Groups of type $L_2(q)$ acting on polytopes

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Abstract

We prove that if G is a string C-group of rank 4 and $G \cong L_2(q)$ with q a prime power, then q must be 11 or 19. The polytopes arising are Grünbaum's 11-cell of type $\{3,5,3\}$ for $L_2(11)$ and Coxeter's 57-cell of type $\{5,3,5\}$ for $L_2(19)$, each a locally projective regular 4-polytope.

1 Introduction

In this paper we determine the projective linear groups $L_2(q)$, q a prime power, which occur as full automorphism groups of abstract regular polytopes of rank 4 (or higher).

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For groups of type $L_2(q)$ there is a wealth of interesting constructions of regular polyhedra or maps on surfaces, but it is often assumed that q is a prime (see, for example, McMullen [16] or McMullen, Monson & Weiss [18]). There is also a considerable body of work available on representing $L_2(q)$ as a group of orientation preserving mappings on surfaces, usually as the rotation (even) subgroup of the automorphism group of a regular or chiral polyhedron or map (see, for example, Glover & Sjerve [7] or Conder [2], as well as Section 6). Sjerve and Cherkassoff [22] determined precisely when a group $L_2(q)$ is the full automorphism group of a regular polyhedron, thereby settling the existence problem when the rank is 3. In fact, the group $L_2(q)$ may be generated by three involutions, two of which commute, if and only if $q \neq 2, 3, 7$ or 9. In other words, $L_2(q)$ is the automorphism group of a regular polyhedron if and only if $q \neq 2, 3, 7$ or 9. In [22] it is also shown that the group $PGL_2(q)$ may be generated by three involutions, two of which commute, for any $q > 2$.

Regular polytopes of rank 5 or higher cannot have an automorphism group of type $L_2(q)$ (see Leemans & Vauthier [15]). This, then, leaves the case of rank 4. The purpose of this article is to prove the following theorem, which was conjectured in [15] and verified for $q \leq 61$.

Theorem 1 *If $L_2(q)$ is the automorphism group of a regular polytope of rank 4, then $q = 11$ or 19.*

In particular, it is found that there is just one regular polytope in each case. The 11-*cell* with group $L_2(11)$ was constructed by Grünbaum in [8] by pasting eleven hemi-icosahedra together (see also [4]), and the 57-*cell* with group $L_2(19)$ was discovered by Coxeter in [3]. For more details see Section 5.

2 Preliminaries

Abstract regular polytopes, string C-groups, and thin regular residually connected geometries with a linear diagram are basically the same mathematical objects. The link between these objects has been discussed in, for instance, McMullen & Schulte [17]. Here we take the viewpoint of string C-groups because it is the easiest and most efficient one to describe abstract regular polytopes.

As defined in [17], a *C-group* is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the following property, called the *intersection property*:

$$\forall J, K \subseteq \{0, \ldots, n-1\}, \ \ \langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle.$$

We call n the *rank* of G.

A C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\})$, or simply G, is a *string* C-group if its generators satisfy the following relations:

$$(\rho_j \rho_k)^2 = 1 \ \ \forall j, k \in \{0, \ldots n-1\} \text{ with } \mid j-k \mid \geq 2.$$

Each string C-group G then determines (uniquely) a regular n-polytope $\mathcal{P}$ with automorphism group G. The i*-faces* of $\mathcal{P}$ are the right cosets of the distinguished subgroup $G_i := \langle \rho_j \mid j \neq i \rangle$ for each $i = 0, 1, \ldots, n-1$, and two faces are incident just when they intersect as cosets. Formally, we also adjoin two copies of G itself, as the (unique) (-1)- and n-faces of $\mathcal{P}$. Conversely, the automorphism group of a regular n-polytope is a C-group, whose generators ρ_j map a fixed, or *base*, flag Φ of $\mathcal{P}$ to the j-adjacent flag Φ^j (differing from Φ in the j-face).

