# Reflection Groups and Polytopes over Finite Fields, II

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**Abstract**

When the standard representation of a crystallographic Coxeter group $\Gamma$ is reduced modulo an odd prime $p$, a finite representation in some orthogonal space over $\mathbb{Z}_p$ is obtained. If $\Gamma$ has a string diagram, the latter group will often be the automorphism group of a finite regular polytope. In Part I we described the basics of this construction and enumerated the polytopes associated with the groups of rank 3 and the groups of spherical or Euclidean type. In this paper, we investigate such families of polytopes for more general choices of $\Gamma$, including all groups of rank 4. In particular, we study in depth the interplay between their geometric properties and the algebraic structure of the corresponding finite orthogonal group.

Key Words: reflection groups, abstract regular polytopes

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# 1 Introduction

The regular polytopes are a rich and ongoing source of mathematical ideas. Their combinatorial features, for instance, have been beautifully generalized in the theory of *abstract regular polytopes*.

In [17], the precursor to this paper, we surveyed some of the essential properties of an abstract regular polytope $\mathcal{P}$, referring to [13] for details. Then, reframing the key results

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in [25], we outlined an abbreviated classification of finite, irreducible groups generated by reflections in $n$-space $V$, over a field of odd characteristic $p$ (see [17, Thm. 3.1]).

When $G$ is a (possibly infinite) crystallographic Coxeter group with string diagram, reduction modulo an odd prime $p$ of the standard real representation yields a finite reflection group $G^p$, which we could then classify and which is often the automorphism group of a finite, abstract regular $n$-polytope $\mathcal{P}$. (If this is so, we say that $G^p$ is a *string C-group.*)

Next we established two useful criteria for $G^p$ to be a string C-group: Theorems 4.1 and 4.2 of [17] concern the features of $V$ as an orthogonal space, as well as the action of standard subgroups of $G^p$ on $V$. With this, we were able to classify all groups $G^p$, and their polytopes, whenever $n \leq 3$, as well as when $G$ is of spherical or Euclidean type, for all ranks $n$.

Here, we begin by summarizing in Section 2 some key notation. Next, in Section 3, we extend our criteria for $G^p$ to be a string C-group. Finally, in Sections 4 and 5, we discuss and completely classify all 4-polytopes which arise from our construction.

# 2 Notation

We refer the reader to the notation and basic set up in [17]. Throughout, $G = \langle r_0, \ldots, r_{n-1} \rangle$ will be a crystallographic Coxeter group $[p_1, p_2, \ldots, p_{n-1}]$ with a string Coxeter diagram $\Delta_c(G)$ (with branches labeled $p_1, p_2, \ldots, p_{n-1}$, respectively), obtained from the corresponding abstract Coxeter group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ via the standard representation on real $n$-space $V$. For any odd prime $p$, we may reduce $G$ modulo $p$ to obtain a subgroup $G^p$ of $GL_n(\mathbb{Z}_p)$ generated by the modular images of the $r_i$'s. We shall abuse notation by referring to the modular images of objects by the same name (such as $r_i$, $b_i$, $B = [b_{ij}]$, $V$, etc.). In particular, $\{b_i\}$ will denote the standard basis for $V = \mathbb{Z}_p^n$. In any event, $G^p$ is a subgroup of the orthogonal group $O(\mathbb{Z}_p^n)$ of isometries for the (possibly singular) symmetric bilinear form $x \cdot y$, the latter being defined on $\mathbb{Z}_p^n$ by means of the Gram matrix $B$; in particular, $r_i$ is the orthogonal reflection with *root* $b_i$ if $b_i^2 \neq 0$.

Next we make a convenient definition: if $p \geq 5$, or $p = 3$ but no branch of $\Delta_c(G)$ is marked 6, then we say that $p$ is *generic* for $G$. In such cases, no node label of the diagram $\Delta(G)$ (for a basic system) is zero mod $p$ and a change in the underlying basic system for $G$ has the effect of merely conjugating $G^p$ in $GL_n(\mathbb{Z}_p)$. On the other hand, in the non-generic case, in which $p = 3$ and $\Delta_c(G)$ has branches marked 6, the group $G^p$ may depend essentially on the actual diagram $\Delta(G)$ taken for the reduction mod $p$. (Note that $p$ generic does not necessarily mean that $p \nmid |G|$, or that certain subspaces of $V$ are non-singular, etc.)

Recall from [17, Thm. 3.1] that an irreducible group $G^p$ of the above sort, generated by $n \geq 3$ reflections, must necessarily be one of the following:

- an orthogonal group $O(n, p, \epsilon) = O(V)$ or $O_j(n, p, \epsilon) = O_j(V)$, excluding the cases $O_1(3, 3, 0)$, $O_2(3, 5, 0)$, $O_2(5, 3, 0)$ (supposing for these three that $\mathrm{disc}(V) \sim 1$), and also excluding the case $O_j(4, 3, -1)$; or
- the reduction mod $p$ of one of the finite linear Coxeter groups of type $A_n$ ($p \nmid n + 1$), $B_n$, $D_n$, $E_6$ ($p \neq 3$), $E_7$, $E_8$, $F_4$, $H_3$ or $H_4$.

We shall say in these two cases that $G^p$ is of *orthogonal* or *spherical type*, respectively, although there is some overlap for small primes. Our description rests on the classification of the finite irreducible reflection groups over any field, obtained in Zalesskiĭ & Serežkin [25] (see also [9, 20, 21, 24]). It is only a slight abuse of notation to let $[p_1, \ldots, p_{n-1}]^p$ denote the modular representation of a group $[p_1, \ldots, p_{n-1}]$, so long as $p$ is generic for the group.

The generators $r_i$ of $G^p$ satisfy the Coxeter-type relations inherited from $G$. Our main problem is to determine when $G^p$ has the intersection property (1) for its standard subgroups. For any $J \subseteq \{0, \ldots, n-1\}$, we let $G^p_J := \langle r_j \,|\, j \notin J\rangle$; in particular, for $k, l \in \{0, \ldots n-1\}$ we let $G^p_k := \langle r_j \,|\, j \neq k\rangle$ and $G^p_{k,l} := \langle r_j \,|\, j \neq k, l\rangle$. Then $G^p$ is a *string C-group* if and only if $G^p$ satisfies the *intersection property*

$$\langle \rho_i \,|\, i \in I\rangle \cap \langle \rho_i \,|\, i \in J\rangle = \langle \rho_i \,|\, i \in I \cap J\rangle; \tag{1}$$

and in this case $G^p$ is the automorphism group of a finite regular polytope denoted by $\mathcal{P}(G^p)$ (see [13, §2E]). Note as well that $G^p$ is a string C-group if and only if $G^p_0$ and $G^p_{n-1}$ are string C-groups and $G^p_0 \cap G^p_{n-1} = G^p_{0,n-1}$. We also let $V_J$ be the subspace of $V = \mathbb{Z}^n_p$ spanned by $\{b_j \,|\, j \notin J\}$, and similarly for $V_k, V_{k,l}$. Note that $V_J$ is $G^p_J$-invariant.

