The mix of a regular polytope with a face *

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Abstract

Mixing is an operation which yields subgroups generated by involutions of a larger group of the same kind. When it is applied to the product of the automorphism groups of two regular polytopes, one talks about the mix of the polytopes. This paper is concerned with conditions under which the mix of two regular polytopes is again a regular polytope. In the important special case of a mix of a polytope with one of its faces, fairly general results about the polytopality of the mix are obtained. In particular, the case when the mix is isomorphic to the original polytope is characterized.

1 Introduction

In recent years, the classical notion of a regular polytope has been generalized to abstract regular polytopes (see Coxeter [4], Grünbaum [8], Danzer-Schulte [6] and McMullen-Schulte [13]). These polytopes are combinatorial structures with a distinctive geometric and topological flavour, which resemble the classical regular polytopes.

In the modern abstract theory, the mixing of polytopes or groups are important operations. The technique is extremely versatile, and many constructions of regular polytopes which have appeared in the literature can be subsumed under its heading (for example, see [11, 12, 13]).

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In the present paper, we introduce the notion of a mix of two regular polytopes, and study conditions under which this mix is again a regular polytope. In the important special case of a mix of a polytope with one of its faces, we obtain some fairly general results about the polytopality of this mix. In particular, we characterize the case when the mix is isomorphic to the original polytope.

After reviewing basic notions in Section 2, we illustrate mixing operations and define the mix of two regular polytopes in Section 3. In Section 4, we then discuss properties of the mix of a regular polytope with a face. Finally, in Section 5, we end with some remarks about the connexions between mixing of abstract polytopes, and blending of realizations of abstract polytopes in euclidean spaces (see [9]).

2 Basic notions

We follow the terminology of [13] (and [12]). An (*abstract*) *polytope of rank* n, or simply an n-*polytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function whose range is $\{-1, 0, \ldots, n\}$. The elements of rank j are called the j-*faces* of $\mathcal{P}$, or *vertices*, *edges* and *facets* of $\mathcal{P}$ if $j = 0$, 1 or $n-1$, respectively. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ each contain exactly $n+2$ faces, including the unique minimal face F_{-1} and unique maximal face F_n of $\mathcal{P}$. Further, $\mathcal{P}$ is strongly flag-connected. Finally, if F and G are an $(j-1)$-face and an $(j+1)$-face with $F < G$, then there are exactly *two* j-faces H such that $F < H < G$.

An n-polytope $\mathcal{P}$ is *regular* if its (*automorphism*) *group* $\Gamma(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$ be a fixed or *base* flag of $\mathcal{P}$. The group $\Gamma(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \ldots, \rho_{n-1}$ (*with respect to* Φ), where ρ_j is the unique automorphism which keeps all but the j-face of Φ fixed. These generators satisfy relations

$$(\rho_i \rho_j)^{p_{ij}} = \varepsilon \quad (i, j = 0, \ldots, n-1), \tag{2.1}$$

with

$$p_{ii} = 1, \quad p_{ij} = p_{ji} \geqslant 2 \ \ (i \neq j), \quad p_{ij} = 2 \ \text{ if } |i-j| \geqslant 2. \tag{2.2}$$

The numbers $p_j := p_{j-1,j}$ $(j = 1, \ldots, n-1)$ determine the (*Schläfli*) *type* $\{p_1, \ldots, p_{n-1}\}$ of $\mathcal{P}$. Further, $\Gamma(\mathcal{P})$ has the *intersection property* (with respect to the distinguished generators), namely

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \ \text{ for all } I, J \subset \{0, \ldots, n-1\}. \tag{2.3}$$

By a (*string*) *C-group*, we mean a group which is generated by involutions such that (2.1), (2.2) and (2.3) hold. The group of a regular polytope is a C-group. Conversely, given a C-group, there is an associated regular polytope of which it is the automorphism group (see [13, Section 2E]). In verifying that a given group is a C-group, it is usually only the intersection property which causes difficulty. Note that Coxeter groups with string diagrams are examples of C-groups.

3 Mixing

The idea of a mixing operation is very general. Let Δ be a group generated by involutions, say $\Delta = \langle\sigma_0, \ldots, \sigma_{n-1}\rangle$; usually, but not necessarily, Δ will be a C-group. A *mixing operation* then derives a new group Γ from Δ, by taking as generators $\rho_0, \ldots, \rho_{m-1}$ for Γ certain suitably chosen products of the σ_i, so that Γ is a subgroup of Δ; it is denoted by

$$(\sigma_0, \ldots, \sigma_{n-1}) \mapsto (\rho_0, \ldots, \rho_{m-1}).$$

We naturally wish Γ to be a string C-group, but unfortunately there are few general circumstances which will guarantee this, and it is often the case that the intersection property has to be addressed directly.