Recall from [17] that $\{\mathcal{P}_1, \mathcal{P}_2\}$ denotes the universal regular polytope (if it exists) with facets $\mathcal{P}_1$ and vertex-figures $\mathcal{P}_2$. This covers every regular polytope with facets $\mathcal{P}_1$ and vertex-figures $\mathcal{P}_2$.

3 The subgroup structure of $L_2(q)$

We frequently require properties of the subgroup lattice of $L_2(q)$. The subgroup structure of $L_2(q)$ may be found in Dickson [6] (and was first obtained in papers by Moore and Wiman). See also Huppert [12] for a weaker version. The theorem reproduced here was obtained by Patricia Vanden Cruyce in her PhD Thesis [24]; we mention her results on the sizes of conjugacy classes of subgroups only for those subgroups employed in the proof of our main theorem.

Theorem 2 *The group $L_2(q)$ of order $\frac{q(q^2-1)}{(2,q-1)}$, where $q = p^r$ with p a prime and r a positive integer, contains only:*

1. *elementary abelian subgroups of order q, denoted by E_q.*
2. *cyclic subgroups of order d, denoted by d, for all divisors d of $\frac{(q\pm1)}{(2,q-1)}$.*

3. $\frac{q(q^2-1)}{2d(2,q-1)}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for all divisors* d *of* $\frac{(q\pm1)}{(2,q-1)}$ *with* $d > 2$*. The number of conjugacy classes of these subgroups is* 1 *if* $\frac{(q\pm1)}{d(2,q-1)}$ *is odd, and* 2 *if it is even.*

4. *For* q *odd,* $\frac{q(q^2-1)}{12(2,q-1)}$ *dihedral groups of order 4 (Klein 4-groups), denoted by* 2^2*. The number of conjugacy classes of these groups is* 1 *if* $q \equiv \pm3(8)$ *and* 2 *if* $q \equiv \pm1(8)$*. For* q *even, the groups* 2^2 *are listed under family 5.*

5. *elementary abelian subgroups of order* p^s*, denoted by* E_{p^s}*, for all natural number* s *such that* $1 \leq s \leq r-1$*.*

6. *subgroups* $E_{p^s} : h$*, each a semidirect product of an elementary abelian group* E_{p^s} *and a cyclic group of order* h*, for all natural numbers* s *such that* $1 \leq s \leq r$ *and all divisors* h *of* $\frac{p^k-1}{(2,1,1)}$*, where* $k = (r,s)$ *and* $(2,1,1)$ *is defined as 2, 1 or 1 according as* $p > 2$ *and* $\frac{r}{k}$ *is even,* $p > 2$ *and* $\frac{r}{k}$ *is odd, or* $p = 2$*.*

7. *For* q *odd or* $q = 4^m$*, alternating groups* A_4*, of order 12.*

8. *For* $q \equiv \pm1(8)$*, symmetric groups* S_4*, of order 24.*

9. *For* $q \equiv \pm1(5)$ *or* $q = 4^m$*, alternating groups* A_5*, of order 60. For* $q \equiv 0(5)$*, the groups* A_5 *are listed under family 10.*

10. $\frac{q(q^2-1)}{p^w(p^{2w}-1)}$ *groups* $L_2(p^w)$*, for all divisors* w *of* r*. The number of conjugacy classes of these groups is* 2*,* 1 *or* 1 *according as* $p > 2$ *and* $\frac{r}{w}$ *is even,* $p > 2$ *and* $\frac{r}{w}$ *is odd, or* $p = 2$*.*

11. *groups* $PGL_2(p^w)$*, for all* w *such that* $2w$ *is a divisor of* r*.*

Observe that when q is even, family 11 of Theorem 2 is a subfamily of family 10.

The following Lemma is also used very often in the proof of our result.