# 3 The Intersection Property

The goal of this section is to assess when the modular reduction $G^p$ of a crystallographic string Coxeter group $G$ satisfies the intersection property (1). In [17] we established a number of sufficient conditions and verified that $G^p$, with $p \geq 3$, has the intersection property whenever $G$ has rank at most 3, or whenever $G$ is of spherical or Euclidean type. However, the situation changes drastically for more general groups of higher ranks, with obstructions already occurring for rank 4. We already know that $G^p$ is a string C-group if one of the subgroups $G^p_0$ or $G^p_{n-1}$ is spherical and the other is a string C-group (see [17, Thm. 4.2]). Moreover, $G^p$ also is a string C-group if both $G^p_0$ and $G^p_{n-1}$ are C-groups, $V_{0,n-1}$ is a non-singular subspace of $V$, and $G^p_{0,n-1}$ is the full orthogonal group $O(n-2, p, \epsilon)$ on $V_{0,n-1}$ (see [17, Thm. 4.1]).

The criteria established here will settle the Coxeter groups $[k, l, m]$ of rank 4 completely. Before we move on, note two simple cases. If $l = 2$ and $k, m < \infty$, then

$$G \cong [k] \times [m] \cong G^p,$$

so $G^p$ certainly is a C-group. Similarly, if say $m = 2$ (but $k$ or $l = \infty$ is allowed), then $G \simeq [k, l] \times C_2$, and

$$G^p \simeq [k, l]^p \times C_2 \,.$$

Thus the intersection property of $G^p$ follows directly from that of its subgroup $[k, l]^p$ (see [17, Thm. 5.1]). More generally, if $[p_1, \ldots, p_{n-1}]^p$ is a string C-group, then so is

$$[p_1, \ldots, p_{n-1}, 2]^p \simeq [p_1, \ldots, p_{n-1}]^p \times C_2 \;\;.$$

After a preliminary lemma, we shall continue to build upon the known results concerning $C$-groups mentioned above.

**Lemma 3.1** *Suppose that $G$ has rank $n$ and that $(\mathrm{rad}\,V) \cap V_j = \{o\}$. Then the subgroup $G_j^p$ is, by restriction to the invariant subspace $V_j$, isomorphic to $H^p$, the reduction modulo $p$ of the group of rank $n-1$ defined from the subdiagram of $\Delta(G)$ which results from the deletion of node $j$. In particular, when $j = 0$ or $n-1$ this holds if $p$ is generic for $G$.*

**Proof**. This result is well known in characteristic 0 [8, §5.5]. Here we restrict $g \in G_j^p$ to the invariant subspace $V_j$, and so obtain a homomorphism

$$\begin{array}{rcl} \varphi : \; G_j^p & \longrightarrow & O(V_j) \\ g & \longmapsto & g_{|V_j} \; . \end{array}$$

Of course, as a subspace of $V$, $V_j$ is isometric to $\mathbb{Z}_p^{n-1}$, with the metric structure obtained from the subdiagram of $\Delta(G)$ obtained by deleting node $j$. Clearly the image group $\varphi(G_j^p)$ is isomorphic to the reflection group $H^p$ of rank $n-1$ defined directly from the subdiagram.

Suppose $g \in \ker \varphi$. Then $g(b_k) = b_k$ for all $k \neq j$, whereas $g(b_j) = b_j + x$ for some $x \in V_j$. Thus for any $k \neq j$

$$b_j \cdot b_k = g(b_j) \cdot g(b_k) = (b_j + x) \cdot b_k = b_j \cdot b_k + x \cdot b_k \; ,$$

so that $x \cdot b_k = 0$, and $x \in \mathrm{rad}\, V_j$. But then $x \cdot x = 0$ and so

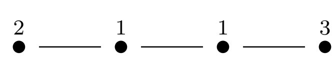
$$b_j \cdot b_j = g(b_j) \cdot g(b_j) = b_j \cdot b_j + 2x \cdot b_j + x \cdot x \; ,$$

whence $x \cdot b_j = 0$. Thus $x \in (\mathrm{rad}\, V) \cap V_j$, so that $x = o$ when this subspace is trivial. Hence $\varphi$ is injective. When $p$ is generic for $G$, a direct calculation in coordinates along the string diagram shows that $(\mathrm{rad}\, V) \cap V_j = \{o\}$ for $j = 0, n-1$. □

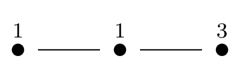
**Remark**. Informally, the Lemma asserts that reduction by a generic prime commutes with the deletion of a node from $\Delta(G)$. Note that

$$G_j^p \simeq [p_1, \ldots, p_{j-1}]^p \times [p_{j+2}, \ldots, p_{n-1}]^p \simeq [p_1, \ldots, p_{j-1}, 2, p_{j+2}, \ldots, p_{n-1}]^p \; .$$

Concerning the non-generic cases, there are examples showing the necessity of the hypotheses. For example, the group $G \simeq [4, 3, 6]$ with diagram




yields, as we observe below, a $C$-group $G^3$. Here the subgroup $G_0^3$ is the automorphism group of order 108 for the toroidal polyhedron $\{3, 6\}_{(3,0)}$. However, the subdiagram




yields the smaller group of order 36 for $\{3, 6\}_{(1,1)}$. Thus the map $\varphi$ of the Lemma is here not injective.

**Theorem 3.1** *Suppose that $G \simeq [k, l, m]$ is crystallographic and that the subgroup $[k, l]$ or $[l, m]$ is spherical. Then $G^p$ is a $C$-group for any prime $p \geq 3$.*

**Proof.** Let $[k, l]$ (say) be spherical, so that $G_3^p \simeq G_3 = [k, l]$. First suppose that $p$ is generic for $G$. Since $G_0^p$ is a C-group by Lemma 3.1, the proof follows directly from [17, Thm. 4.2]. Moreover, even in non-generic cases of rank 4 (so that $p = 3$), $G^3$ turns out to be a C-group when $[k, l]$ is spherical. This is routinely verified using the computer algebra system GAP [7]. The pertinent examples are $G \simeq [3, 3, 6], [3, 4, 6]$ or $[4, 3, 6]$, each with two essentially distinct diagrams $\Delta(G)$ (for the basic systems). □

We now establish two general results, of which the first allows us to reject large classes of groups $G^p$ as C-groups because of the size of their subgroups $G_0^p \cap G_{n-1}^p$. First we deal with the *fully non-singular* case.

