphic to $[6,3]_{\mathbf{s}}$ and $[3,p]$, respectively. In this case, $\Gamma({}_p\mathcal{T}^4_{\mathbf{s}}) = {}_p\Gamma^4_{\mathbf{s}}$.

On the other hand, for ${}_6\mathcal{T}^4_{\mathbf{s},\mathbf{t}}$ there are two extra relations, each with three generators. The first involves ρ_0, ρ_1, ρ_2, and is of the same kind as before. The second involves ρ_1, ρ_2, ρ_3, and is dual to a relation of (5.3); thus it is obtained from (5.3) by replacing ρ_i by ρ_{3-i}, k by l, and s by t. These extra relations turn the facet $\{6,3\}$ and vertex-figure $\{3,6\}$ of the honeycomb $\{6,3,6\}$ (again, see [5]) into the polyhedra $\{6,3\}_{\mathbf{s}}$ and $\{3,6\}_{\mathbf{t}}$, respectively. Now the corresponding group ${}_6\Gamma^4_{\mathbf{s},\mathbf{t}}$ is defined to be the factor group of $[6,3,6]$ determined by these two extra relations. In particular, ${}_6\mathcal{T}^4_{\mathbf{s},\mathbf{t}}$ exists if and only if ${}_6\Gamma^4_{\mathbf{s},\mathbf{t}}$ is a C-group, whose subgroups $\langle \rho_0, \rho_1, \rho_2 \rangle$ and $\langle \rho_1, \rho_2, \rho_3 \rangle$ are isomorphic to $[6,3]_{\mathbf{s}}$ and $[3,6]_{\mathbf{t}}$, respectively. In this case, $\Gamma({}_6\mathcal{T}^4_{\mathbf{s},\mathbf{t}}) = {}_6\Gamma^4_{\mathbf{s},\mathbf{t}}$.

We now describe the technique applied to enumerate the finite polytopes. Let $\mathcal{P} := {}_p\mathcal{T}^4_{\mathbf{s}}$ or ${}_6\mathcal{T}^4_{\mathbf{s},\mathbf{t}}$, and let $\Gamma := {}_p\Gamma^4_{\mathbf{s}}$ or ${}_6\Gamma^4_{\mathbf{s},\mathbf{t}}$, respectively. We proceed by the following three steps. First, we find a "suitable" normal subgroup W of finite index in Γ, such that Γ is a semi-direct product of W by a subgroup of its automorphisms. Second, for this subgroup W, we construct a complex "locally unitary" representation $r\colon W \to \mathrm{GL}_m(\mathbb{C})$, as in the previous sections and in [22], with m determined by $\mathcal{P}$. In particular, r will preserve a hermitian form $\langle\cdot,\cdot\rangle$ on complex m-space $\mathbb{C}^m$, such that the image group $G := Wr$ consists of isometries with respect to this form. Last, we study the hermitian form to analyse the structure of $\mathcal{P}$ and Γ. For much of the effort required in the second and third step we can appeal to the results of the previous sections.

Our approach works only under the mild restriction that $\mathbf{s}, \mathbf{t} \neq (1,1)$, which we shall assume from now on. The case $\mathbf{s} = (1,1)$ or $\mathbf{t} = (1,1)$ must be treated separately.

The construction of W and its representation r is based on the following simple observation, which relates $[6,3]_{\mathbf{s}}$ to the generalized triangle group $[1\,1\,1]^s$ of Figure 5.1. It shows that the above technique can already be applied in rank 3 to obtain $\{6,3\}_{\mathbf{s}}$. Recall that $[1\ 1\ 1]^s$ (with $s \geqslant 2$) is (isomorphic to) the group generated by involutions $\sigma_1, \sigma_2, \sigma_3$, and abstractly defined by the presentation

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = (\sigma_1\sigma_3)^3 = (\sigma_1\sigma_2\sigma_3\sigma_2)^s = \varepsilon. \tag{5.4}$$

Using the group automorphisms τ_1 and τ_2, which act on the generators σ_i as indicated, we can extend $[1\ 1\ 1]^s$ (with $s \geqslant 2$) in two ways; both are simple examples of twisting operations. First, we can extend by τ_1 and take $(\rho_0, \rho_1, \rho_2) = (\tau_1, \sigma_2, \sigma_3)$, to obtain $[6,3]_{(s,0)} = [1\ 1\ 1]^s \ltimes C_2$. Second, if we also extend by τ_2 and take $(\rho_0, \rho_1, \rho_2) = (\sigma_1, \tau_1, \tau_2)$, then we obtain $[6,3]_{(s,s)} = [1\ 1\ 1]^s \ltimes S_3$. With appropriate interpretation, this also remains true if $s = 1$; in particular, $[1\ 1\ 1]^1 \cong S_3$ and $[6,3]_{(1,1)} \cong S_3 \times S_3$.

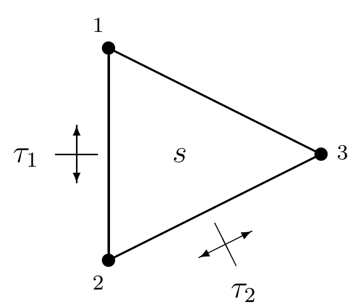


Figure 5.1: The group $[1\ 1\ 1]^s$.

We also need the following lemma (see [18]).

Lemma 5.1 *Let $\mathcal{K}$ be a regular polyhedron of type $\{k,3\}$ with $k \geqslant 2$. Then the universal polytope $\mathcal{P} := \{\mathcal{K}, \{3,6\}_{(2,0)}\}$ exists if and only if the universal polytope $\mathcal{Q} := \{\mathcal{K}, \{3,3\}\}$ exists. In this case, $\Gamma(\mathcal{P}) = \Gamma(\mathcal{Q}) \times C_2$.*

6 Polytopes with facets $\{6,3\}_{(s,s)}$

In [18], we described various methods of twisting reflexion groups generated by involutory reflexions, and we applied these twisting techniques to the problem of classifying certain types of polytopes. Since then, we have found that some of the constructions work in a more general context and allow significant generalizations to other classes of polytopes. The new approach also clarifies the overall picture.

In this section, we shall investigate the finite regular 4-polytopes whose facets are toroidal polyhedra $\{6,3\}_{(s,s)}$ with $s \geqslant 1$, and classify those universal polytopes (with any appropriate vertex-figure) which are finite. For the most part, we apply the technique described in Section 5; this will deal with the case $s \geqslant 2$. The case $s = 1$ requires a slightly different treatment. In particular, we shall enumerate all the finite polytopes ${}_p\mathcal{T}^4_{(s,s)} = \{\{6,3\}_{(s,s)}, \{3,p\}\}$, with $p = 3$, 4 or 5, and ${}_6\mathcal{T}^4_{(s,s),\mathbf{t}} = \{\{6,3\}_{(s,s)}, \{3,6\}_{\mathbf{t}}\}$, with $\mathbf{t} = (t^k, 0^{2-k})$, $t \geqslant 2$ if $k = 1$ or $t \geqslant 1$ if $k = 2$. Now let $s \geqslant 2$.