What we shall do in this section is illustrate mixing operations by means of a number of examples, and introduce the notion of the mix of two regular polytopes.

In our initial example, Δ is the group of a regular n-polytope $\mathcal{P}$. The operation

$$(\sigma_0, \ldots, \sigma_{n-1}) \mapsto (\sigma_0\sigma_2\sigma_4\cdots, \sigma_1\sigma_3\sigma_5\cdots) =: (\rho_0, \rho_1),$$

which results in the two products of alternate generators of Δ, yields a dihedral group, which is that of the *Petrie polygon* of $\mathcal{P}$ (see [4, p.223] and [13, Section 6C]). Recall that, inductively, a Petrie polygon of an abstract n-polytope, is an edge-path such that any $n-1$ consecutive edges, but no n, belong to a Petrie polygon of a facet; here we begin the induction by declaring that the Petrie polygon of a polygon (2-polytope) is the polygon itself.

For the second example, let $\Delta = [4, 3^{n-2}]$ be the group of the regular n-cube $\{4, 3^{n-2}\}$ (see [4, p.123]). The n-th operation

$$(\sigma_0, \ldots, \sigma_{n-1}) \mapsto (\sigma_0, \sigma_1\sigma_3\sigma_5\cdots, \sigma_2\sigma_4\sigma_6\cdots) =: (\rho_0, \rho_1, \rho_2)$$

yields Coxeter's regular polyhedron $\{4, n \,|\, 4^{\lfloor \frac{n}{2} \rfloor - 1}\}$ (see [3, p.57] and [13, Section 7.6]). The polyhedron shares its 2^n vertices and $2^{n-1}n$ edges with the n-cube, and its $2^{n-2}n$ square faces occur among those of the cube; its vertex-figure is the Petrie polygon $\{n\}$ of the vertex-figure $\{3^{n-2}\}$ of the cube.

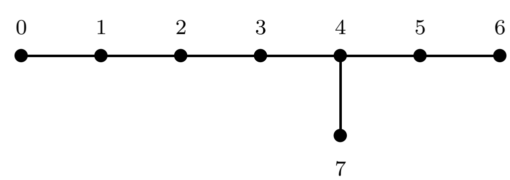

Figure 1: The Coxeter diagram of E_8.

Our next examples arise from the Coxeter group $\Delta = E_8 = [3^{4,2,1}]$, and certain of its subgroups (see [4, p.200]). Here, we have Δ generated by $\sigma_0, \ldots, \sigma_7$, with relations $(\sigma_i\sigma_j)^{p_{ij}} = \varepsilon$, where $p_{ii} = 1$ for $i = 0, \ldots, 7$, $p_{ij} = 3$ for $j = i + 1$ with $i = 0, \ldots, 6$ and

$\{i,j\} = \{4,7\}$, and $p_{ij} = 2$ otherwise. (See Figure 1 for the Coxeter diagram of E_8, in which a label j indicates the generator σ_j.) The mixing operation is

$$(\sigma_0, \ldots, \sigma_7) \mapsto (\sigma_0\sigma_6, \sigma_1\sigma_5, \sigma_2\sigma_4, \sigma_3\sigma_7) =: (\rho_0, \rho_1, \rho_2, \rho_3).$$

It may be verified (see Monson [14]) that $\langle \rho_0, \ldots, \rho_3 \rangle$ is isomorphic to the group $[3,3,5]$ of the regular 600-cell $\{3,3,5\}$ (see [4, p.153]); excluding $\rho_0 = \sigma_0\sigma_6$ exhibits how to obtain the group $[3,5]$ of the icosahedron $\{3,5\}$ from $B_6 = [3^{3,1,1}]$ by an analogous operation.

We next consider a construction which is often useful, but which unfortunately does not always yield a polytope. Let $m \leqslant n$, and let $\Gamma = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$ and $\Delta = \langle \tau_0, \ldots, \tau_{m-1} \rangle$ be string C-groups. If we define

$$\rho_j := (\sigma_j, \tau_j) \in \Gamma \times \Delta,$$

for $j = 0, \ldots, n-1$, with $\tau_j := \varepsilon$ for $j \geqslant m$, then the group

$$\Gamma \diamond \Delta := \langle \rho_0, \ldots, \rho_{n-1} \rangle \tag{3.1}$$

is called the *mix* of Γ and Δ. In [12, p.270], we introduced a more general construction in the context of amalgamations; in that terminology, the mix is actually the 0-mix. Of course, we can express the mix as a mixing operation

$$(\sigma_0, \ldots, \sigma_{n-1}, \tau_0, \ldots, \tau_{m-1}) \mapsto (\sigma_0\tau_0, \ldots, \sigma_{n-1}\tau_{n-1}) =: (\rho_0, \ldots, \rho_{n-1}).$$

This certainly yields a group satisfying the relations $(\rho_i\rho_j)^2 = \varepsilon$ for $0 \leqslant i < j-1 \leqslant n-2$, but it will not generally be a C-group.