**Theorem 3.2** *Let $G = \langle r_0, \ldots, r_{n-1} \rangle$ be a crystallographic linear Coxeter group with string diagram. Suppose that $n \geq 3$ and that the prime $p$ is generic for $G$. Let the subspaces $V$, $V_0$, $V_{n-1}$ and $V_{0,n-1}$ be non-singular, and let $G_0^p$, $G_{n-1}^p$ be of orthogonal type. Suppose as well that there is a square among the labels of the nodes $1, \ldots, n-2$ of the diagram $\Delta(G)$ (this can be achieved by readjusting the node labels).*
*(a) Then $G_0^p \cap G_{n-1}^p$ acts trivially on $V_{0,n-1}^{\perp}$, and*

$$O_1(V_{0,n-1}) \leq G_0^p \cap G_{n-1}^p \leq O(V_{0,n-1}),$$

*where we have identified the $(n-2)$-dimensional groups $O(V_{0,n-1})$ and $O_1(V_{0,n-1})$ with the pointwise stabilizers of $V_{0,n-1}^{\perp}$ in the $n$-dimensional groups $O(V)$ and $O_1(V)$, respectively.*
*(b) If $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$, with a similar interpretation as stabilizers, then*

$$G_0^p \cap G_{n-1}^p = O(V_{0,n-1}).$$

**Proof.** Since all four subspaces of $V = \langle b_0, \ldots, b_{n-1} \rangle$ are non-singular, we have the orthogonal sums

$$V = V_{n-1} \oplus \langle v \rangle = V_0 \oplus \langle v' \rangle,\ V_{n-1} = V_{0,n-1} \oplus \langle w \rangle,\ V_0 = V_{0,n-1} \oplus \langle w' \rangle,$$

for non-isotropic vectors $v, v', w, w'$. Then,

$$\langle v, v' \rangle = V_{0,n-1}^{\perp} = \langle w, w' \rangle.$$

Since $p$ is generic for $G$, each reflection $r_j$ actually has $b_j$ as a root, and $v \perp b_j$ for $j \leq n-2$, while $v' \perp b_j$ for $j \geq 1$. Hence the subgroups $G_{n-1}^p$, $G_0^p$ and $G_0^p \cap G_{n-1}^p$ stabilize the vectors $v$, $v'$ or $v, v'$, respectively. In particular,

$$G_0^p \cap G_{n-1}^p \leq O(V_{0,n-1}),$$

with $O(V_{0,n-1})$ identified with the pointwise stabilizer of $V_{0,n-1}^{\perp}$ in $O(V)$. Note that the restrictions of $G_{n-1}^p$, $G_0^p$ and $G_0^p \cap G_{n-1}^p$ to the subspaces $V_{n-1}$, $V_0$ or $V_{0,n-1}$, respectively, are faithful, by Lemma 3.1.

Since $G_0^p$, $G_{n-1}^p$ are of orthogonal type and there is a square among the labels of the nodes $1, \ldots, n-2$, we must have $O_1(V_{n-1}) \leq G_{n-1}^p$ and $O_1(V_0) \leq G_0^p$ (that is, a group merely of type $O_2$ cannot occur). Now, if $g \in O_1(V_{0,n-1})$, then $g(v) = v$, so that $g \in O(V_{n-1})$; but the spinor norm is invariant under orthogonal embedding ([1, Thm. 5.13]), so actually

$g \in O_1(V_{n-1})$. Similary, $g \in O_1(V_0)$, and hence $g \in G_0^p \cap G_{n-1}^p$. This completes the proof of part (a).

Now let $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$. Once again, if $g \in O(V_{0,n-1})$, then $g(v) = v$, so now $g \in O(V_{n-1}) = G_{n-1}^p$. Similarly, $g \in O(V_0) = G_0^p$, and hence $g \in G_0^p \cap G_{n-1}^p$, as required. □

We note an immediate corollary to Theorem 3.2. It shows that many groups $G^p$ of rank 4 fail to satisfy the intersection property for large primes $p$. However, those primes for which $G^p$ actually is a C-group lead to interesting polytopes, which we investigate in later sections.

**Corollary 3.1** *Suppose the prime $p$ is generic for the crystallographic group $G = [k, l, m]$. Let $V$, $V_0$, $V_3$, $V_{0,3}$ be non-singular, and let $G_0^p$, $G_3^p$ be of orthogonal type.*
*(a) Then $G^p$ is not a C-group if $p > 2l + \epsilon(V_{0,3})$, where $\epsilon(V_{0,3}) = \pm 1$ is the parameter associated with the plane $V_{0,3}$.*
*(b) If $G_0^p = O(V_0)$ and $G_3^p = O(V_3)$, then $G^p$ is a C-group if and only if $p = l + \epsilon(V_{0,3})$.*

**Proof.** We apply Theorem 3.2 with $n = 4$. The subgroups $G_0^p$ and $G_3^p$ are known to be C-groups ([17, Thm. 5.1]), so it suffices to determine when

$$G_0^p \cap G_3^p = G_{0,3}^p.$$

Now $G_{0,3}^p = \langle r_1, r_2 \rangle$ is a dihedral group of order $2l = 6$, 8 or 12; the case $l = \infty$ is excluded, as $V_{0,3}$ is then a non-singular plane. Note that we may assume that there is a square (in fact, a 1) among the labels of the nodes 1 or 2 of the diagram; this can be achieved by readjusting the node labels as described earlier. Then, by Theorem 3.2 we have

$$O_1(V_{0,3}) \leq G_0^p \cap G_3^p,$$

so $G_0^p \cap G_3^p$ is larger than $G_{0,3}^p$ if the order of $O_1(V_{0,3})$, which is $p - \epsilon(V_{03})$, exceeds $2l$. Hence the intersection property certainly fails if $p > 2l + \epsilon(V_{0,3})$. Moreover, by Theorem 3.2, if $G_0^p = O(V_0)$ and $G_3^p = O(V_3)$, then

$$G_0^p \cap G_3^p = O(V_{0,3}),$$

so $G_0^p \cap G_3^p = G_{0,3}^p$ if and only if $2(p - \epsilon(V_{03})) = 2l$, or equivalently, $p = l + \epsilon(V_{0,3})$. □

Corollary 3.1 immediately implies (in fully non-singular cases) that $G^p$ is not a C-group if $p > 13$. However, for the primes $p = 5, 7, 11, 13$ (and 3), the outcome is less predictable and actually depends on the group $G = [k, l, m]$ as well as the diagram $\Delta(G)$ chosen for the reduction modulo $p$. For example, $G^{13}$ can only be a C-group if $l = 6$ and $G_0^{13} = O_1(V_0)$ or $G_3^{13} = O_1(V_3)$. Similarly, if $G_0^p = O(V_0)$ and $G_3^p = O(V_3)$, then $G^p$ is not a C-group if $p > 7$; moreover, $G^7$ can then be a C-group only if $l = 6$.

Next we study the case when the middle section of the diagram for $G$ determines a singular space $V_{0,n-1}$, while $V$, $V_0$ and $V_{n-1}$ still are non-singular, again with $G_0^p, G_{n-1}^p$ of orthogonal type. In a singular space $W$ over a field $\mathbb{K}$, the isometry group $O(W)$ leaves invariant the radical subspace $\operatorname{rad} W$, thereby providing a natural epimorphism $\eta : O(W) \to O(W/\operatorname{rad} W)$. Since $W/\operatorname{rad} W$ is non-singular, we may define a 'spinor norm' $\theta$ on $W$, sufficient for our needs, by

$$\theta(g) := \theta_{W/\operatorname{rad} W}(\eta(g)), \quad g \in O(W) \; .$$

Now let $\widehat{O}(W)$ denote the subgroup of $O(W)$ consisting of those isometries $g$ which act trivially on $\mathrm{rad}\, W$. It is not hard to show that $\widehat{O}(W)$ contains all reflections (with non-isotropic roots in $W$) and is even generated by them. The key observation here is that any transvection in the kernel of the action of $O(W)$ on $\mathrm{rad}\, W$ can be factored as a product of reflections (cf. [1, Thm. 3.20 and p. 133]).