First, let $\mathcal{K}$ be a regular polyhedron which is a lattice (which here means that two of its vertices determine at most one edge), such that the class $\langle \{6,3\}_{(s,s)}, \mathcal{K}\rangle$ is non-empty. Of course, $\mathcal{K}$ must have triangular faces. Hence there is a universal polytope $\{\{6,3\}_{(s,s)}, \mathcal{K}\}$, with group $\Gamma = \langle \rho_0, \ldots, \rho_3\rangle$, say.

Writing $\Gamma_0 := \langle \rho_1, \rho_2, \rho_3\rangle$ ($\cong \Gamma(\mathcal{K})$), consider the subset

$$V := \{\gamma^{-1}\rho_0\gamma \mid \gamma \in \Gamma_0\},$$

which consists of involutions (conjugates of ρ_0), and the corresponding subgroup $N_0 := \langle V\rangle$. Then it is easy to see that N_0 is the normal closure of ρ_0 in Γ, and $\Gamma = N_0 \cdot \Gamma_0$ (see [23, Lemma 4E7]). Now, if $\Gamma_{01} := \langle \rho_2, \rho_3\rangle$ and $\beta \in \Gamma_{01}\gamma$, then $\beta^{-1}\rho_0\beta = \gamma^{-1}\rho_0\gamma$; it thus follows that there is a natural map $\kappa\colon \mathcal{K}_0 \to V$, with $\mathcal{K}_0$ the vertex-set of $\mathcal{K}$, which takes the vertex associated with $\Gamma_{01}\gamma$ (for $\gamma \in \Gamma_0$) onto the conjugate $\gamma^{-1}\rho_0\gamma$. Notice that κ commutes with the action of Γ_0; that is, if we identify (for a moment) the vertices in $\mathcal{K}$ with the cosets of Γ_{01}, then we have

$$((\Gamma_{01}\gamma)\beta)\kappa = \beta^{-1} \cdot ((\Gamma_{01}\gamma)\kappa) \cdot \beta,$$

for all $\gamma, \beta \in \Gamma_0$.

Suppose that $\lambda, \mu, \nu \in V$ correspond to the vertices of a face of $\mathcal{K}$; without loss of generality, we can take

$$\lambda = \rho_0, \quad \mu = \rho_1\rho_0\rho_1, \quad \nu = \rho_2\rho_1\rho_0\rho_1\rho_2.$$

Observe that conjugation by elements of $\langle \rho_1, \rho_2\rangle$ merely permutes λ, μ, ν. Then we have

$$\lambda\mu = (\rho_0\rho_1)^2,$$

so that $(\lambda\mu)^3 = \varepsilon$ (and hence $\lambda\nu$ and $\mu\nu$ also have period 3), and

$$\lambda\mu\lambda\nu = \rho_0 \cdot \rho_1\rho_0\rho_1 \cdot \rho_0 \cdot \rho_2\rho_1\rho_0\rho_1\rho_2 = (\rho_0\rho_1\rho_0\rho_1\rho_2)^2,$$

so that (5.3) yields $(\lambda\mu\lambda\nu)^s = \varepsilon$.

Now suppose that $\mathcal{K}$ has v vertices. We can identify $\mathcal{K}_0$ with $\mathcal{N} := \{1, \ldots, v\}$, and then identify the sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of edges and faces of $\mathcal{K}$ with the sets of pairs $\{i, j\}$ and triples $\{i, j, k\}$ in $\mathcal{N}$, respectively. Define the group W by

$$\begin{aligned} W \;:=\; \langle \sigma_1, \ldots, \sigma_v \mid \;& \sigma_i^2 = \varepsilon \; (i \in \mathcal{K}_0),\; (\sigma_i \sigma_j)^3 = \varepsilon \; (\{i, j\} \in \mathcal{K}_1), \\ & (\sigma_i \sigma_j \sigma_k \sigma_j)^s = \varepsilon \; (\{i, j, k\} \in \mathcal{K}_2) \rangle. \end{aligned} \tag{6.1}$$

Once again, recall that the generators in the last relation can be taken in any order (see Section 4). Then N_0 is clearly a quotient of W (under the homomorphism which sends σ_i to the conjugate of ρ_0 corresponding to the vertex i).

We can now twist W. We let $\Gamma(\mathcal{K}) = \langle \tau_0, \tau_1, \tau_2 \rangle$ act on $\{\sigma_i \mid i \in \mathcal{K}_0\}$ in the natural way as a group of automorphisms, to obtain a group

$$\overline{\Gamma} := W \ltimes \Gamma(\mathcal{K}),$$

under the operation

$$(\{\sigma_i \mid i \in \mathcal{K}_0\}; \tau_0, \tau_1, \tau_2) \mapsto (\sigma_1, \tau_0, \tau_1, \tau_2), \tag{6.2}$$

where we associate σ_1 with the initial vertex of $\mathcal{K}$; in other words, we take $\overline{\Gamma}$ with distinguished generators $\sigma_1, \tau_0, \tau_1, \tau_2$. Since we recover Γ from N_0 by the exactly analogous operation, we deduce that Γ is a quotient of $\overline{\Gamma}$; in other words, since κ commutes with the action of Γ_0, the homomorphism from W onto N_0 extends to one from $\overline{\Gamma}$ onto Γ. But because Γ was the group of the universal polytope $\{\{6,3\}_{(s,s)}, \mathcal{K}\}$, and since, from the quotient criterion of [20, Lemma 2.1] applied to the group $\Gamma(\mathcal{K})$ of the vertex-figure, $\overline{\Gamma}$ is the group of a polytope in $\langle \{6,3\}_{(s,s)}, \mathcal{K} \rangle$, we conclude that $\overline{\Gamma} = \Gamma$, so that Γ itself is obtained by means of the twisting operation of (6.2). Moreover, $N_0 \cong W$, and $\Gamma \cong N_0 \ltimes \Gamma_0$.