Lemma 1 *([12], Sätze 8.3 and 8.4) Let* $G = L_2(q)$ *with* $q = p^r$*,* p *a prime and* r *a positive integer. Let* M *be a cyclic subgroup of order* m *in* G*. If* M *is a subgroup of a cyclic subgroup* $H < G$ *of order* $h = \frac{q\pm1}{(q-1,2)}$*, then* $N_G(M) = N_G(H) \cong D_{2h}$ *and therefore* $N_G(M)$ *is a maximal subgroup of* G*.*

4 $L_2(q)$ acting flag-transitively

In this section, we assume that G is a group isomorphic to $L_2(q)$, with $q = p^r$, p a prime and r a positive integer. Moreover, we assume that $(G, \{\rho_0, \ldots, \rho_3\})$ is a string C-group of type $\{t, l, s\}$ (i.e. the orders of $\rho_0\rho_1$, $\rho_1\rho_2$ and $\rho_2\rho_3$ are t, l and s, respectively). Clearly, $t, l, s \geq 3$, since G is not a direct product of two non-trivial groups. As before we set

$$G_i = \langle \rho_j \mid j \in \{0, \ldots, 3\} \backslash \{i\} \rangle,$$

for $i = 0, \ldots, 3$. Our aim is to prove that q must be 11 or 19.

We say that a subgroup H of G is an (irreducible) *rank* 3 *subgroup* of G if H is a rank 3 string C-group with a connected Coxeter diagram.

Observe that rank 3 subgroups must, by definition, contain dihedral subgroups of order at least 6.

The next lemma is folklore but we provide a proof for completeness.

Lemma 2 *The dihedral group D_{2n} of order $2n$ is not a rank 3 group.*

Proof : Suppose D_{2n} is a rank 3 group with generators $\alpha_0, \alpha_1, \alpha_2$ (say). Then $\alpha_0\alpha_2$ is an involution contained in the cyclic subgroup C_{2n} of D_{2n} and hence is the central element in D_{2n}. Let $C_{2n} = \langle \beta \rangle$. Then $\alpha_0\alpha_2 = \beta^n$, $\alpha_0\alpha_1 = \beta^k$ for some k (with $1 < k < n$), and $\alpha_1\alpha_2 = \beta^{n-k}$. Hence

$$\beta^{2k} = (\alpha_0\alpha_1)^2 \in \langle \alpha_0, \alpha_1 \rangle$$

and

$$\beta^{2k} = \beta^{-2(n-k)} = (\alpha_1\alpha_2)^{-2} \in \langle \alpha_1, \alpha_2 \rangle,$$

but $\beta^{2k} \notin \langle \alpha_1 \rangle$. This contradicts the intersection property for C-groups. □

Note that it is the intersection property that makes dihedral groups fail to be an irreducible rank 3 string C-group. On the other hand, a dihedral group can be the automorphism group of a regular map on a surface; however, these maps are not polyhedra in our sense (see [5, p.139]).

By Theorem 2, the observation, and the previous lemma, the rank 3 subgroups of G are isomorphic to S_4, A_5, or $L_2(q')$ or $PGL_2(q')$ for some q'. These are the only possible types of subgroups for G_0 and G_3.

We begin with a sequence of lemmas aimed at eliminating $L_2(q')$ and $PGL_2(q')$ as possibilities.

By inspection of the list in Theorem 2, we have:

Lemma 3 *The centre of a nonabelian subgroup of G has size at most 2. Moreover, if G has a nonabelian subgroup H whose centre has size 2, then q is odd and H must be a dihedral group.*

Lemma 4 *The prime p must be odd.*

Proof: The subgroup $G_2 = \langle \rho_0, \rho_1, \rho_3 \rangle$ is a subgroup of the form $2 \times D_{2t}$, with t as above. This subgroup is nonabelian and has a nontrivial centre. Hence by Lemma 3, it must be contained in a maximal subgroup of dihedral type of G and q must be odd. $\square$

From now on we may assume that p is *odd*.