It is easy to see that if $g$ is the product of reflections with non-isotropic roots $a_1, \ldots, a_k$, then $\theta(g) = a_1^2 \cdots a_k^2 \dot{\mathbb{K}}^2$. Naturally, by $\widehat{O}_1(W)$ (or $\widehat{O}_2(W)$) we mean the subgroup of $\widehat{O}(W)$ generated by the reflections in $O(W)$ whose spinor norm is a square (or non-square, respectively).

**Theorem 3.3** *Let $G = \langle r_0, \ldots, r_{n-1} \rangle$ be a crystallographic linear Coxeter group with string diagram, and suppose the prime $p$ is generic for $G$. Let $V$, $V_0$, $V_{n-1}$ be non-singular, let $V_{0,n-1}$ be singular, and let $G_0^p, G_{n-1}^p$ be of orthogonal type. Suppose there is a square among the labels of the nodes $1, \ldots, n-2$ of the diagram $\Delta(G)$ (this can be achieved by readjusting the node labels).*
*(a) Then $G_0^p \cap G_{n-1}^p$ acts trivially on $V_{0,n-1}^\perp$, and*

$$\widehat{O}_1(V_{0,n-1}) \leq G_0^p \cap G_{n-1}^p \leq \widehat{O}(V_{0,n-1}),$$

*where $\widehat{O}(V_{0,n-1})$ has been identified with the pointwise stabilizer of $V_{0,n-1}^\perp$ in $O(V)$, and $\widehat{O}_1(V_{0,n-1})$ with the subgroup of $O_1(V)$ generated by the reflections with roots in $V_{0,n-1}$ and square spinor norm.*
*(b) If $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$, then also*

$$\widehat{O}(V_{0,n-1}) = G_0^p \cap G_{n-1}^p.$$

**Proof.** As before we have

$$V = V_{n-1} \oplus \langle v \rangle = V_0 \oplus \langle v' \rangle$$

with non-isotropic vectors $v, v'$. The subspace $V_{0,n-2}^\perp$ still is 2-dimensional, so necessarily $V_{0,n-1}^\perp = \langle v, v' \rangle$. Moreover,

$$V_{0,n-1} \cap V_{0,n-1}^\perp = \mathrm{rad}\, V_{0,n-1} \neq \{o\},$$

so the vectors in $V_{0,n-1} \cup V_{0,n-1}^\perp$ span a singular hyperplane $U$ in $V$ with 1-dimensional radical $\mathrm{rad}\, U = \mathrm{rad}\, V_{0,n-1}$. For the same reason as before, $G_{n-1}^p$, $G_0^p$ and $G_0^p \cap G_{n-1}^p$ stabilize the vectors $v$, $v'$ or $v, v'$, respectively, and yield faithful restrictions to the subspaces $V_{n-1}$, $V_0$ and $V_{0,n-1}$. In particular,

$$G_0^p \cap G_{n-1}^p \leq H := \{g \in O(V) \,|\, g(x) = x,\ \forall x \in V_{0,n-1}^\perp\}.$$

Note also that $G_0^p \cap G_{n-1}^p$ leaves $U$ invariant because it leaves $V_{0,n-1}$ invariant.

We claim that we may identify $H$ with $\widehat{O}(V_{0,n-1})$; this would settle the inclusion on the right in part (a) of the theorem. Now, since each element of $H$ leaves $V_{0,n-1}$ invariant, while fixing $\mathrm{rad}\, V_{0,n-1}$, we can consider the restriction mapping to $V_{0,n-1}$,

$$\begin{array}{rccl} \kappa: & H & \longrightarrow & \widehat{O}(V_{0,n-1}) \\ & g & \longrightarrow & g_{|V_{0,n-1}}. \end{array} \tag{2}$$

We prove that $\kappa$ is an isomorphism. If $g \in \ker(\kappa)$, then $g$ acts trivially on $V_{0,n-1}$, and hence also on $U$ because $g \in H$. It follows that $g = e$, the identity mapping on $V$. Here we have used the fact that an isometry of a non-singular space is uniquely determined by its effect on a hyperplane, $H$ in this case, if this hyperplane is singular ([1, Thm. 3.17]). This shows that $\kappa$ is injective. Now let $h \in \widehat{O}(V_{0,n-1})$. Then $h$ acts trivially on $\operatorname{rad} V_{0,n-1}$, so we can extend $h$ to an isometry $h'$ (say) of $U = V_{0,n-1} \oplus \langle v \rangle$ by setting $h'(v) := v$. We now apply Witt's extension theorem for isometries between subspaces of a non-singular space ([1, Thm. 3.9]) and conclude that $h'$ extends further to an isometry $g$ of the entire space $V$. Then $g$ must be in $H$ and so $\kappa$ is also surjective. By our earlier remarks, $\widehat{O}(V_{0,n-1})$ is generated by all reflections $r_a$, for non-isotropic roots $a \in V_{0,n-1}$; pulling back, a similar claim is true for $H$.

Continuing along these lines, we now prove the inclusion on the left in part (a) of the theorem. For a non-isotropic vector $a \in V_{0,n-1}$, let $r_{a,V}$, $r_{a,V_{n-1}}$, $r_{a,V_0}$ and $r_{a,V_{0,n-1}}$ denote the reflections with root $a$ in $V$, $V_{n-1}$, $V_0$ or $V_{0,n-1}$, respectively. Then $r_{a,V} \in H$ because $a \perp V_{0,n-1}^{\perp}$, and

$$r_{a,V_{0,n-1}} = \kappa(r_{a,V}).$$

It follows that the subgroup $H_1$ of $H$ generated by the reflections $r_{a,V}$, with $a \in V_{0,n-1}$ and $a^2$ a square, is isomorphic, under $\kappa$, to $\widehat{O}_1(V_{0,n-1})$. We have to show that $H_1 \leq G_0^p \cap G_{n-1}^p$.

Now, by assumption, the subgroups $G_0^p, G_{n-1}^p$ of $G$ are of orthogonal type, and there is a square among the labels of the nodes $1, \ldots, n-2$ of the diagram. It follows that $G_0^p$ and $G_{n-1}^p$, when restricted to the subspaces $V_0$ or $V_{n-1}$, respectively, must contain the groups $O_1(V_0)$ or $O_1(V_{n-1})$. Hence, if we identify the restricted groups with the stabilizers of $v'$ or $v$, respectively, in $O_1(V)$, then we see that $O_1(V_0)$ and $O_1(V_{n-1})$ are actually subgroups of $G_0^p$ and $G_{n-1}^p$. In particular, if $a \in V_{0,n-1}$ and $a^2$ is a square, then $r_{a,V}$ belongs to $O_1(V_j)$, for $j = 0, n-1$, so $r_{a,V} \in G_0^p \cap G_{n-1}^p$. Now it follows that $H_1$ is a subgroup of $G_0^p \cap G_{n-1}^p$. This settles part (a).