We first give an example which illustrates the latter point. Let $\mathcal{P} := \{3,3,3\}$ be the 4-simplex, and let $\mathcal{Q} := \{\{5,3\}_5, \{3,5\}_5\}$ be the self-dual 4-polytope with 57 hemi-dodecahedral facets $\{5,3\}_5$ and 57 hemi-icosahedral vertex-figures $\{3,5\}_5$ described in [5]. Define $\Gamma := \Gamma(\mathcal{P})$ and $\Delta := \Gamma(\mathcal{Q})$. Now we have

$$(\sigma_0\sigma_1\sigma_2)^4 = \varepsilon = (\tau_0\tau_1\tau_2)^5.$$

Within Γ, we have

$$\begin{aligned} \sigma_2\sigma_1\sigma_0 \cdot \sigma_2\sigma_1 \cdot \sigma_0\sigma_1\sigma_2 &= \sigma_2\sigma_1\sigma_2 \cdot \sigma_0\sigma_1\sigma_0\sigma_1 \cdot \sigma_2 \\ &= \sigma_2\sigma_1\sigma_2\sigma_1 \cdot \sigma_0\sigma_2 \\ &= \sigma_1\sigma_2\sigma_0\sigma_2 \\ &= \sigma_1\sigma_0. \end{aligned}$$

Similarly,

$$\sigma_2\sigma_1\sigma_0 \cdot \sigma_0\sigma_1 \cdot \sigma_0\sigma_1\sigma_2 = \sigma_2\sigma_0\sigma_1\sigma_2 = \sigma_0\sigma_1\sigma_2\sigma_1.$$

Thus, if $\alpha := \sigma_0\sigma_1\sigma_2 = (\sigma_0\sigma_1\sigma_2)^5$, then

$$\begin{aligned} (\sigma_2\sigma_1)^{\alpha^2}(\sigma_1\sigma_2)^{\alpha} &= (\sigma_1\sigma_0)^{\alpha} \cdot (\sigma_1\sigma_0)^{-1} \\ &= \sigma_1\sigma_2\sigma_1\sigma_0 \cdot \sigma_0\sigma_1 \\ &= \sigma_1\sigma_2. \end{aligned}$$

However, within Δ, the analogous calculations with $\beta := (\tau_0\tau_1\tau_2)^5 = \varepsilon$ yield

$$(\tau_2\tau_1)^{\beta^2}(\tau_1\tau_2)^{\beta} = \tau_2\tau_1 \cdot \tau_1\tau_2 = \varepsilon.$$

Hence, if $\gamma := (\rho_0\rho_1\rho_2)^5$, then

$$(\rho_2\rho_1)^{\gamma^2}(\rho_1\rho_2)^{\gamma} = (\sigma_1\sigma_2, \varepsilon),$$

so that $(\sigma_1\sigma_2, \varepsilon) \in \langle\rho_0, \rho_1, \rho_2\rangle$. Analogous calculations (in effect, the same ones applied to the dual) show that $(\sigma_1\sigma_2, \varepsilon) \in \langle\rho_1, \rho_2, \rho_3\rangle$. But it is clear that $(\sigma_1\sigma_2, \varepsilon) \notin \langle\rho_1, \rho_2\rangle$, and hence we have the required violation of the intersection property in $\Gamma \diamond \Delta$.

Finally, we introduce the notion of the mix of two regular polytopes. When $\Gamma := \Gamma(\mathcal{P})$ and $\Delta := \Gamma(\mathcal{Q})$ for some regular polytopes $\mathcal{P}$ and $\mathcal{Q}$, then we define

$$\mathcal{P} \diamond \mathcal{Q} := \mathcal{P}(\Gamma \diamond \Delta), \tag{3.2}$$

whether or not $\Gamma \diamond \Delta$ is a C-group, and call it the *mix* of $\mathcal{P}$ and $\mathcal{Q}$. Here, $\mathcal{P}(\Gamma \diamond \Delta)$ denotes the poset (indeed, pre-polytope) determined by $\Gamma \diamond \Delta$ (see [13, Section 2E]). (A pre-polytope may not be strongly flag-connected, but the other defining properties are the same as for a polytope; see [13, Section 2D].) Clearly, $\mathcal{P} \diamond \mathcal{Q}$ is polytopal if and only if $\Gamma \diamond \Delta$ is a C-group. But, as we have just seen, the mix of two regular polytopes need not itself be a polytope.