Lemma 5 *The orders t of $\rho_0\rho_1$ and s of $\rho_2\rho_3$ must be odd.*

Proof: Let us prove, without loss of generality, that t must be odd. Suppose t is even. Then $G_2 \cong 2^2 \times D_t$ is nonabelian and $\mid Z(G_2) \mid > 2$, a contradiction with Lemma 3. Therefore, t (and s) must be odd. $\square$

Lemma 6 *Let H and K be two subgroups of type $L_2(q')$ in $L_2(q)$, with ${q'}^m = q$ for some positive integer m. Then $H \cap K$ cannot be a dihedral group D_{2k} with $k > 2$ (and k a divisor of $\frac{(q'\pm1)}{2}$).*

Proof: Let $k > 2$, and let k be a divisor of $\frac{q'\pm1}{2}$. By Theorem 2, we know that

- in $L_2(q)$, there are $\frac{q(q^2-1)}{q'(q'^2-1)}$ subgroups isomorphic to $L_2(q')$ and $\frac{q(q^2-1)}{4k}$ subgroups isomorphic to D_{2k};
- in $L_2(q')$, there are $\frac{q'(q'^2-1)}{4k}$ subgroups isomorphic to D_{2k}.

Let $n := \frac{(q'-1)}{2}$ if $k \mid \frac{(q'-1)}{2}$ and $n := \frac{(q'+1)}{2}$ if $k \mid \frac{(q'+1)}{2}$. By Theorem 2, there are $\frac{q(q^2-1)}{4n}$ subgroups D_{2n} in $L_2(q)$. Each subgroup D_{2n} contains $\frac{n}{k}$ subgroups D_{2k}. Therefore, each subgroup D_{2k} is contained in exactly one subgroup D_{2n}. The same kind of arguments show that each D_{2n} is contained in exactly one $L_2(q')$. Therefore, each subgroup D_{2k} of $L_2(q)$ (with $k > 2$ and k a divisor of $\frac{(q'\pm1)}{2}$) is contained in a subgroup $L_2(q')$ of $L_2(q)$, and the number of subgroups $L_2(q')$ containing a given subgroup D_{2k} is precisely one. Now the lemma follows. $\square$

Lemma 7 *The subgroups G_0 and G_3 of G cannot be isomorphic to $L_2(q')$, with ${q'}^m = q$ for some positive integer m.*

Proof: Suppose, without loss of generality, that $G_3 \cong L_2(q')$. Then the basic involution ρ_3 is such that it conjugates two subgroups $L_2(q')$ of G whose intersection contains a dihedral group of order $2t$; in terms of the underlying polytope, of which G is the automorphism group, each of these two subgroups is the stabilizer of one of the two facets which meet at the t-gonal 2-face in the base flag of the polytope and are interchanged by the "involutory reflection" ρ_3. Then this intersection itself, being a subgroup of $L_2(q)$, must be a dihedral group. However, this contradicts Lemma 6. □

Lemma 8 *Let H and K be two subgroups of type $L_2(q')$ in $L_2(q)$, with ${q'}^m = q$ for some positive integer m. If $H \cap K$ contains a cyclic group Z_k, $k > 2$, whose normalizer $N_{L_2(q)}(Z_k)$ in $L_2(q)$ is a maximal subgroup of $L_2(q)$ of dihedral type, then $H \cap K \cong Z_{\frac{(q'-1)}{2}}$ when $k \mid \frac{(q'-1)}{2}$ and $H \cap K \cong Z_{\frac{(q'+1)}{2}}$ when $k \mid \frac{(q'+1)}{2}$.*

Proof: As before, let $n := \frac{(q'-1)}{2}$ if $k \mid \frac{(q'-1)}{2}$ and $n := \frac{(q'+1)}{2}$ if $k \mid \frac{(q'+1)}{2}$. For the normalizers in H and K, by Lemma 1, we have $N_H(Z_k) = N_H(Z_n) = D_{2n}$ and $N_K(Z_k) = N_K(Z'_n) = D'_{2n}$, where D_{2n} and D'_{2n} are dihedral subgroups of H and K of order $2n$, respectively, contained in the (dihedral) normalizer $D := N_{L_2(q)}(Z_k)$ (of order $q \pm 1$), and Z_n and Z'_n are their cyclic subgroups of order n. However, Z_n and Z'_n are cyclic subgroups of D of the same order, hence $Z_n = Z'_n \leq H \cap K$. By Lemma 6, $H \cap K$ cannot be dihedral, so $H \cap K = Z_n$. □