Finally, suppose that $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$. Much of the same analysis carries over, but now it is applied to all reflections, including those whose spinor norm is a non-square. In particular, if $a$ is any non-isotropic vector in $V_{0,n-1}$, then $r_{a,V} \in G_0^p \cap G_{n-1}^p$. Hence we also have

$$\widehat{O}(V_{0,n-1}) \leq G_0^p \cap G_{n-1}^p \ ,$$

since $\widehat{O}(V_{0,n-1})$ is generated by its reflections. Now part (a) gives the equality of these groups. □

**Remark**. It is not difficult to show that $\widehat{O}_1(V_{0,n-1})$ can also be identified with the pointwise stabilizer of $V_{0,n-1}^{\perp}$ in $O_1(V)$.

For a crystallographic Coxeter group $[k, l, m]$, the middle section of the diagram determines a singular subspace if and only if $l = \infty$. In this case, the reduced group $G^p$ is always a $C$-group:

**Corollary 3.2** *Let $G \simeq [k, \infty, m]$ be crystallographic. Then $G^p$ is a $C$-group for any prime $p \geq 3$.*

**Proof.** We know that $G_0^p$ and $G_3^p$ are C-groups ([17, Thm. 5.1]), so again it suffices to check that

$$G_0^p \cap G_3^p = G_{0,3}^p.$$

Suppose for the moment that $p$ is generic for $G$, so that we may apply Theorem 3.3 with $n = 4$. Then, with few exceptions, $V$ is still non-singular. Moreover, $V_0$ and $V_3$ correspond to the subgroups $[\infty, m]$ and $[k, \infty]$ of $G$, and hence are known to be non-singular as well (except in the non-generic case with $p = 3$, $k, m = 6$; see [17, Sect. 5]). However, $V_{0,3}$ is a singular plane, so Theorem 3.3 implies that

$$\widehat{O}_1(V_{0,3}) \leq G_0^p \cap G_3^p \leq \widehat{O}(V_{0,3}),$$

provided $V$ is non-singular. If the labels of the nodes 1 and 2 of the diagram are 1 and 4, respectively, as we may assume, then $V_{0,3}$ is a singular plane in which the squared norm of each non-isotropic vector is a square. In particular,

$$\widehat{O}_1(V_{0,3}) = \widehat{O}(V_{0,3}) \cong [p] \cong G_{0,3}^p,$$

and hence $G_0^p \cap G_3^p = G_{0,3}^p$. In fact, since $2b_1 + b_2$ spans $\operatorname{rad} V_{0,3}$, a matrix representing an element of $\widehat{O}(V_{0,3})$ in the basis $b_1, 2b_1 + b_2$ must necessarily have the form

$$\begin{bmatrix} \pm 1 & 0 \\ \mu & 1 \end{bmatrix} \quad (\mu \in \mathbb{Z}_p),$$

so there are at most $2p$ of them. On the other hand, the restrictions of $r_1$ and $r_2$ to $V_{0,3}$ already generate a dihedral group $[p]$ contained in $\widehat{O}_1(V_{0,3})$, so the three groups must coincide.

The space $V$ is singular for the following groups $G$ (up to duality) and primes $p\,(> 3)$: $[4, \infty, 3]$ and $[6, \infty, 6]$, with $p = 5$; $[3, \infty, 3]$ and $[4, \infty, 6]$, with $p = 7$; and $[3, \infty, 6]$, with $p = 13$. In each of these cases, as well as for $p = 3$ for any of the crystallographic groups $[k, \infty, m]$, computations in GAP confirm that $G^p$ also is a C-group. □

We now concentrate entirely on the groups $G = [k, l, m]$ which are not yet covered by the previous results. These groups have a Euclidean subgroup $[k, l]$ or $[l, m]$. The following theorem settles the case when $l = 4$ or 6.

**Theorem 3.4** *Let $G \simeq [k, l, m]$ be crystallographic. Suppose the subgroup $[k, l]$ or $[l, m]$ is Euclidean, and that $l = 4$ or 6. Then $G^p$ is a C-group for any prime $p \geq 3$.*

**Proof.** Suppose $G_3 = [k, l]$ (say) is Euclidean. The case $m = 2$ was already settled, so let $m \neq 2$. For the moment, let $p$ be generic for $G$. Then $V = \langle b_0, \ldots, b_3 \rangle$ is non-singular and $V_3$ is singular, so that every isometry of $V$ is uniquely determined by its effect on $V_3$. Let $\operatorname{rad} V_3 = \langle c \rangle$ (say).

Let $g \in G_0^p \cap G_3^p$. Then $g$ leaves $V_3$ and $V_{0,3}$ invariant, and fixes $c$. Since $l = 4$ or 6, we necessarily have $G_3 = [3, 6]$ or $[4, 4]$, so $G_3^p$ is a semi-direct product of its "translation subgroup" $T^p$ (of order $p^2$) by $G_{0,3}^p$. In particular, since we are allowed to multiply $g$ by an element in $G_{0,3}^p$, we may assume that $g \in T^p$. We prove that this forces $g = e$, hence the desired conclusion.

When $G_3 = [4, 4]$, we may assume that the labels of the nodes 0, 1, 2 of the diagram of $G$ are 1, 2, 1, respectively. Then $c = b_0 + b_1 + b_2$ and

$$T^p = \langle r_0 r_1 r_2 r_1, r_1 r_0 r_1 r_2 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

Let $M^p$ denote the group of $p^2$ matrices of the form

$$M(\lambda,\mu) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & 1 \end{bmatrix} \qquad (\lambda,\mu \in \mathbb{Z}_p). \tag{3}$$

In the basis $b_0, b_1, c$ of $V_3$, each element of $T^p$, when restricted to $V_3$, is represented by a matrix in $M^p$. ($T^p$ acts faithfully on $V_3$ by Lemma 3.1.) For example, we have

$$(r_0r_1r_2r_1)^j(r_1r_0r_1r_2)^k \mapsto M(2j, 2(k-j))$$

so $T^p$ and $M^p$ are clearly isomorphic. Now suppose that $M(\lambda,\mu)$ is the matrix for $g$. Then

$$g(b_1) = b_1 + \mu c = \mu b_0 + (1+\mu)b_1 + \mu b_2,$$

so we must have $\mu = 0$ because $V_{0,3}$ is invariant under $g$. Similarly,

$$\begin{aligned} g(b_2) = g(c - b_0 - b_1) &= g(c) - g(b_0) - g(b_1) \\ &= c - (b_0 + \lambda c) - b_1 \quad = -\lambda b_0 - \lambda b_1 + (1-\lambda)b_2, \end{aligned}$$

so also $\lambda = 0$, for the same reason. It follows that $g$ acts trivially on $V_3$ and hence also on $V$, that is, $g = e$.