A very similar approach deals with the case when the vertex-figure $\mathcal{K}$ is not a lattice. Then there is a natural quotient $\mathcal{L}$ of $\mathcal{K}$, obtained by identifying edges and faces with the same vertices. Here, we must allow $\mathcal{L} = \{3, 2\}$ (a ditope) and $\mathcal{L} = \{3, 1\}$ (a poset of rank 3 with a single triangular face) as possible quotients. (We slightly abuse the Schläfli symbol notation, and allow 1 as an entry.) More precisely, we have the following

Lemma 6.1 *Let $\mathcal{K}$ be a regular polyhedron of type $\{3, k\}$ with $k \geqslant 2$, but let $\mathcal{K} \neq \{3, 4\}/2$. Let $\Gamma(\mathcal{K}) = \langle \tau_0, \tau_1, \tau_2 \rangle$, and let $N := \langle (\tau_2 \tau_1)^p \rangle$, where $p \geqslant 1$ is the smallest integer such that $\tau_0 (\tau_2 \tau_1)^p \tau_0 \in \langle \tau_1, \tau_2 \rangle$. Then $p > 1$, $p \mid k$, and N is a normal subgroup of $\Gamma(\mathcal{K})$ of order k/p contained in $\langle \tau_1, \tau_2 \rangle$. The quotient $\mathcal{L} := \mathcal{K}/N$ is a regular polyhedron of type $\{3, p\}$, whose vertex-set can naturally be identified with that of $\mathcal{K}$. Moreover, $\mathcal{L}$ is a lattice unless $\mathcal{L} = \{3, 2\}$, and $\mathcal{L} \cong \mathcal{K}$ if $\mathcal{K}$ is a lattice.*

Proof Suppose that we have two vertices in the vertex-figure at the base vertex x (say) which coincide. We know that this must occur when $\mathcal{K}$ is not a lattice. Then the element $(\tau_2 \tau_1)^q$, for some q, takes one of these vertices onto the other, and hence $\tau_0 (\tau_2 \tau_1)^q \tau_0$ fixes x. Clearly, $p \neq 1$, $p \mid q$ and $p \mid k$. It follows that the vertex-figure has p distinct vertices through which it cycles k/p times. If $\mathcal{K}$ is a lattice, then $p = k$, and the lemma holds trivially. The same is true if $k = 2$, and so we may assume that $k \geqslant 3$.

The subgroup N is normal in $\Gamma(\mathcal{K})$. Indeed, $\tau := \tau_0(\tau_2\tau_1)^p\tau_0$ fixes the two vertices x and y (say) in the base edge, and so we must have $\tau = {\tau_2}^i(\tau_2\tau_1)^q$, where $i = 0, 1$ and again $p \mid q$. Then $\tau \in N$ if $i = 0$, and hence N is invariant under conjugation by the generators in this case. We can rule out $i = 1$ as a possibility as follows. First observe that $i = 1$ forces $k = 2p$ (and $q = p$), because now τ is an involution; in the vertex-figure at y, it acts like a half-turn, and in the vertex-figure at x, it acts like a reflexion. Now, if z (say) is the third vertex of the base face (distinct from x and y), then we have $z\tau = z$, because $z\tau$ is the vertex opposite to z in the vertex-figure at y; on the other hand, being the image of z under the reflexion $\tau_2(\tau_2\tau_1)^p$, the vertex $z\tau$ is $p-2$ steps away from z on the vertex-figure at x. This contradicts the definition of p unless $p = 2$ (and $k = 4$). But $p = 2$ would imply that $\mathcal{K} = \{3, 4\}/2$, which is excluded by assumption. (Recall that $\{3, 4\}/2$ is the projective polyhedron obtained from the octahedron $\{3, 4\}$ by identifying antipodal faces.)

Finally, the vertices, edges and faces of $\mathcal{L}$ are the orbits of the vertices, edges and faces of $\mathcal{K}$ under N, respectively. Since N is normal, and contained in $\langle\tau_1, \tau_2\rangle$, it fixes every vertex of $\mathcal{K}$, and so we can identify the vertices of $\mathcal{L}$ with those of $\mathcal{K}$. Furthermore, N maps every edge and every face of $\mathcal{K}$ onto an edge or a face with the same vertices. Indeed, if two edges of $\mathcal{K}$ share the same vertices, then they are equivalent under N, and hence are identified. (This may not be true for the faces.) It follows that two vertices of $\mathcal{L}$ determine at most one edge in $\mathcal{L}$. Hence $\mathcal{L}$ is a lattice unless $\mathcal{L} = \{3, 2\}$. □

Observe that the case $\mathcal{L} = \{3, 2\}$ occurs for the toroidal polyhedron $\mathcal{K} = \{3, 6\}_{(1,1)}$ (and of course for $\{3, 2\}$ itself).

Notice also that the quotient $\mathcal{L}$ in Lemma 6.1 is "minimal", meaning that N is the largest normal subgroup of $\Gamma(\mathcal{K})$ for which polytopality and the vertex-set are preserved for the resulting quotient.

The projective polyhedron $\mathcal{K} = \{3, 4\}/2$ is special and was excluded in the lemma. Here we have $\tau_0(\tau_2\tau_1)^2\tau_0 = \tau_2(\tau_2\tau_1)^2$ (that is, the case $i = 1$ of the proof occurs here), and so τ_2 is a product of conjugates of $(\tau_2\tau_1)^2$. These conjugates generate an abelian normal subgroup N of order 4 which contains τ_2. Now the quotient $\mathcal{L} := \mathcal{K}/N$ is a poset of rank 3 with a single face; that is, $\mathcal{L} = \{3, 1\}$. Note that $\mathcal{K} = \{3, 4\}/2$ is the only polyhedron for which the associated quotient $\mathcal{L}$ is not itself a polyhedron.

At this stage, we have a natural quotient $\mathcal{L}$ for every polyhedron $\mathcal{K}$ with triangular faces. Now, if $\langle\{6, 3\}_{(s,s)}, \mathcal{L}\rangle \neq \emptyset$, then also $\langle\{6, 3\}_{(s,s)}, \mathcal{K}\rangle \neq \emptyset$, by the quotient criterion of [20, Lemma 2.1] applied to the group of the facet $\{6, 3\}_{(s,s)}$. The quotient mapping $\mathcal{K} \searrow \mathcal{L}$ induces a quotient mapping $\{\{6, 3\}_{(s,s)}, \mathcal{K}\} \searrow \{\{6, 3\}_{(s,s)}, \mathcal{L}\}$ determined by the same subgroup N, which remains normal in the larger group. (When $\mathcal{L} = \{3, 1\}$, the quotient $\{\{6, 3\}_{(s,s)}, \mathcal{L}\}$ is not actually a polytope but instead a poset of rank 4 with a single facet $\{6, 3\}_{(s,s)}$.) Moreover, if Γ and Γ' are the groups of $\{\{6, 3\}_{(s,s)}, \mathcal{K}\}$ and $\{\{6, 3\}_{(s,s)}, \mathcal{L}\}$, respectively, then the corresponding homomorphism $\Gamma \to \Gamma'$ takes the product $N_0 \cdot \Gamma_0 = \Gamma$ into the semi-direct product $N'_0 \cdot \Gamma'_0 = \Gamma'$ (say), where $N_0 \cong N'_0 \cong W$, $\Gamma_0 \cong \Gamma(\mathcal{K})$ and $\Gamma'_0 \cong \Gamma(\mathcal{L})$. Since the homomorphism is one-to-one on N_0, we must have $N_0 \cap \Gamma_0 = \{\varepsilon\}$, and so $N_0 \cdot \Gamma_0$ is semi-direct as well.

Let us summarize the above analysis.