Lemma 9 *The subgroups G_0 and G_3 of G cannot be isomorphic to $PGL_2(q')$, with ${q'}^{2m} = q$ for some positive integer m.*

Proof: Suppose, without loss of generality, that $G_3 \cong PGL_2(q')$. Then, by the same argument as in Lemma 7, there are two subgroups isomorphic to $PGL_2(q')$, namely G_3 and $G_3^{\rho_3}$, which are conjugated by ρ_3, and their intersection D must contain the dihedral group $D_{2t} := \langle \rho_0, \rho_1 \rangle$ of order $2t$.

First observe that if $m > 1$, there exists a subgroup $L \cong L_2({q'}^2)$ such that $G_3 < L < G$. The element ρ_3 fixes D by conjugation, namely $D^{\rho_3} = D$, and hence $D \leq L \cap L^{\rho_3}$. Therefore, $L \cap L^{\rho_3}$ itself must be a dihedral group, which contradicts Lemma 6. Hence we may assume that $m = 1$.

Each of G_3 and $G_3^{\rho_3}$ contains a subgroup of index 2 isomorphic to $L_2(q')$. Since the intersection of these two subgroups H and $K\,(= H^{\rho_3})$ cannot be dihedral by Lemma 3, it must coincide with a cyclic subgroup Z_k of D of some order k. Then the normalizer of Z_k in G is a maximal subgroup of dihedral type, since it contains D (and ρ_3). By Lemma 8, we then know that k must be $\frac{(q'-1)}{2}$ or $\frac{(q'+1)}{2}$; that is, $k = n$, in our previous notation. On the other hand, since t is odd by Lemma 5 and the square of every element in D (and hence of $\rho_0\rho_1$) is necessarily in $H \cap K$ (the index of $L_2(q')$ in $PGL_2(q')$ is 2), we also have $t \mid n$, that is, $t \mid n = k$.

We claim that D is a dihedral group of order $4n$. First recall that the maximal dihedral subgroups of $PGL_2(q')$ are of order $2(q' \pm 1)$ (see Moore [20]), so certainly D is of order $2n$ or $4n$. Next observe that every subgroup D_{2n} of G is contained in a unique subgroup D_{4n} of G. This can be seen as follows. Clearly, every dihedral subgroup D_{4n} of G contains exactly two dihedral subgroups D_{2n}. Moreover, by item (3) of Theorem 2, the number of subgroups D_{2n} in G is exactly twice the number of subgroups D_{4n} in G. Hence, since every subgroup D_{2n} of G actually is contained in a subgroup D_{4n} of G, the latter must necessarily be uniquely determined, proving our claim. But now we can argue as follows. Assume that D is only of order $2n$. Then D, viewed as a dihedral subgroup of G_3 of order $2n$, is contained in a maximal dihedral subgroup D' of G_3 of order $4n$. Similarly, D, viewed as a dihedral subgroup of $G_3^{\rho_3}$, is also contained in a maximal dihedral subgroup D'' of $G_3^{\rho_3}$ of order $4n$. Hence, by uniqueness in G,

$$D \leq D' = D'' \leq G_3 \cap G_3^{\rho_3} = D;$$

that is, $D = D'$, of order $4n$. This is a contradiction, so D must be of order $4n$.

Next we proceed by constructing a dihedral subgroup E in D of order $2t$ which is contained in H and K. This, then, forces $H \cap K$ to be dihedral and once again provides a contradiction to Lemma 6, thereby completing the proof.