When $G_3 = [3,6]$, we may take the labels of nodes 0, 1, 2 of the diagram to be 1, 1, 3, respectively. Then $c = b_0 + 2b_1 + b_2$ and

$$T^p = \langle r_0r_1(r_2r_1)^2, r_1r_0(r_1r_2)^2 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

The elements of $T^p$ still are represented by the matrices $M(\lambda,\mu)$ in $M^p$ (in the basis $b_0, b_1, c$ of $V_3$), so we can proceed in a similar fashion as above. In particular, if $M(\lambda,\mu)$ is the matrix for $g$, then we obtain $\mu = 0$ from

$$g(b_1) = b_1 + \mu c = \mu b_0 + (1+2\mu)b_1 + \mu b_2,$$

and $\lambda = 0$ from

$$\begin{aligned} g(b_2) = g(c - b_0 - 2b_1) &= g(c) - g(b_0) - 2g(b_1) \\ &= c - (b_0 + \lambda c) - 2b_1 \quad = -\lambda b_0 - 2\lambda b_1 + (1-\lambda)b_2, \end{aligned}$$

in each case using the invariance of $V_{0,3}$ under $g$. Hence $g = e$, as required.

Only a few non-generic cases remain, which are not covered by previous theorems: $p = 3$ for $[4,4,6]$, $[3,6,3]$, $[3,6,4]$, $[3,6,6]$ or $[3,6,\infty]$. But for each of the eleven possibly distinct basic systems here, we easily verify the intersection condition with the help of *GAP*. □

This, then, leaves us with the groups $G = [6,3,m]$. If $m = 3$ or 4, then $G_0$ is spherical, so Theorem 3.1 applies and proves that $G^p$ is a C-group. It remains to investigate the cases $m = 6$ or $\infty$.

**Theorem 3.5** *Let $G = [6,3,m]$ with $m = 6$ or $\infty$. Then for $m = 6$, $G^p$ is a C-group only for $p = 3$; and for $m = \infty$, $G^p$ is a C-group if and only if $p = 3$ or $p \equiv \pm 5 \pmod{12}$.*

**Proof.** As in the previous theorem, we use *GAP* to verify the intersection condition in all cases with $p = 3$. So suppose $p > 3$. We may assume that the nodes 0, 1, 2, 3 have the set of labels 3, 1, 1, 3, or 3, 1, 1, 4 according as $m = 6$ or $\infty$. Since $V$ is non-singular, each isometry of $V$ is uniquely determined by its effect on the singular subspace $V_3$. In particular,

$$G_3^p \cong [6,3]^p \cong T^p \rtimes \langle r_0, r_1 \rangle,$$

with

$$T^p = \langle r_1 r_2 (r_1 r_0)^2, r_2 r_1 (r_0 r_1)^2 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

In the basis $b_1, b_2, c$ of $V_3$, with $c := b_0 + 2b_1 + b_2$ (generating $\operatorname{rad} V_3$), each element of $T^p$, when restricted to $V_3$, is represented by a matrix $M(\lambda, \mu)$ as in (3). In particular,

$$r_2 r_1 (r_0 r_1)^2 \mapsto M(-1, 2), \quad r_1 r_2 (r_1 r_0)^2 \mapsto M(1, 1).$$

Each element in $G_3^p$ is of the form $r_1^i (r_0 r_1)^j t$ with $i = 0, 1$, $j = 0, \ldots, 5$ and $t \in T^p$. Inspection shows that the elements with $i = 0$ have the following matrix representation in the basis $b_1, b_2, c$: if a matrix $M(a, b)$ represents an element $t = t(a, b)$ of $T^p$, then the matrix of $(r_0 r_1)^j t(a, b)$ is obtained from the matrix of $(r_0 r_1)^j$ by simply adding $a$ or $b$, respectively, to the first or second entry in its last row.

Now suppose an element $g \in G_3^p$ leaves $V_{0,3}$ invariant. Multiplying by $r_1$ if need be, we may assume that $i = 0$, that is, $g = (r_0 r_1)^j t$ with $j = 0, \ldots, 5$ and $t = t(a, b) \in T^p$. The invariance of $V_{0,3}$ considerably restricts the possibilities for $j$ and $t$. In fact, inspection of the matrices for $(r_0 r_1)^j t(a, b)$ shows that we must have $(j, a, b) = (0, 0, 0)$, $(1, 1, -1)$, $(2, 1, -2)$, $(3, 0, -2)$, $(4, -1, -1)$ or $(5, -1, 0)$. Bearing in mind an initial multiplication by $r_1$, this leaves 12 choices for $g$.

In particular, since $G_0^p \cap G_3^p$ leaves $V_{0,3}$ invariant, it has order at most 12 and contains $G_{0,3}^p$ as a subgroup of order 6. Comparison of the matrices shows that $(r_0 r_1)^j t(a, b)$ with $(j, a, b) = (1, 1, -1)$, $(3, 0, -2)$ or $(5, -1, 0)$ is not contained in $G_{0,3}^p$. It follows that $G^p$ fails to be a C-group if and only if one of these (and then all) three elements also lies in $G_0^p$. Now consider

$$g := (r_0 r_1)^3 t(0, -2)$$

obtained for $(j, a, b) = (3, 0, -2)$; its matrix in the basis $b_1, b_2, c$ is

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Note that

$$g = (r_0 r_1)^3 (r_1 r_2 (r_1 r_0)^2)^\ell (r_2 r_1 (r_0 r_1)^2)^\ell, \quad 3\ell \equiv -2 \bmod p. \tag{5}$$

First, if $G = [6, 3, 6]$, then $V_0$ is also singular, $d := b_1 + 2b_2 + b_3$ generates $\operatorname{rad} V_0$, and $b_1, b_2, d$ is a basis of $V_0$. Now it is straightforward to check that $g(d) = d$; in fact, the equations

$$g(b_3) \cdot g(b_j) = b_3 \cdot b_j \quad (j = 0, \ldots, 3)$$

have the unique solution $g(b_3) = 2b_1 + 4b_2 + b_3$. We now exploit duality. In fact, by the symmetry of the diagram of $G$, we also have an element

$$g' = (r_3 r_2)^3 t'(0, -2)$$

(say) in $G_0^p$, which is obtained from $G_0^p$ in the same way as $g$ from $G_3^p$. Since $g$ and $g'$ have the same $(4\times 4)$-matrices in the full basis $b_1, b_2, c, d$ of $V$, we must have $g = g'$. Hence $g \in G_0^p \cap G_3^p$ but $g \notin G_{0,3}^p$, so $G^p$ is not a C-group. In particular, from (5) and a similar expression for $g'$ we obtain the relation

$$(r_0r_1)^3(r_1r_2(r_1r_0)^2)^l(r_2r_1(r_0r_1)^2)^l = (r_3r_2)^3(r_2r_1(r_2r_3)^2)^l(r_1r_2(r_3r_2)^2)^l,$$

with $3\ell \equiv -2 \bmod p$.