Lemma 6.2 *Let $\mathcal{K}$ be a regular polyhedron with triangular faces, and let $\mathcal{L}$ be the associated regular polyhedron (poset) for which the covering map $\mathcal{K} \searrow \mathcal{L}$ preserves vertices. If $s \geqslant 2$, and if $\mathcal{L}$ is such that $\langle \{6,3\}_{(s,s)}, \mathcal{L} \rangle \neq \emptyset$, then also $\langle \{6,3\}_{(s,s)}, \mathcal{K} \rangle \neq \emptyset$, and the group of the universal polytope $\{\{6,3\}_{(s,s)}, \mathcal{K}\}$ is obtained from the group W of (6.1) by means of the twisting operation of (6.2).*

To complete the solution of the classification problem, we must now show that, if we start with an arbitrary regular polyhedron $\mathcal{K}$, then the group W of (6.1) does not collapse to one with fewer than v generators. (Notice that it certainly will if $s = 1$, which is why this case is excluded from the present discussion.) Of course, W depends on the quotient polyhedron $\mathcal{L}$ (if we need to pass to it), rather than on the original vertex-figure $\mathcal{K}$. Indeed, its definition involves $\mathcal{L}_0 \,(= \mathcal{K}_0)$, $\mathcal{L}_1$ and $\mathcal{L}_2$, with an appropriate interpretation of $\mathcal{L}_2$ if $\mathcal{L} = \{3,2\}$ (for $\mathcal{L} = \{3,2\}$, we can pick any one of the two faces to represent $\mathcal{L}_2$). Once all the properties of W have been established, then we are basically done. The twisting operation in (6.2) will give us a regular polytope in $\langle \{6,3\}_{(s,s)}, \mathcal{L} \rangle$ to start with, and the rest is taken care of by Lemma 6.2. Therefore it suffices to concentrate on W.

Suppose first that $\mathcal{L}$ (or, equivalently, $\mathcal{K}$) is not neighbourly. (Recall that a polytope is *neighbourly* if every pair of vertices determines an edge.) We now have exactly the situation of Theorem 4.2, with v instead of n. The group we constructed there genuinely has v generators. Since it is a quotient (in the obvious way) of the group W of (6.1), we conclude

Lemma 6.3 *The group W defined by (6.1) has v generators. In particular, it does not collapse.*

The last step involves a deeper investigation of W. If the underlying polyhedron $\mathcal{L}$ is non-neighbourly, then the period of a product $\sigma_i \sigma_j$ with $\{i,j\} \notin \mathcal{L}_1$ is infinite, again as in (the proof of) Theorem 4.2; hence W is infinite. There remains the case that $\mathcal{L}$ is neighbourly, and, with a further appeal to Theorem 4.2, we know that, for finiteness, we must have $s = 2$ if $v \geqslant 4$, whereas any finite s will give a finite group if $v = 3$. We need a subsidiary result (see also [16]).

Lemma 6.4 *The only neighbourly regular polyhedra with triangular faces which are lattices are the tetrahedron $\{3,3\}$ and the hemi-icosahedron $\{3,5\}_5$.*

Proof Let $\mathcal{L}$ be such a polyhedron, of type $\{3,p\}$, say; then $\mathcal{L}$ has $p+1$ vertices. We label its initial vertex ∞, and the vertices adjacent to ∞ (that is, those of the vertex-figure) $0, 1, \ldots, p-1$ in cyclic order. We take $\{\infty, 0\}$ as the initial edge, and $\{\infty, 0, 1\}$ as initial triangular face; further, let τ_0, τ_1, τ_2 be the associated distinguished generators of its group $\Gamma(\mathcal{L})$. Let r be the third vertex of the other face which contains $\{0,1\}$.

Then r lies in the vertex-figure of $\mathcal{L}$ at ∞, and is fixed by τ_1 (which fixes ∞ and interchanges 0 and 1), and so must be the vertex of the vertex-figure opposite to $\{0,1\}$. It follows that p is odd, and that $r = \frac{1}{2}(p+1)$. Now, by symmetry, the edge $\{0,r\}$ of $\{0,1,r\}$ belongs to the face $\{0, r-1, r\}$ as well; since the edges like $\{0,r\}$ are the only ones which can be free (in a planar drawing), we see that $\mathcal{L}$ closes up. There are just two possibilities. Either $p = 3$, and $\mathcal{L}$ is closed up by one face not containing ∞, or $p = 5$, and we have 5 extra faces, giving the hemi-icosahedron (the projective polyhedron obtained from the icosahedron $\{3, 5\}$ by identifying antipodal faces). □

We now tie up the loose ends. If $\mathcal{L} = \{3, 3\}$ and $s = 2$, then $W = S_5$, and so W is finite (see again Theorem 4.2). We are thus reduced to the case $\mathcal{L} = \{3, 5\}_5$ (with $s = 2$), with the labelling of its vertices as in Lemma 6.4.

For $\{3, 5\}_5$, we directly construct a representation of W as a reflexion group G in $\mathbb{C}^6$. We wish to mark the triangles of a diagram (of G) on 6 nodes and all 15 branches unmarked, so that those corresponding to faces in $\mathcal{L}$ get marks 2, while (some at least of) the non-faces are marked ∞. We can obtain the required marking by a specific description of the Gram matrix for the normal vectors of the generating reflexions. Its off-diagonal entries are given by

$$2\alpha_{jk} = \begin{cases} 1, & \text{if } j = 0, \ldots, 4 \text{ and } k = j + 1 \pmod 5, \\ -1, & \text{otherwise.} \end{cases}$$

Here, ∞ is permitted as a suffix. But now we see that a typical triangle such as $\{\infty, 0, 2\}$ is unmarked (its turn is 0); it thus yields an infinite subgroup $[1\,1\,1]^\infty$ of G, which is therefore always itself infinite. (In fact, all 10 triangles corresponding to non-faces are unmarked, although we did not initially require this.) It follows that W is infinite as well.

We have now come full circle. Whether finite or not, we can twist W as in (6.2) and obtain a regular polytope in $\langle \{6, 3\}_{(s,s)}, \mathcal{L} \rangle$ (its group is a semi-direct product, and so the intersection property is trivial here). The rest is taken care of by Lemma 6.2.

In summary, we conclude that we have proved the following theorem.

Theorem 6.5 *Let $\mathcal{K}$ be a regular polyhedron with triangular faces, and let $s \geqslant 2$. Then the universal regular 4-polytope $\mathcal{P} := \{\{6, 3\}_{(s,s)}, \mathcal{K}\}$ exists. Its group is $\Gamma(\mathcal{P}) = W \ltimes \Gamma(\mathcal{K})$, where W is abstractly defined by (6.1). Moreover, if $\mathcal{L}$ is the regular polyhedron (poset) for which the covering map $\mathcal{K} \searrow \mathcal{L}$ preserves vertices, then $\mathcal{P}$ is finite only when*
a) $\mathcal{L} = \{3, 1\}$ *or* $\{3, 2\}$ *with any* $s \geqslant 2$, *with group* $[1\ 1\ 1]^s \ltimes \Gamma(\mathcal{K})$;
b) $\mathcal{L} = \{3, 3\}$ *and* $s = 2$, *with group* $S_5 \ltimes \Gamma(\mathcal{K})$.