Let τ_0, τ_1 be a pair of involutory generators of D with $(\tau_0\tau_1)^{2n} = 1$. Since D contains $\langle \rho_0, \rho_1 \rangle = D_{2t}$, we may assume that $\tau_0 = \rho_0$. Then $(\tau_0\tau_1)^2 \in H \cap K$ but $\tau_0\tau_1 \notin H \cap K$. Hence $\tau_0\tau_1 \notin H$ or $\tau_0\tau_1 \notin K$. On the other hand, $\rho_0\rho_1 \in H \cap K$. However, $\rho_0 \notin H$ and $\rho_0 \notin K$; in fact, if ρ_0 is in H or K, respectively, then $\rho_0 = \rho_0^{\rho_3}$ is in $K\,(= H^{\rho_3})$ or $H\,(= K^{\rho_3})$ and hence $\tau_0 = \rho_0 \in H \cap K = \langle (\tau_0\tau_1)^2 \rangle$, which is impossible. Now suppose, without loss of generality, that $\tau_0\tau_1 \notin H$. Then, since $\tau_0 = \rho_0 \notin H$ and H has index 2 in G_3, we must have $\tau_1 \in H$. It follows that the subgroup $E := \langle \rho_0\rho_1, \tau_1 \rangle$ of D must be a dihedral group of order $2t$ contained in H. We claim that E is also contained in K and hence in $H \cap K$. For the proof we need to verify that $\tau_1 \in K$. Assume to the contrary that $\tau_1 \notin K$. Then, since $\tau_0 = \rho_0 \notin K$ and K has index 2 in $G_3^{\rho_3}$, we must have $\tau_0\tau_1 \in K$. However, then K would contain the cyclic subgroup $\langle \tau_0\tau_1 \rangle$ of order $2n$, contradicting the fact that the maximal cyclic subgroups of a group $L_2(q')$ are of order $\frac{(q' \pm 1)}{2}$. It follows that $\tau_1 \in K$ and hence $E \leq K$, as required. This completes the proof. □

We finally have all the tools to prove the following theorem, which in turn implies Theorem 1.

Theorem 3 *Let $(G, \{\rho_0, \ldots, \rho_3\})$ be a string C-group. Suppose that $G \cong L_2(q)$. Then $q = 11$ or 19.*

Proof: Lemmas 6 and 9 reduce the possible subgroups for G_0 and G_3 to only two kinds, S_4 and A_5.

The only rank 3 polytopes with group S_4 are $\{3,3\}$ $(= \{3,3\}_4)$, $\{3,4\}_3$ and $\{4,3\}_3$, and those with group A_5 are $\{3,5\}_5$, $\{5,3\}_5$ and $\{5,5\}_3$. Recall here from [5] that $\{m,n\}_k$ is obtained from the regular tessellation $\{m,n\}$ by identifying any two vertices that are separated by k steps along a Petrie polygon of $\{m,n\}$. Its group $\langle \tau_0, \tau_1, \tau_2 \rangle$ has a presentation consisting of the standard Coxeter type relations for $\{m,n\}$ and the single extra relation $(\tau_0\tau_1\tau_2)^k = 1$.)

We can now check which pairs of polyhedra can be combined to form the facets and vertex-figures, respectively, of a regular rank 4 polytope. Table 1 gives the possible combinations and the structure of the corresponding "universal" groups; these groups are obtained by taking as defining relations just those of the facet group and vertex-figure group as well as $(\rho_0\rho_3)^2 = 1$. Only one from a pair of dual combinations is listed, since dual combinations yield the same groups (with the orders of the generators reversed). The results in this table can easily be obtained using a Computational Algebra package like Magma [1] (or, if necessary, by hand). Finally, by inspection we readily see that the only possibilities for $(G, \{\rho_0, \ldots, \rho_3\})$ to be a string C-group of rank 4 occur when $q = 11$ or $q = 19$. □