4 The mix with a face

A particularly interesting case is that of a mix of a regular n-polytope with an m-face. The cases $m = 1$ and $m = n - 1$ are straightforward.

Theorem 4.1 *Let $\mathcal{P}$ be a regular n-polytope, and let $\mathcal{Q}$ be the m-face of $\mathcal{P}$. Then, at least in the cases $m = 1$ and $m = n - 1$, the mix $\mathcal{P} \diamond \mathcal{Q}$ is polytopal.*

Proof: The proof is easy. Using the notation for the groups employed above, we just consider the surjective homomorphism π of $\Lambda := \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ onto $\Gamma(\mathcal{P})$ induced by the mappings

$$\rho_j = (\sigma_j, \tau_j) \mapsto \sigma_j$$

for $j = 0, \ldots, n - 1$, and appeal to the quotient criterion of [13, Theorem 2E17]. Note that π is the restriction to Λ of the natural projection of the direct product $\Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$ onto its first factor $\Gamma(\mathcal{P})$. If $m = 1$, the homomorphism is one-to-one on the group of the vertex-figure, and if $m = n - 1$, then it is one-to-one on the group of the facet. By [13, Theorem 2E17], this proves that the mix is polytopal. □

We can say more about the case $m = 1$, where $\mathcal{Q}$ is a segment (1-polytope).

Theorem 4.2 *Let $\mathcal{P}$ be a regular n-polytope and $\mathcal{Q}$ a segment. Then $\Gamma(\mathcal{P} \diamond \mathcal{Q}) \cong \Gamma(\mathcal{P})$ if all edge-circuits of $\mathcal{P}$ are even; otherwise $\Gamma(\mathcal{P} \diamond \mathcal{Q}) \cong \Gamma(\mathcal{P}) \times C_2$.*

Proof: Using the notation of Theorem 4.1, we write each relation in $\Gamma(\mathcal{P})$ in the form

$$\sigma_0\alpha_1\sigma_0\alpha_2\cdots\sigma_0\alpha_k = \varepsilon,$$

with $\alpha_1, \ldots, \alpha_k \in \langle\sigma_1, \ldots, \sigma_{n-1}\rangle$, the group of the vertex-figure of $\mathcal{P}$. It is shown in [11, Theorem 2.5] (see also [13, Theorem 2F4]) that this relation corresponds to an edge-circuit in $\mathcal{P}$ of length k. Since $\rho_i = (\sigma_i, \varepsilon)$ for $i \geqslant 1$, we have

$$\rho_0\beta_1\rho_0\beta_2\cdots\rho_0\beta_k = (\sigma_0\alpha_1\sigma_0\alpha_2\cdots\sigma_0\alpha_k, \tau_0^k) = (\varepsilon, \tau_0^k)$$

in $\Lambda = \Gamma(\mathcal{P} \diamond \mathcal{Q})$, with $\beta_i := (\alpha_i, \varepsilon)$. Clearly, this is the identity in Λ if and only if k is even. It follows at once that $\Lambda \cong \Gamma(\mathcal{P})$ if and only if all edge-circuits of $\mathcal{P}$ are even; otherwise $(\varepsilon, \tau) \in \Lambda$, and $\Lambda \cong \Gamma(\mathcal{P}) \times C_2$. □

The result of Theorem 4.1 no longer holds in full generality when $1 < m < n-1$. Nevertheless, it is still possible to obtain some fairly general results about polytopality of the mix in this case.

We first discuss the case when the mix is isomorphic to the original polytope. Recall from [12, p.266] (or [13, Section 4E]) that a regular n-polytope $\mathcal{P}$, with group $\Gamma(\mathcal{P}) = \langle\sigma_0, \ldots, \sigma_{n-1}\rangle$, has the FAP (flat amalgamation property) with respect to its m-faces if

$$\Gamma(\mathcal{P}) \cong N_m^+ \ltimes \langle\sigma_0, \ldots, \sigma_{m-1}\rangle,$$

where $N_m^+ := \langle\varphi^{-1}\sigma_i\varphi \mid i \geqslant m,\, \varphi \in \Gamma(\mathcal{P})\rangle$. Then we have

Theorem 4.3 *Let $\mathcal{P}$ be a regular n-polytope, and let $\mathcal{Q}$ be the m-face of $\mathcal{P}$. Then $\mathcal{P} \diamond \mathcal{Q}$ is isomorphic to $\mathcal{P}$ if and only if $\mathcal{P}$ has the FAP with respect to its m-faces.*

Proof: Recall that two regular n-polytopes are isomorphic if and only if there is an isomorphism between the groups mapping the distinguished generators of the first group onto those of the second.