Note that $\Gamma(\mathcal{K})$ acts on W in the natural way, whether or not $\mathcal{K}$ is a lattice. The projective polytope $\mathcal{K} = \{3, 4\}/2$ is the only instance of a regular polyhedron which fits into Theorem 6.5(a) with $\mathcal{L} = \{3, 1\}$.

We should recall that this section is intended to treat locally toroidal regular polytopes. As far as these are concerned, we can extract the following results from Theorem 6.5.

Corollary 6.6 *Let $s \geqslant 2$. The only finite universal locally toroidal regular polytopes $\mathcal{P} = \{\{6, 3\}_{(s,s)}, \mathcal{K}\}$ are given by the following:*
a) $\mathcal{K} = \{3, 6\}_{(1,1)}$ *for any* s, *with group* $\Gamma(\mathcal{P}) = [1\ 1\ 1]^s \ltimes (S_3 \times S_3)$;
b) $\mathcal{K} = \{3, 3\}$ *for* $s = 2$, *with group* $\Gamma(\mathcal{P}) = S_5 \times S_4$;
c) $\mathcal{K} = \{3, 6\}_{(2,0)}$ *for* $s = 2$, *with group* $\Gamma(\mathcal{P}) = S_5 \times (S_4 \times C_2)$.

Notice that the products in the second and third part of Corollary 6.6 are direct. The second part covers the polytopes of type ${}_3\mathcal{T}^4_{(s,s)} = \{\{6, 3\}_{(s,s)}, \{3, 3\}\}$ with tetrahedral vertex-figures, whose group is ${}_3\Gamma^4_{(s,s)}$. Here we can embed $\Gamma(\mathcal{P}) = W \ltimes \Gamma(\mathcal{K})$ in S_9 if $s = 2$, taking $\sigma_i = (i\ 5)$ for $i = 1, \ldots, 4$, and $\tau_i = (i\ i+1)(5+i\ 5+i+1)$ for $i = 1, 2, 3$; thus $\Gamma(\mathcal{P}) = S_5 \times S_4$. From Lemma 5.1, we then also have the direct product in the third part.

The same arguments we have used here will work in a more general context. Let $\mathcal{K}$ be a regular $(n-1)$-polytope with triangular 2-faces. We slightly abuse our usual notation, and write $\langle \{6,3\}_{(s,s)}, \mathcal{K} \rangle$ for the class of regular n-polytopes with 3-faces isomorphic to $\{6,3\}_{(s,s)}$ and vertex-figures isomorphic to $\mathcal{K}$. (Strictly speaking, we should set up a recursive definition, and prove first that, if $n > 4$ and $\mathcal{J}$ is the facet of $\mathcal{K}$, then $\langle \{6,3\}_{(s,s)}, \mathcal{J} \rangle$ contains a universal member $\mathcal{Q} := \{\{6,3\}_{(s,s)}, \mathcal{J}\}$, say. We then discuss the class $\langle \mathcal{Q}, \mathcal{K} \rangle$.)

The same considerations lead us to the existence of an abstract diagram (for W) on v nodes, with v the number of vertices of $\mathcal{K}$ (see Section 4). Branches corresponding to edges of $\mathcal{K}$ are unmarked, while those associated with non-edges are marked ∞ (note that two vertices of $\mathcal{K}$ may be joined by more than one edge). Similarly, triangular circuits corresponding to 2-faces of $\mathcal{K}$ are marked s, while all others are unmarked.

Theorem 4.2 leads at once to the following; we shall not give the straightforward proof.

Theorem 6.7 *Let $s \geqslant 2$, and for $n \geqslant 5$ let $\mathcal{K}$ be a finite regular $(n-1)$-polytope with triangular 2-faces. Then the class $\langle \{6,3\}_{(s,s)}, \mathcal{K} \rangle$ is non-empty. Its universal member $\{\{6,3\}_{(s,s)}, \mathcal{K}\}$ is infinite if $\mathcal{K}$ has at least four vertices and*
a) $s \geqslant 3$,
b) $s = 2$ *and $\mathcal{K}$ is non-neighbourly.*

It is probable that $\{\{6,3\}_{(s,s)}, \mathcal{K}\}$ is infinite even if $s = 2$ and $\mathcal{K}$ is neighbourly, unless every triangular circuit arises from a 2-face of $\mathcal{K}$, so that the 2-skeleton of $\mathcal{K}$ must collapse onto that of some simplex by natural identification of its edges and faces with the same vertices. However, to prove this would require a stronger version of Theorem 4.2 than we have been able to establish. The condition does give finite universal polytopes with $\mathcal{K} = \{3^{n-2}\}$ (the $(n-1)$-simplex) or $\{3^{n-3}, 4\}/2$ (the hemi-$(n-1)$-cross-polytope).

If $\mathcal{K}$ has only three vertices, then $\langle \{6,3\}_{(s,s)}, \mathcal{K} \rangle$ is finite for every $s \geqslant 2$, and its group is $[1\ 1\ 1]^s \ltimes \Gamma(\mathcal{K})$. An example of a regular 4-polytope with only three vertices is $\{\{3,6\}_{(1,1)}, \{6,3\}_{(1,1)}\}$ (see [18, p.123]).

Before we move on, note that the dual of the polytope ${}_3\mathcal{T}^4_{\mathbf{s}} = \{\{6,3\}_{\mathbf{s}}, \{3,3\}\}$ (with $\mathbf{s} \neq (1,1), (2,0)$) is actually a 3-dimensional simplicial complex whose vertex links are isomorphic to the toroidal polyhedron $\{3,6\}_{\mathbf{s}}$. A general result due to Altshuler [1] says that, given a finite set of abstract polyhedra which are simplicial 2-complexes, there always exists a finite 3-dimensional simplicial complex whose vertex links belong to the set (with each polyhedron in the set actually occurring as a link). This complex will be an abstract 4-polytope, but will generally not be regular.

The regular polyhedron $\{6,3\}_{(1,1)}$ (that is, the case $s = 1$) was excluded from the previous discussion, because of the lack of appropriate hermitian forms. We refer to [18, §8] for the enumeration of the finite polytopes $\{\{6,3\}_{(1,1)}, \mathcal{K}\}$, with $\mathcal{K}$ any regular polyhedron of type $\{3,p\}$ for some $p \geqslant 3$.

7 Other types of polytopes

Twisting operations on reflexion groups can also be applied to enumerate other types of finite universal regular polytopes. A number of types have already been discussed in [18]. Our new techniques enable us to describe their enumeration with a lot more clarity and detail. However, to avoid unnecessary duplication we shall restrict ourselves to a brief summary of some important results.

The polytopes $\{\{2l,3\}_{2s},\{3,p\}\}$

We now consider finite regular 4-polytopes with facets isomorphic to regular maps $\{2l,3\}_{2s}$ and vertex-figures isomorphic to Platonic solids $\{3,p\}$, with $l,s \geqslant 2$ and $p = 3,4,5$. This will include the finite polytopes ${}_p\mathcal{T}^4_{(s,0)}$ obtained for $l=3$.