Finally, we consider $G = [6, 3, \infty]$. Here it is a little easier to work with $r := r_1g$, still with $g$ defined in (5). $G^p$ will be a C-group exactly when $r \notin G_0^p$. But $r$ acts on $V_3$ as a reflection with root $a := b_1 + 2b_2$; thus, since $V_3$ is a singular subspace of $V$, $r$ actually is a reflection in $G^p$. On the other hand, $G_0^p = O_1(V_0) \simeq O_1(3, p, 0)$, for $p > 3$ (see [17, 5.7]); and $a^2 = 3$. Hence, for $p > 3$, $G^p$ is a C-group if and only if 3 is a non-square modulo $p$, i.e. if and only if $p \equiv \pm 5 \pmod{12}$. □

We conclude this section with a look back at the peculiar role of the non-generic prime $p = 3$, which is a frequent irritant in proofs but never an obstruction to polytopality. We have seen for $p = 3$ that when one of $k, l, m$ is 6, different basic systems for the crystallographic group $G = [k, l, m]$ can result in non-isomorphic reduced groups $G^3$. In all cases, however, $G^3$ happens to be a C-group. Although this fact can be checked by hand, we have often resorted to computer verification.

In fact, we can refine our description of the group $G^p$ in singular cases. Indeed, if $n = \dim(V)$ and $r = \dim(\mathrm{rad(V)})$, then it is easy to see that

$$\widehat{O}(V) \simeq \mathbb{Z}_p^{r(n-r)} \rtimes O(V/\mathrm{rad}\ V)\ .$$

In particular, if $n = 4$ and $r = 1, 2$ or 3, then it follows from the above isomorphism that (in non-generic cases), $G^3$ must have order of the form $2^m3^n$ (see [17, p. 301] for the orders of the groups of orthogonal type). Several such groups appear in the following sections.

# 4 Groups $[k, l, m]$ with a spherical or Euclidean subgroup $[l, m]$

In the previous section we determined the crystallographic groups $G = [k, l, m]$ and primes $p \geq 3$ for which the modular reduction $G^p$ is C-group. We now investigate the corresponding regular 4-polytope $\mathcal{P} = \mathcal{P}(G^p)$, whose automorphism group $\Gamma(\mathcal{P})$ is $G^p$. When $p$ is generic for $G$, Lemma 3.1 implies that $G_3^p \simeq [k, l]^p$ and $G_0^p \simeq [l, m]^p$. (In fact, this often holds in non-generic situations, too.) Thus the facets and vertex-figures of $\mathcal{P}(G^p)$ are usually isomorphic to the regular maps associated with the reduced groups $[k, l]^p$ and $[l, m]^p$, respectively, as described in [17, § 5]. These maps are orientable, because the even subgroups of $G_0^p$ and $G_3^p$ have index 2.

The groups $G$ with disconnected diagrams $\Delta(G)$ were described early in Section 3, and the corresponding polytopes are easily classified. Thus we shall assume from here on that $\Delta(G)$ is connected. It then follows from Lemma 3.2 of [17] that $G^p$ acts irreducibly on $V$, so long as $\det(m_{ij}) \not\equiv 0 \pmod p$. (Recall that the $m_{ij}$ are the Cartan integers appearing in [17, §4]. We take this opportunity to correct an oversight in part (a) of [17, Lemma 3.2], where we should have stated that the roots $a_j$ form a basis for $V$. This actually is the case in all applications here and in [17].)

Before proceeding to specifics, we indicate how to decide whether $G^p$ is of orthogonal or spherical type (see Section 2). If, for example, $[k,l]$ is Euclidean, it is very easy to check that a basic translation is a product of two reflections in $[k,l]^p$ and has period $p$. By scanning the parameter $d(G)$ in Table 1 of [17], we see that the product of two reflections has period at most 5 in spherical cases. Thus $G^p$ is of orthogonal type for $p > 5$, and perhaps also for $p = 3$ or 5, cases which can be directly checked in *GAP*. We will employ this sort of analysis without much comment in what follows.

We break the discussion down into three cases according as the (vertex-figure) subgroup $[l,m]$ of $G$ is spherical, Euclidean or hyperbolic, respectively. In this section we treat the groups $G$ with a spherical or Euclidean subgroup $[l,m]$; groups with hyperbolic subgroups $[l,m]$ are studied in the next section. We begin with the spherical case.

### 4.1. Groups with a spherical subgroup $[l,m]$

Let $G = [k,l,m]$ be crystallographic, let $[l,m]$ be spherical, and let $p \geq 3$. Then $G^p$ is a C-group by Theorem 3.1. Moreover, $[l,m]^p \cong [l,m]$, so the vertex-figures of the corresponding polytope $\mathcal{P}$ are isomorphic to Platonic solids $\{l,m\}$. The finite or Euclidean groups $G$ were already discussed in [17, §5-6], so we may assume here that $[k,l]$ is not spherical.

If $[k,l]$ is Euclidean, then $G = [4,4,3]$, $[6,3,3]$ or $[6,3,4]$, and $\mathcal{P}$ is locally toroidal. Recall that a regular 4-polytope is *locally toroidal* if its facets and vertex-figures are toroidal or spherical, but not all spherical.

For $G = [4,4,3]$ with diagram

$$\overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\bullet},$$

the facets of $\mathcal{P}$ are toroidal maps $\{4,4\}_{(p,0)}$ and the vertex-figures are 3-cubes $\{4,3\}$. Now $\mathrm{disc}(V) \sim -1$, independent of $p$, so $\epsilon(V) = 1$ if and only if $p \equiv 1 \bmod 4$. Moreover, $G^p = O_1(V)$ or $O(V)$ according as 2 is a square or non-square, that is, $p \equiv \pm 1 \bmod 8$ or $p \equiv \pm 3 \bmod 8$. Hence $\mathcal{P}$ has automorphism group

$$\Gamma(\mathcal{P}) = G^p = \begin{cases} O_1(4,p,1)\,, & \text{if } p \equiv 1 \pmod 8 \\ O_1(4,p,-1)\,, & \text{if } p \equiv 7 \pmod 8 \\ O(4,p,1)\,, & \text{if } p \equiv 5 \pmod 8 \\ O(4,p,-1)\,, & \text{if } p \equiv 3 \pmod 8. \end{cases} \tag{6}$$

In particular, for $p = 3$ we obtain the group

$$\Gamma(\mathcal{P}) \;=\; O(4,3,-1) \;\cong\; O_1(4,3,-1) \times C_2 \;\cong\; S_6 \times C_2 \tag{7}$$

([17, §3]). In this case,

$$\mathcal{P} = \{\{4,4\}_{(3,0)}, \{4,3\}\}, \tag{8}$$

the universal regular 4-polytope with facets $\{4,4\}_{(3,0)}$ and vertex-figures $\{4,3\}$ (see [13, Thm. 10B3], which also implies that the product in (7) indeed is direct). Recall that $\{\mathcal{P}_1, \mathcal{P}_2\}$

denotes the universal regular $(n+1)$-polytope (if it exists) with facets isomorphic to $\mathcal{P}_1$ and vertex-figures isomorphic to $\mathcal{P}_2$ (see [13, Ch. 4]).

For $G = [6, 3, m]$, with $m = 3$ or 4, we take the diagram

$$\overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet} \qquad \text{or} \qquad \overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{2}{\bullet}, \tag{9}$$

respectively. Now the vertex-figures of $\mathcal{P}$ are tetrahedra $\{3, 3\}$ or octahedra $\{3, 4\}$, respectively, and the facets are toroidal maps $\{6, 3\}_{(p,0)}$ for all $p \geq 3$ (see [17, §5.6]). Note that $\mathrm{disc}(V) \sim -3$, so $V$ is non-singular if $p > 3$, and $\epsilon(V) = 1$ if and only if $-3$ is a square; furthermore, $G^p = O_1(V)$ or $O(V)$ depending on the quadratic character of 3 (or 2 and 3).