Consider again the surjective homomorphism $\pi\colon \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}) =: \Lambda \to \Gamma(\mathcal{P})$, which is the restriction to Λ of the natural projection of $\Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$ onto $\Gamma(\mathcal{P})$. We now investigate when π is an isomorphism.

Let $\rho := \rho_{i_1}\rho_{i_2}\cdots\rho_{i_k} = (\varphi, \tau)$ be an element in Λ, where $\varphi := \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ and $\tau := \tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}$. Now consider φ modulo N_m^+. First note that a generator σ_j of $\Gamma(\mathcal{P})$ either belongs to $\langle\sigma_0, \ldots, \sigma_{m-1}\rangle$ if $j \leqslant m-1$, or to N_m^+ if $j \geqslant m$. Then we have

$$\varphi N_m^+ = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k} N_m^+ = \textstyle\prod_l \sigma_{i_l} N_m^+,$$

where the product is taken over all l such that $i_l \leqslant m-1$. Since $\tau_j = \varepsilon$ if $j \geqslant m$, we also have $\tau = \prod_l \tau_{i_l}$, where the product is taken over the same indices l.

Now suppose that $\mathcal{P}$ has the FAP with respect to its m-faces, so that $\Gamma(\mathcal{P})$ is the semi-direct product of N_m^+ by $\langle\sigma_0, \ldots, \sigma_{m-1}\rangle$. If ρ is in the kernel of π, then $\varphi = \varepsilon$, and hence $\prod_l \sigma_{i_l} = \varepsilon$, since the latter is the component of φ in $\langle\sigma_0, \ldots, \sigma_{m-1}\rangle$. But then also $\tau = \prod_l \tau_{i_l} = \varepsilon$, since this product represents the same element in $\Gamma(\mathcal{Q})$ as the former product. It follows that $\rho = \varepsilon$, and hence π is an isomorphism.

Conversely, let π be an isomorphism. Then its kernel is trivial; that is, if $(\varepsilon, \tau) \in \Lambda$, then $\tau = \varepsilon$. To prove the FAP we show that the trivial element in $\Gamma(\mathcal{P})$ admits only the trivial decomposition as a product of an element in $\langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ with an element in N_m^+. Let $\varepsilon = \varphi_1\varphi_2$, where $\varphi_1 \in \langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ and $\varphi_2 \in N_m^+$. Write φ_1 as a product of generators σ_j with $j \leqslant m-1$, and φ_2 as a product of conjugates of generators σ_j with $j \geqslant m$ by elements in $\langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ (every element of $\Gamma(\mathcal{P})$ has a decomposition of this kind; see [12, Lemma 3.6] or [13, Lemma 4E7]). Let $\varphi_1 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$ and $\varphi_2 = \sigma_{i_{r+1}}\sigma_{i_{r+2}}\cdots\sigma_{i_k}$ (say), and consider the element $\rho := \rho_{i_1}\rho_{i_2}\cdots\rho_{i_k} = (\varphi, \tau)$ in Λ. Then $\varphi = \varphi_1\varphi_2 = \varepsilon$, and hence also $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k} = \tau = \varepsilon$, because π is an isomorphism. On the other hand, we also have $\tau_{i_{r+1}}\tau_{i_{r+2}}\cdots\tau_{i_k} = \varepsilon$; indeed, if we replace each generator σ_{i_s} in φ_2 by τ_{i_s}, and use the fact that $\tau_j = \varepsilon$ if $j \geqslant m$, then we arrive at the trivial element. But then also $\tau_{i_1}\cdots\tau_{i_r} = \varepsilon$, and hence $\sigma_{i_1}\cdots\sigma_{i_r} = \varepsilon$, because both words in the generators represent the same element in $\Gamma(\mathcal{Q})$. It follows that $\varphi_1 = \varepsilon$, and hence also $\varphi_2 = \varepsilon$. This proves the FAP. □

The sufficiency of the FAP in Theorem 4.3 can also be obtained by applying [12, Theorem 4.3] with $\mathcal{P}_2 := \mathcal{P}$, $\mathcal{K} := \mathcal{Q}$, and $\mathcal{P}_1 := \left(2^{\mathcal{K}^*}\right)^*$, the dual of the power complex $2^{\mathcal{K}^*}$ of the dual $\mathcal{K}^*$ of $\mathcal{K}$ (see [7] and [13, Sections 8C, 8D]). Then $\mathcal{P}$ is the vertex-figure of the polytope in [12, Theorem 4.3], and inspection of the proof shows that this is just the mix $\mathcal{P} \diamond \mathcal{Q}$. Hence, if the FAP holds, then the mix is isomorphic to the original polytope.