We begin with a general remark. Let $\mathcal{P}$ be any regular n-polytope with 3-faces of the form $\{2l,3\}_{2s}$ for some $l,s \geqslant 2$. If its group $\Gamma := \Gamma(\mathcal{P})$ is, as usual, $\langle \rho_0,\ldots,\rho_{n-1}\rangle$, then the "mixing operation"

$$(\rho_0,\ldots,\rho_{n-1}) \mapsto (\rho_0\rho_1\rho_0,\rho_1,\ldots,\rho_{n-1}) =: (\sigma_1,\ldots,\sigma_n)$$

yields a subgroup $W := \langle \sigma_1,\ldots,\sigma_n\rangle$ of index at most 2 in Γ (with coset representative ρ_0 if the index is 2—the notation has been chosen to accord with what we do immediately below.) We can recover Γ from W by means of the involutory twisting operation τ $(=\rho_0)$, acting by

$$\tau\sigma_1\tau = \sigma_2, \qquad \tau\sigma_j\tau = \sigma_j \quad (j = 3,\ldots,n).$$

Examples (if they exist) with index 1 are of less interest here.

As we saw in [22, Theorem 5.1], the subgroup $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ of W is isomorphic to the generalized triangle group $\Gamma^3(l,s;3,3)$. Indeed, we have $\sigma_1\sigma_2 = (\rho_0\rho_1)^2$ and $\sigma_1\sigma_3\sigma_2\sigma_3 = (\rho_0\rho_1\rho_2)^2$, and hence

$$(\sigma_1\sigma_2)^l = \varepsilon, \quad (\sigma_1\sigma_3\sigma_2\sigma_3)^s = \varepsilon.$$

In general, this observation is of little value. However, when the vertex-figure of $\mathcal{P}$ is a spherical or toroidal polyhedron, there results a criterion for the finiteness (or otherwise) of the corresponding universal polytope.

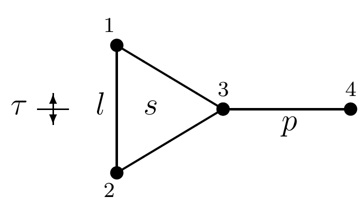


Figure 7.1:

For the polytopes $\{\{2l,3\}_{2s},\{3,p\}\}$, the observation above leads us to begin with the abstract group $W := \Gamma^4(l,s;3,3;p) = \langle\sigma_1,\ldots,\sigma_4\rangle$ corresponding to the diagram in Figure 7.1. Here, τ indicates the group automorphism of W which corresponds to the symmetry of the diagram in its horizontal axis. In more concrete terms, W has a presentation which consists of the relations for $\Gamma^3(l,s;3,3)$ and

$$\sigma_4^2 = (\sigma_1\sigma_4)^2 = (\sigma_2\sigma_4)^2 = (\sigma_3\sigma_4)^p = \varepsilon. \tag{7.1}$$

We now extend W using the operation

$$(\sigma_1, \ldots, \sigma_4; \tau) \mapsto (\tau, \sigma_2, \sigma_3, \sigma_4) =: (\rho_0, \rho_1, \rho_2, \rho_3). \tag{7.2}$$

The new group $\Gamma := \langle \rho_0, \ldots, \rho_3 \rangle = W \ltimes C_2$, with C_2 generated by τ, is in fact the group of the universal regular polytope $\mathcal{P} := \{\{2l, 3\}_{2s}, \{3, p\}\}$, if the latter should exist. Clearly, for a finite polytope we must have a finite group $\Gamma^3(l, s; 3, 3)$, and then $\Gamma^3(l, s; 3, 3) = [1\ 1\ 1^l]^s$. If we now impose the corresponding conditions on l and s (see Theorem 2.6 and Table 2.2), then the methods of Sections 2 and 3 become available, and we can produce the required representation of W as a reflexion group on $\mathbb{C}^4$. Theorem 3.3 shows that W is a C-group, and then Γ must also be a C-group. For all other properties, we can appeal to Theorems 3.1 and 3.2.

Theorem 7.1 *Suppose that the pair (l, s) satisfies the following condition: $l \geqslant 2$, $s = 3$; or $l = 3$, $s \geqslant 2$; or $(l, s) = (4, 4)$, $(4, 5)$ or $(5, 4)$. Let $p = 3$, 4 or 5. Then the universal regular 4-polytope $\{\{2l, 3\}_{2s}, \{3, p\}\}$ exists. Its group is $W \ltimes C_2$, where W is the group abstractly defined by the diagram in Figure 7.1. In particular, the polytope is finite precisely in the following cases:*
a) $l = p = 3$ *and* $s = 2$, 3 *or* 4, *with group* $S_5 \times C_2$, $[1\ 1\ 2]^3 \ltimes C_2$ *or* $[1\ 1\ 2]^4 \ltimes C_2$, *of order* 240, 1296 *or* 15360, *respectively;*
b) $l = 3$, $p = 4$ *and* $s = 2$, *with group* $[3, 3, 4] \times C_2$, *of order* 768*;*
c) $l = 3$, $p = 5$ *and* $s = 2$, *with group* $[3, 3, 5] \times C_2$, *of order* 28800*.*
d) $l \geqslant 2$ *and* $s = p = 3$, *with group* $[1\ 1\ 2^l]^3 \ltimes C_2$, *of order* $48l^3$*.*

In part (d), if $l = 4$, the facets are isomorphic to Dyck's map $\{8, 3\}_6$ of genus 3 (see [11, 13, 14]).

The polytopes $\{\{2p, 3\}_{2s}, \{3, 2r\}_{2t}\}$

Our preliminary considerations run much along the lines of those for the previous subsection. Let $\mathcal{P}$ be a (not necessarily universal) regular polytope with facets $\{2p, 3\}_{2s}$ and vertex-figures $\{3, 2r\}_{2t}$ for some $s, t \geqslant 2$, and let $\Gamma = \langle \rho_0, \ldots, \rho_3 \rangle$ be its group. Thus Γ has a presentation which includes the relations

$$(\rho_0\rho_1\rho_2)^{2s} = (\rho_1\rho_2\rho_3)^{2t} = \varepsilon, \tag{7.3}$$

adjoined to the standard presentation of the Coxeter group $[2p, 3, 2r]$.