Note that the groups occurring in rows 1, 2, 5 and 9 of Table 1 are trivial. In row 4, the group actually is a group $L_2(q)$ but is too small to be a C-group of rank 4. In row 7 we also do not have a C-group of rank 4. Finally, in

Facet	Vertex-figure	Order of G	Structure of G
$\{5,5\}_3$	$\{5,5\}_3$	1	
$\{5,5\}_3$	$\{5,3\}_5$	1	
$\{5,3\}_5$	$\{3,5\}_5$	3420	$L_2(19)$
$\{5,3\}_5$	$\{3,4\}_3$	60	$A_5 \cong L_2(5)$
$\{5,3\}_5$	$\{3,3\}_4$	1	
$\{4,3\}_3$	$\{3,4\}_3$	96	$2^4:S_3$
$\{4,3\}_3$	$\{3,3\}_4$	24	S_4
$\{3,5\}_5$	$\{5,3\}_5$	660	$L_2(11)$
$\{3,4\}_3$	$\{4,3\}_3$	1	
$\{3,3\}_4$	$\{3,3\}_4$	120	S_5

Table 1: Combinations of rank 3 polytopes

row 6 we obtain a C-group of rank 4 isomorphic to $2^4:S_3$, namely the group of the universal locally projective regular 4-polytope $\{\{4,3\}_3, \{3,4\}_3\}$.

Given any type of group, the problem of determining whether or not it is the full automorphism group of a regular polytope, is generally difficult to solve due to the complexity of the subgroup lattice of the group at hand. However, for small groups, an atlas of their regular polytopes is available (see Leemans & Vauthier [15], Hartley [10]). For example, the Suzuki group $Sz(8)$ admits regular polytopes, all of rank 3, while its automorphism group $Aut(Sz(8)) = Sz(8):3$ does not. In fact, the following more general result about almost simple groups of Suzuki type was obtained in Leemans [14]. Let G be a group such that $Sz(q) \leq G \leq Aut(Sz(q))$, where $q = 2^{2e+1}$ with e a positive integer. Then G is a C-group if and only if $G = Sz(q)$. Moreover, if $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a string C-group, then $n = 3$. Rephrased in terms of regular polytopes this result says that, among such groups G, only the Suzuki group $Sz(q)$ itself can occur as automorphism group of a regular polytope of any rank, but does so only for regular polytopes of rank 3, that is, for abstract regular polyhedra.

5 The 11-cell and 57-cell

In terms of regular polytopes our main theorem can be rephrased as follows.

Theorem 4 *The only regular polytopes of rank* 4 *with automorphism groups* *of type* $L_2(q)$ *are the* 11*-cell* $\{\{3,5\}_5, \{5,3\}_5\}$ *with group* $L_2(11)$ *of order* 660*, and the* 57*-cell* $\{\{5,3\}_5, \{3,5\}_5\}$ *with group* $L_2(19)$ *of order* 3420*.*

The two polytopes of the theorem are self-dual and locally projective (see [17]); their facets and vertex-figures are regular maps in the projective plane. The 11-cell has 11 hemi-icosahedral facets, 11 vertices with hemi-dodecahedral vertex-figures, 55 edges, and 55 triangular 2-faces. (The hemi-icosahedron and hemi-dodecahedron, respectively, are obtained from the icosahedron or dodecahedron by identifying pairs of antipodal vertices, edges and 2-faces.) The edge-graph of the 11-cell is the complete graph on 11 vertices. The 57-cell has 57 hemi-dodecahedral facets, 57 vertices with hemi-icosahedral vertex-figures, 171 edges, and 171 pentagonal 2-faces. Each polytope is universal among the regular polytopes with the same kind of facets and vertex-figures. Moreover, in terms of their basic generators $\rho_0, \ldots, \rho_3$, each of the groups $L_2(11)$ and $L_2(19)$ has a presentation consisting of the standard Coxeter type relations for the 3-dimensional regular hyperbolic honeycomb $\{3,5,3\}$ or $\{5,3,5\}$, respectively, as well as the two extra relations

$$(\rho_0\rho_1\rho_2)^5 = (\rho_1\rho_2\rho_3)^5 = 1$$

(the same two extra relations in each case).