We next discuss conditions which guarantee polytopality of the mix. We begin with the following

Lemma 4.4 *Let $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$. Let $\mathcal{P}$ be a regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, and let $\mathcal{Q}$ be the m-face of $\mathcal{P}$. For $0 \leqslant i \leqslant m$ define $\Gamma_{i,m} := \langle \sigma_i, \ldots, \sigma_m \rangle$, where again $\Gamma(\mathcal{P}) = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$. If each group $\Gamma_{i,m}$ with $1 \leqslant i \leqslant m$ is generated by the conjugates of σ_m in $\Gamma_{i,m}$, then the mix $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. If the condition also holds for $i = 0$, then $\Gamma(\mathcal{P} \diamond \mathcal{Q}) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$.*

Proof: We first prove the polytopality statement by induction on n, assuming that the condition on $\Gamma_{i,m}$ holds for $i = 1, \ldots, m$. Then, if $m = 1$ or $m = n-1$, we can directly appeal to Theorem 4.1. In particular, this settles the case $n = 3$.

Now let $n \geqslant 4$ and $1 < m < n-1$. Then we can make the obvious inductive assumption that the facet and vertex-figure of the mix $\mathcal{P} \diamond \mathcal{Q}$ are polytopal. Indeed, the condition on the groups $\Gamma_{i,m}$ is hereditary, and thus holds for the facet and vertex-figure of $\mathcal{P}$, so that the inductive assumption for rank $n-1$ yields polytopality of the facet and vertex-figure of the mix.

We now verify a restricted intersection property for the group $\Lambda := \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ of the mix. Set $\Lambda_{n-1} := \langle \rho_0, \ldots, \rho_{n-2} \rangle$, $\Lambda_0 := \langle \rho_1, \ldots, \rho_{n-1} \rangle$, and $\Lambda_{0,n-1} := \langle \rho_1, \ldots, \rho_{n-2} \rangle$, where again $\rho_j := (\sigma_j, \tau_j)$, $\Gamma(\mathcal{Q}) = \langle \tau_0, \ldots, \tau_{m-1} \rangle$, and $\tau_j := \varepsilon$ for $j \geqslant m$. Then we know that Λ is a C-group if $\Lambda_{n-1} \cap \Lambda_0 = \Lambda_{0,n-1}$ (see [13, Proposition 2E16]).

Now, from the definition of the mix, it is clear that the two groups Λ_{n-1} and Λ_0 are contained in the direct products $\langle \sigma_0, \ldots, \sigma_{n-2} \rangle \times \langle \tau_0, \ldots, \tau_{m-1} \rangle$ and $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle \times \langle \tau_1, \ldots, \tau_{m-1} \rangle$, respectively, and so $\Lambda_{n-1} \cap \Lambda_0$ is a subgroup of $\langle \sigma_1, \ldots, \sigma_{n-2} \rangle \times \langle \tau_1, \ldots, \tau_{m-1} \rangle$, by the intersection property of $\Gamma(\mathcal{P})$. We now use our hypothesis on $\Gamma_{i,m}$, with $i = 1$, to conclude that $\Gamma_{1,m} \times \langle \varepsilon \rangle$ is a subgroup of $\Lambda_{0,n-1}$. First, note that $\Lambda_{0,n-1}$ contains

$\rho_m = (\sigma_m, \varepsilon)$ and all its conjugates by elements in $\langle \rho_1, \dots, \rho_{m-1} \rangle$, and that each such conjugate is of the form (φ, ε) with $\varphi \in \Gamma_{1,m}$. But by assumption $\Gamma_{1,m}$ is generated by the conjugates of σ_m in $\Gamma_{1,m}$, and so we can achieve every $\varphi \in \Gamma_{1,m}$.

We now know that $\Lambda_{0,n-1}$ contains all the elements (σ_i, ε) for $i = 1, \dots, n-2$, and hence also (ε, τ_i) for $i = 1, \dots, m-1$, since the latter is the product with ρ_i. It follows that $\Lambda_{0,n-1} = \langle \sigma_1, \dots, \sigma_{n-2} \rangle \times \langle \tau_1, \dots, \tau_{m-1} \rangle$. But then $\Lambda_{n-1} \cap \Lambda_0$ is clearly a subgroup of $\Lambda_{0,n-1}$, because it is contained in the direct product. Since the opposite inclusion is trivial, this now proves that the mix is polytopal.