Consider the mixing operation

$$(\rho_0, \ldots, \rho_3) \mapsto (\rho_0\rho_1\rho_0, \rho_1, \rho_2, \rho_3\rho_2\rho_3) =: (\sigma_1, \ldots, \sigma_4). \tag{7.4}$$

As before, we see that $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \Gamma^3(p, s; 3, 3)$; moreover, since $\langle \sigma_1, \sigma_2, \sigma_4 \rangle$ is the conjugate of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ by ρ_3, we have $\langle \sigma_1, \sigma_2, \sigma_4 \rangle \cong \Gamma^3(p, s; 3, 3)$ also. In a similar way, $\langle \sigma_1, \sigma_3, \sigma_4 \rangle \cong \Gamma^3(r, t; 3, 3) \cong \langle \sigma_2, \sigma_3, \sigma_4 \rangle$. Thus $W = \langle \sigma_1, \ldots, \sigma_4 \rangle$ has a presentation which includes the relations

$$\begin{cases} \sigma_i^2 = (\sigma_1\sigma_2)^p = (\sigma_1\sigma_3)^3 = (\sigma_1\sigma_4)^3 = (\sigma_2\sigma_3)^3 = (\sigma_2\sigma_4)^3 = (\sigma_3\sigma_4)^r = \varepsilon, \\ (\sigma_1\sigma_2\sigma_3\sigma_2)^s = (\sigma_1\sigma_2\sigma_4\sigma_2)^s = (\sigma_1\sigma_3\sigma_4\sigma_3)^t = (\sigma_2\sigma_3\sigma_4\sigma_3)^t = \varepsilon. \end{cases} \tag{7.5}$$

Again, we can recover the original group Γ by means of the two involutory twisting operations (group automorphisms) τ_1 $(= \rho_0)$ and τ_2 $(= \rho_3)$, which act on W by

$$\tau_1\sigma_1\tau_1 = \sigma_2, \qquad \tau_1\sigma_j\tau_1 = \sigma_j \quad (j = 3, 4),$$

and

$$\tau_2\sigma_3\tau_2 = \sigma_4, \qquad \tau_2\sigma_j\tau_2 = \sigma_j \quad (j = 1, 2).$$

Thus Γ is given by the operation

$$(\sigma_1, \ldots, \sigma_4; \tau_1, \tau_2) \mapsto (\tau_1, \sigma_2, \sigma_3, \tau_2) =: (\rho_0, \ldots, \rho_3). \tag{7.6}$$

Analogous arguments to those of Section 6 may now be pursued. In order for the original polytope $\mathcal{P}$ to be universal, W must satisfy only the relations of (7.5). We thus consider the abstract group $W = \langle\sigma_1, \ldots, \sigma_4\rangle$ corresponding to the tetrahedral diagram of Figure 7.2, with triangles $\{1, 2, 3\}$ and $\{1, 2, 4\}$ marked s and $\{1, 3, 4\}$ and $\{2, 3, 4\}$ marked t, and branches $\{1, 2\}$ and $\{3, 4\}$ marked p and r, respectively. Then W admits the two automorphisms which correspond to the permutations (1 2) and (3 4) of the nodes, respectively. The operation of (7.6) leads back to the group $\Gamma = \langle\rho_0, \ldots, \rho_3\rangle$, which is thus a semi-direct product of W by $C_2 \times C_2$.

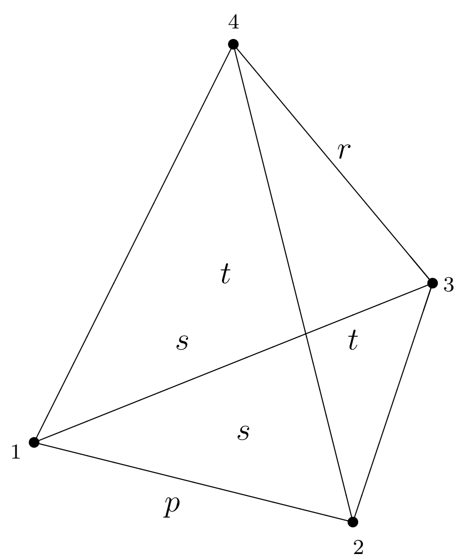


Figure 7.2:

Hence, if Γ is a C-group, then the polytope $\mathcal{P}$ exists and $\Gamma = \Gamma(\mathcal{P})$. As in the previous case, under the assumption that the parameters p, r, s, t are such that the groups $[1\,1\,1^p]^s$ and $[1\,1\,1^r]^t$ are finite, we can appeal to our results about reflection groups, notably Theorem 3.7, where W occurs as the group with diagram $\mathcal{S}_4(p, s; r, t)$. Then we have the following

Theorem 7.2 *Assume that each pair (p, s) and (r, t) is either of the form $(l, 3)$ or $(3, l)$ with $2 \leqslant l < \infty$, or one of $(4, 4)$, $(4, 5)$ or $(5, 4)$. Then the universal regular 4-polytope*

$\mathcal{P} := \{\{2p,3\}_{2s},\{3,2r\}_{2t}\}$ *exists. Its group is* $W \ltimes (C_2 \times C_2)$*, where* W *is the abstract group defined by the diagram in Figure 7.2. In particular,* $\mathcal{P}$ *is finite if and only if*

$$(p,s,r,t) = \begin{cases} (3,a,3,2),(3,2,3,a) & \text{with } a = 2,3,4, \text{ or} \\ (3,2,b,3),(b,3,3,2) & \text{with } b \geqslant 2. \end{cases} \tag{7.7}$$

In this case, $\Gamma(\mathcal{P}) = [1\,1\,2]^a \ltimes (C_2 \times C_2)$ *or* $[1\,1\,2^b]^3 \ltimes (C_2 \times C_2)$*, respectively.*

The theorem says that there are only two kinds of finite polytopes $\mathcal{P}$, namely the polytopes $\{\{2p,3\}_6,\{3,6\}_{(2,0)}\}$, with $p \geqslant 2$, and the finite polytopes ${}_6\mathcal{T}^4_{(s,0),(t,0)}$, obtained for $(p,s,r,t) = (3,a,3,2)$ or $(3,2,3,a)$ with $a = 2,3,4$. The first kind is related to the (simple) polytopes $\mathcal{Q} := \{\{2p,3\}_6,\{3,3\}\}$; in particular, $\Gamma(\mathcal{P}) = \Gamma(\mathcal{Q}) \times C_2$ (see Lemma 5.1).

The type $\{3,6,3\}$

For the type $\{3,6,3\}$, the enumeration of the finite univeral locally toroidal regular polytopes is not yet complete, and only partial results are known. The general method of the previous sections seems to fail except for certain parameter values. Here we describe some new results which complement those previously known. We write

$${}_7\mathcal{T}^4_{\mathbf{s},\mathbf{t}} := \{\{3,6\}_{\mathbf{s}},\{6,3\}_{\mathbf{t}}\}, \tag{7.8}$$

with $\mathbf{s} = (s^k, 0^{2-k})$, $\mathbf{t} = (t^l, 0^{2-l})$; the general assumptions on the parameters are as before: $s \geqslant 2$ if $k = 1$ and $s \geqslant 1$ if $k = 2$; $t \geqslant 2$ if $l = 1$ and $t \geqslant 1$ if $l = 2$. To set the scene, we list in Table 7.1 those universal locally toroidal polytopes of type $\{3,6,3\}$ which are known to be finite (see also [4, 18, 26, 37]); we shall only list one of each dual pair, that with fewer vertices.