It is interesting to note the effect of dropping one, the first (say), of the two extra relations just mentioned. For the type $\{5,3,5\}$ we then obtain the universal regular 4-polytope $\{\{5,3\}, \{3,5\}_5\}$ with automorphism group

$$J_1 \times L_2(19),$$

where J_1 denotes the first Janko group, a sporadic simple group of order 175560 (see [11]). This polytope covers the 57-cell.

In [8], Grünbaum showed that there is no polytope of type $\{3,5,3\}$ with icosahedral facets and hemi-dodecahedral vertex figures (see also [9]). Hence, in that case, dropping one of the two extra relations still gives $L_2(11)$.

Recall that the even (or rotation) subgroup of a C-group consists of the elements which can be expressed as products of an even number of generators ρ_i; it has index at most 2 in the full group. Note that, when considered as C-groups, $L_2(11)$ and $L_2(19)$ coincide with their even subgroups, because they are simple groups.

The April 2007 Issue of Discover Magazine runs a popular article about the 11-cell written by Jaron Lanier (see [13]).

6 $L_2(q)$ acting with two flag orbits

Our main theorem can be rephrased as saying that there are just two abstract polytopes of rank 4 on which a group of type $L_2(q)$ admits a flag-transitive action as a group of automorphisms. By contrast, there is a wealth of abstract polytopes of rank 4 on which a group $L_2(q)$ acts with precisely two flag orbits. In fact, this is already true in rank 3, the most prominent example being $L_2(7)$, acting as the even subgroup of the automorphism group $PGL_2(7)$ of Klein's regular map $\{3,7\}_8$ of genus 3 (see [5], as well as [7])).

The known examples of polytopes $\mathcal{P}$ with two flag orbits generally have the property that adjacent flags are in distinct orbits. This can arise in one of two ways. Either the polytope $\mathcal{P}$ is regular and $L_2(q)$ is the even subgroup of its automorphism group (as in the case of the Klein map), or $\mathcal{P}$ is chiral and $L_2(q)$ is its full automorphism group. (Recall here that an abstract polytope is *chiral* if its automorphism group has two orbits on the flags, such that adjacent flags are in distinct orbits.)

One type of construction of examples of rank 4 begins with a 3-dimensional regular hyperbolic honeycomb and a faithful representation of its symmetry group as a group of complex Möbius transformations, generated by the inversions in four circles cutting one another at the same angles as the corresponding reflection planes in hyperbolic space (see [23]). For example, for the regular honeycomb $\{4,4,3\}$, the even subgroup of its symmetry group is isomorphic to $L_2(\mathbb{Z}[i]):2$, where $\mathbb{Z}[i]$ is the ring of Gaussian integers. Then interesting finite regular or chiral polytopes $\mathcal{P}$ of rank 4 can be obtained by modular reduction of this group. In each case, the resulting group is of type $L_2(R)$, with R a finite ring, and acts as a group of automorphisms of $\mathcal{P}$ with two flag orbits. In certain cases, R is a field.

For example, if p is a prime with $p \equiv 3(4)$, then $\mathcal{P}$ is regular, the even subgroup is $L_2(p^2)$, and $\mathcal{P}$ has toroidal facets $\{4,4\}_{(p,0)}$ and cubical vertex-figures $\{4,3\}$ (see [23, p.238]). Similarly, if p is a prime with $p \equiv 1(8)$, then $\mathcal{P}$ is chiral, the automorphism group is $L_2(p)$, and $\mathcal{P}$ has toroidal facets $\{4,4\}_{(b,c)}$, with $b^2 + c^2 = p$, and again cubical vertex-figures $\{4,3\}$ (see [23, p.239]). There are similar such results for other Schläfli symbols (see also [21]).

Yet more examples of polytopes with two flag orbits can be found in, for instance, [19].

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