Now assume that the condition on $\Gamma_{i,m}$ also holds for $i = 0$. In the above we can then replace $\Lambda_{0,n-1}$ by the group Λ_{n-1} of the facet and conclude similarly that it must contain $\Gamma_{0,m} \times \langle \varepsilon \rangle$ as a subgroup. But then Λ contains all the elements (σ_i, ε) for $i = 0, \dots, n-1$, and hence also (ε, τ_i) for $i = 0, \dots, m-1$. It follows that $\Lambda = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$. □

Theorem 4.5 *Let $n \geqslant 4$, let $\mathcal{P}$ be a regular n-polytope of type $\{p_1, \dots, p_{n-1}\}$, and let $\mathcal{Q}$ be the m-face of $\mathcal{P}$. If $p_2, \dots, p_m$ are odd, then the mix $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. If p_1 is also odd, then $\Gamma(\mathcal{P} \diamond \mathcal{Q}) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$.*

Proof: We apply Lemma 4.4, and use the simple observation that the distinguished generators of a dihedral group D_p are conjugate in D_p if p is odd. Then, if $p_2, \dots, p_m$ are odd, the two generators of $\langle \sigma_{j-1}, \sigma_j \rangle$ $(= D_{p_j})$ are conjugate in $\langle \sigma_{j-1}, \sigma_j \rangle$ for $j = 2, \dots, m$, and hence any two consecutive generators of $\Gamma_{i,m} = \langle \sigma_i, \dots, \sigma_m \rangle$ are conjugate in $\Gamma_{i,m}$ for $1 \leqslant i \leqslant m$. But then each generator of $\Gamma_{i,m}$ is conjugate to σ_m, and so $\Gamma_{i,m}$ is generated by the conjugates of σ_m in $\Gamma_{i,m}$. By Lemma 4.4, this proves polytopality of the mix.

If p_1 is odd as well, then we can apply the same argument to the larger group $\Gamma_{0,m} := \langle \sigma_0, \dots, \sigma_m \rangle$, to show that it is also generated by the conjugates of σ_m. Then appealing again to Lemma 4.4 yields $\Gamma(\mathcal{P} \diamond \mathcal{Q}) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$. □

Before we give an example of a polytope whose mix with a face is not polytopal, we mention the following interesting special case. (This also admits a generalization to higher ranks m).

Theorem 4.6 *Let $\mathcal{P}$ be a regular n-polytope of type $\{p_1, \dots, p_{n-1}\}$, and let p_2 be odd if $n \geqslant 4$. Let r be a multiple of p_1 (including $r = \infty$). Then the mix $\mathcal{P} \diamond \{r\}$ is polytopal.*

Proof: We appeal to the quotient criterion. There is a natural covering

$$\mathcal{P} \diamond \{r\} \searrow \mathcal{P} \diamond \{p_1\}.$$

The latter is polytopal by Theorem 4.1 (with $n = 3$) or Theorem 4.5 (with $m = 2$), and the covering mapping is one-to-one on the vertex-figure, which is the mix of the vertex-figure of $\mathcal{P}$ with a segment; see Theorem 4.1. Hence $\mathcal{P} \diamond \{r\}$ is polytopal. □

We now describe a regular 4-polytope $\mathcal{P}$ whose mix with a 2-face is not polytopal. Then the entry p_2 in its Schläfli symbol must necessarily be even. (Note that, when realized as a suitable section, this polytope will also give examples with $n > 4$ and $m > 2$.)

Indeed, if $\mathcal{P}$ is a regular 4-polytope with facets $\{3,6\}_{(s,0)}$ and vertex-figures $\{6,3\}_{(t,0)}$ with s, t odd, then the intersection property fails and thus the mix $\mathcal{P} \diamond \{3\}$ is not polytopal.