$\mathbf{s}$	$\mathbf{t}$	v	f	g
$(1,1)$	$(1,1)$	3	3	108
$(1,1)$	$(3,0)$	3	9	324
$(2,0)$	$(2,0)$	5	5	240
$(2,0)$	$(2,2)$	5	15	720
$(3,0)$	$(3,0)$	27	27	2916
$(3,0)$	$(2,2)$	288	384	41472
$(3,0)$	$(4,0)$	1260	2240	241920

Table 7.1: The known finite polytopes ${}_7\mathcal{T}^4_{\mathbf{s},\mathbf{t}}$.

The new results will only concern the case $\mathbf{s} = (2,0)$. Initially, at least, let us consider a general polyhedron $\mathcal{K}$ with facets $\{6\}$, and ask whether it is suited to be a vertex-figure of a polytope of type $\{\{3,6\}_{(2,0)},\mathcal{K}\}$. Now $\mathcal{Q} := \{3,6\}_{(2,0)}$ has four vertices, and covers the tetrahedron $\{3,3\}$ twice. The natural identification $\mathcal{Q} \searrow \{3,3\}$ then forces an analogous identification of the hexagonal faces of $\mathcal{K}$ onto triangles. (This is even more obvious, if we think of realizing the vertex-figure $\mathcal{K}$, with vertices the other vertices of edges through an initial vertex.)

When $\mathcal{K} = \{6,3\}_{\mathbf{t}}$ (or even $\{6,3\}$), the only possible identification leads to $\mathcal{K} \searrow \{3,3\}$. We conclude

Lemma 7.3 *If $\mathcal{P} := \{\{3,6\}_{(2,0)}, \{6,3\}_{\mathbf{t}}\}$ exists, then the covering $\{6\} \searrow \{3\}$ of its section $\{6\}$ induces a covering $\{6,3\}_{\mathbf{t}} \searrow \{3,3\}$, and hence a covering $\mathcal{P} \searrow \{3,3,3\}$ which preserves vertices.*

Actually, Lemma 7.3 considerably understates what really happens. The census of Colbourn & Weiss [4] lists $\{\{3,6\}, \{6,3\}_{(2,0)}\}$ (with the tessellation $\{3,6\}$ as facet!) as a polytope with group order 720. Obviously, the facet here cannot in fact be infinite (the Schläfli symbol notation of [4] is slightly more general than ours). We explain this (in the dual formulation) as follows.

Lemma 7.4 *The imposition of the relation $(\rho_0\rho_1\rho_2)^4 = \varepsilon$ on the group $[3,6,3]$ (with standard generators) implies that $(\rho_1\rho_2\rho_1\rho_2\rho_3)^4 = \varepsilon$.*

In other words, $\{\{3,6\}_{(2,0)}, \{6,3\}\}$ collapses to $\{\{3,6\}_{(2,0)}, \{6,3\}_{(2,2)}\}$.

Proof Note first that

$$\varepsilon = (\rho_0\rho_1\rho_2)^4 = (\rho_0\rho_1\rho_0\rho_2\rho_1\rho_2)^2 = (\rho_1\rho_0\rho_1\rho_2\rho_1\rho_2)^2 \sim (\rho_0\rho_1\rho_2\rho_1\rho_2\rho_1)^2,$$

and hence

$$\rho_0 \leftrightharpoons \rho_1\rho_2\rho_1\rho_2\rho_1.$$

Using this fact and the other relations of the groups freely, it then follows that

$$\begin{aligned}
(\rho_1\rho_2\rho_1\rho_2\rho_3)^2 &= \rho_1\rho_2\rho_1\rho_2\rho_1 \cdot \rho_3\rho_2\rho_1\rho_2\rho_3 \\
&\sim \rho_1\rho_2\rho_1\rho_2\rho_1 \cdot \rho_0\rho_3\rho_2\rho_1\rho_2\rho_3\rho_0 \\
&= \rho_1\rho_2\rho_1\rho_2\rho_1 \cdot \rho_3\rho_2\rho_1\rho_0\rho_1\rho_2\rho_3 \\
&\sim \rho_3\rho_2\rho_1\rho_2\rho_3 \cdot \rho_1\rho_2\rho_1\rho_0\rho_1\rho_2\rho_1 \\
&= \rho_3\rho_2\rho_1\rho_2\rho_3\rho_2\rho_1 \cdot \rho_1\rho_2\rho_1\rho_2\rho_1\rho_0\rho_1\rho_2\rho_1 \\
&= \rho_3\rho_2\rho_1\rho_2\rho_3\rho_2\rho_1 \cdot \rho_0\rho_1\rho_2 \\
&\sim \rho_1\rho_2\rho_3\rho_2\rho_1\rho_0\rho_1\rho_2\rho_3\rho_2 \\
&= \rho_1\rho_3\rho_2\rho_3\rho_1\rho_0\rho_1\rho_3\rho_2\rho_3 \\
&\sim \rho_1\rho_2\rho_1\rho_0\rho_1\rho_2.
\end{aligned}$$

We see at once that $(\rho_1\rho_2\rho_1\rho_2\rho_3)^4 \sim (\rho_0\rho_1\rho_2\rho_1\rho_2\rho_1)^2 = \varepsilon$, which establishes our claim. □

As an immediate consequence, we have

Theorem 7.5 *The only universal regular 4-polytopes $\mathcal{P} := \{\{3,6\}_{(2,0)}, \{6,3\}_{\mathbf{t}}\}$ are those with $\mathbf{t} = (2,0)$ and group $S_5 \times C_2$, and $\mathbf{t} = (2,2)$ with group $S_5 \times S_3$.*

Proof Lemma 7.4 implies that $\mathcal{P}$ can only be a quotient of $\{\{3,6\}_{(2,0)}, \{6,3\}_{(2,2)}\}$; apart from this itself, the only possibility is $\mathbf{t} = (2,0)$ (note that the case $\mathbf{t} = (1,1)$ can be eliminated by the results of [18, p.123]). Because both polytopes have 5 vertices by Lemma 7.3, the group orders are easily calculated. From [18, Theorem 6] we know the group to be $S_5 \times S_3$ if $\mathbf{t} = (2,2)$. Finally, if $\mathbf{t} = (2,0)$, we can either determine the group as a quotient of $S_5 \times S_3$, or observe that the element $(\rho_1\rho_2)^3$ of $\Gamma(\mathcal{P})$ which defines the covering $\mathcal{P} \searrow \{3,3,3\}$ is central. $\square$

We would guess that, up to duality, there is just one further possibility for a finite polytope of type $\{3,6,3\}$ in addition to those of Table 7.1, namely $\{\{3,6\}_{(3,0)}, \{6,3\}_{(5,0)}\}$. The reason for this speculation is that, as we have seen in [18, §6], the regular polytope $\{\{3,6\}_{(3,0)}, \{6,3\}_{(3,3)}\}$ is infinite, but only just, in that the associated hermitian form is positive semi-definite. Since $\{6,3\}_{(3,3)}$ has 54 vertices, whereas $\{6,3\}_{(5,0)}$ has only 50, it seems plausible that the slightly smaller vertex-figure might yield a finite polytope. We could also look at this argument from a different point of view using cuts of polytopes (see [19, §5]).

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