Examples of such polytopes $\mathcal{P}$ are obtained by the methods of [10, Section 6] and [13, Theorem 11H7] (see also Colbourn-Weiss [2] and Weiss [15]). The defining relations for facets and vertex-figures now yield

$$(\rho_1\rho_2\rho_3\rho_2)^t = ((\sigma_1\sigma_2\sigma_3\sigma_2)^t, \tau_1^t) = (\varepsilon, \tau_1)$$

and

$$(\rho_0\rho_1\rho_2\rho_1)^s = ((\sigma_0\sigma_1\sigma_2\sigma_1)^s, \tau_0^s) = (\varepsilon, \tau_0),$$

and hence also

$$\rho_0\rho_1(\rho_0\rho_1\rho_2\rho_1)^s\rho_1\rho_0 = (\varepsilon, \tau_0\tau_1\tau_0\tau_1\tau_0) = (\varepsilon, \tau_1).$$

But then $\langle\rho_0, \rho_1, \rho_2\rangle = \langle\sigma_0, \sigma_1, \sigma_2\rangle \times \langle\tau_0, \tau_1\rangle$ and $\langle\rho_1, \rho_2, \rho_3\rangle = \langle\sigma_1, \sigma_2, \sigma_3\rangle \times \langle\tau_1\rangle$, and these groups intersect in $\langle\sigma_1, \sigma_2\rangle \times \langle\tau_0\rangle (= D_6 \times C_2)$. On the other hand, the latter group is twice as large as the dihedral subgroup $\langle\rho_1, \rho_2\rangle (= D_6)$ of the mix. It follows that the mix cannot be polytopal.

Let us also observe that more general mixes of the kind discussed in [12, p.270] need not be polytopal either. A trivial example is provided by $\{3, 3\} \diamond_1 \{\,\}$. Let τ_1 be an involution, set $\tau_j = \varepsilon$ for $j = 0$ and 2, and define $\rho_j := (\sigma_j, \tau_j)$ for $j = 0, 1, 2$. Then

$$(\varepsilon, \tau_1) = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 \in \langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle,$$

but clearly

$$(\varepsilon, \tau_1) \notin \langle\rho_1\rangle.$$

Hence we do not obtain a polytope.

5 Mixing and blending

It is appropriate to end with some remarks about the connexions between mixing of abstract polytopes, and blending of euclidean realizations of abstract polytopes. The point of view in [9] and [13, Chapter 5] was "internal"; that is, we looked at a realization of a regular polytope, and asked whether it was a blend in a non-trivial way, or, in other words, whether its symmetry group was reducible. The subject of this paper, however, is mixing, which may be regarded as an "external" construction. In a sense, blending and mixing are the geometric and combinatorial sides of the same coin; nevertheless, there are important distinctions between the two concepts. It should particularly be borne in mind that mixing is a combinatorial operation, which applies to different regular polytopes, whereas blending, on the other hand, is a geometric operation, which applies to different realizations of the same regular polytope.

As a simple example of this distinction, if $p \geqslant 3$ is an odd integer, then (combinatorially) $\{2p\} = \{p\} \diamond \{\,\}$. This mix, which is not isomorphic to either component, has pure faithful realizations, as well as blended ones (see the description of the realization space of a polygon in [9] and [13, Section 5B]).

We saw above that the mix of two polytopes (even of the same rank) need not be polytopal. However, the example we gave, namely

$$\{3, 3, 3\} \diamond \{\{5, 3\}_5, \{3, 5\}_5\},$$

can be "realized" faithfully in $\mathbb{E}^{60}$, as follows. Take the simplex realizations of the two component polytopes in $\mathbb{E}^4$ and $\mathbb{E}^{56}$, respectively, with corresponding groups $\langle S_0, \ldots, S_3 \rangle$ and $\langle T_0, \ldots, T_3 \rangle$ (thus S_j corresponds to σ_j and T_j to τ_j in the notation used earlier). By this, we mean that we realize the polytopes in such a way that their vertex-sets are those of the corresponding regular simplices with the same numbers of vertices. We then copy the construction of the blend; that is, for $j = 0, \ldots, 3$, we set

$$R_j := S_j \times T_j \subseteq \mathbb{E}^4 \times \mathbb{E}^{56} = \mathbb{E}^{60},$$

and apply Wythoff's construction to (u, v), with u, v the initial vertices of the two polytopes. Since both $\{3, 3, 3\}$ and $\{\{5, 3\}_5, \{3, 5\}_3\}$ are covered by the (infinite) universal polytope $\{15, 3, 15\}$, this construction will yield a realization of the latter which is not itself polytopal (that is, its symmetry group is not a C-group).

In [13, Section 7E], we actually have the following situation. There will be blended realizations whose symmetry groups are C-groups, and the groups of the components will also be C-groups, which are often non-isomorphic homomorphic images of the original. The natural identifications induced by the projection mappings will then give coverings (again not usually isomorphisms) of polytopal components by the original realization. In this case, the realization is then also the mix of these components—certainly in the combinatorial sense that this holds of their symmetry groups. In such circumstances, it is, perhaps, pardonable to confuse mixing and blending, although one should try to avoid doing so. Our attitude is, as we said above, that blending is a geometric operation, while mixing is combinatorial.

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