In particular, if $G = [6, 3, 3]$ and $p > 3$, we have

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4, p, 1)\ , & \text{if } p \equiv 1 \pmod{12} \\ O_1(4, p, -1)\ , & \text{if } p \equiv 11 \pmod{12} \\ O(4, p, 1)\ , & \text{if } p \equiv 7 \pmod{12} \\ O(4, p, -1)\ , & \text{if } p \equiv 5 \pmod{12}. \end{cases} \tag{10}$$

When $p = 3$ we also obtain a C-group $G^3$, which acts on a singular space $V$ and has order 1296. The corresponding subgroup $G^3_3$ has order 108, and the facets are toroidal maps $\{6, 3\}_{(3,0)}$ (see [17, § 5.6]). Note that Lemma 3.1 does not apply; indeed the subdiagram on nodes $0, 1, 2$ of $\Delta(G)$ defines the rank 3 group $H^3$, of order just 36, for the map $\{6, 3\}_{(1,1)}$. In a sense, the subspace $V_3$ of $V$ cannot fully represent the structure of the facet. The alternative diagram

$$\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{3}{\bullet}$$

represents the same group, but without this deficiency. By comparing group orders, we find that we have the universal polytope

$$\mathcal{P} = \{\{6, 3\}_{(3,0)}, \{3, 3\}\} \tag{11}$$

with

$$G^3 = \Gamma(\mathcal{P}) \cong [1\,1\,2]^3 \rtimes C_2,$$

where $[1\,1\,2]^3$ is a certain unitary reflection group in $\mathbb{C}^4$ (see [13, Thm. 11B5]). (The superscript on $[1\,1\,2]^3$ signifies a group relation, not reduction modulo 3).

When $G = [6, 3, 4]$ and $p > 3$, we similarly find that

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4, p, 1)\ , & \text{if } p \equiv 1 \pmod{24} \\ O_1(4, p, -1)\ , & \text{if } p \equiv 23 \pmod{24} \\ O(4, p, 1)\ , & \text{if } p \equiv 7, 13, 19 \pmod{24} \\ O(4, p, -1)\ , & \text{if } p \equiv 5, 11, 17 \pmod{24}. \end{cases} \tag{12}$$

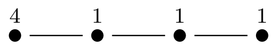
For $p = 3$ and for either of the diagrams

$$\overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{2}{\bullet} \qquad \text{or} \qquad \overset{1}{\bullet}\!-\!\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\!-\!\overset{6}{\bullet}\ , \tag{13}$$

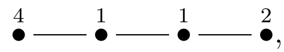
$G^3$ has order 2592; we obtain the same polytope $\mathcal{P}$ of type $\{\{6, 3\}_{(3,0)}, \{3, 4\}\}$. However, $\mathcal{P}$ is not the (infinite!) universal polytope of this type [13, Thm. 11B5].

We now consider the case that the subgroup $[k, l]$ is hyperbolic (and $[l, m]$ is still spherical). The corresponding groups are $G = [\infty, 3, 3]$, $[\infty, 3, 4]$, $[6, 4, 3]$ or $[\infty, 4, 3]$.

For $G = [\infty, 3, m]$, with $m = 3$ or 4, we employ the diagram




or




respectively. In either case, $G_3^p = [\infty, 3]^p$, so that the facets of $\mathcal{P}$ are the regular maps $\mathcal{M}_{p,3}$ of type $\{p, 3\}$ described in [17, §5.7] (see also [12]); the vertex-figures are tetrahedra $\{3, 3\}$ or octahedra $\{3, 4\}$, respectively. In particular, if $p = 3$, we obtain the 4-simplex $\{3, 3, 3\}$ or 4-cross-polytope $\{3, 3, 4\}$, respectively. Now $\mathrm{disc}(V) \sim -1$ or $-2$, respectively, independent of $p$; and $G^p = O_1(V)$ unless $m = p = 3$, or $m = 4$ and 2 is a non-square. Thus, when $m = 3$ we have

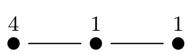
$$\Gamma(\mathcal{P}) = \begin{cases} S_5\ , & \text{if } p = 3 \\ O_1(4, p, 1)\ , & \text{if } p \equiv 1 \pmod 4 \\ O_1(4, p, -1)\ , & \text{if } p \equiv 3 \pmod 4,\ (p \neq 3), \end{cases} \tag{14}$$

and when $m = 4$ and $p > 3$, we have

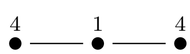
$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4, p, 1)\ , & \text{if } p \equiv 1 \pmod 8 \\ O_1(4, p, -1)\ , & \text{if } p \equiv 7 \pmod 8 \\ O(4, p, 1)\ , & \text{if } p \equiv 3 \pmod 8 \\ O(4, p, -1)\ , & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{15}$$

(When $m = 4$ and $p = 3$, the group is $B_4$, which is of index 3 in $O(4, 3, 1) \cong F_4$.) For both $m = 3$ and $m = 4$, and all $p \geq 3$, we have $G_3^p = O_1(3, p, 0)$, of order $p(p^2 - 1)$. For $p = 5$ the facets are isomorphic to dodecahedra $\{5, 3\} = \mathcal{M}_{5,3}$; and for $p = 7$ they are isomorphic to Klein's map $\{7, 3\}_8 = \mathcal{M}_{7,3}$, of genus 3 (see [6, § 8.6]).

With $m = 3$, we here encounter, for the first time, the classical regular 4-polytopes of 'pentagonal type'. For future reference, let us denote $\mathcal{P}$ by $\mathcal{C}_{p,3,3}$. Likewise, as suggested above, it suits us to denote the regular maps arising from




and




by $\mathcal{M}_{p,3}$ and $\mathcal{M}_{p,p}$, respectively (see [17, §5.7]).

Thus, for $p = 5$, $\mathcal{C}_{5,3,3}$ is the 120-cell $\{5, 3, 3\}$ (isomorphic to both the convex regular polytope of this type and to the star-polytope $\{\frac{5}{2}, 3, 3\}$; see [13, 7D]). For $p = 7$, we find that $\mathcal{C}_{7,3,3}$ is the universal regular polytope

$$\{\{7, 3\}_8, \{3, 3\}\},$$

first described in [16]. Rephrasing the conjectured presentation in [16, 3.1], we now offer

**Conjecture 1**: For primes $p \geq 3$, $\mathcal{C}_{p,3,3}$ is the universal polytope

$$\{\, \mathcal{M}_{p,3}\, ,\ \{3, 3\}\, \}\ .$$

(This has been verified by coset enumeration for $p \leq 31$. We cannot, in general, expect the facets to be determined in some elegant way, say by their Petrie polygons or holes.)

Moving on to the next class of groups, let $G = [k, 4, 3]$, with $k = 6$ or $\infty$, and let the diagram be

$$\overset{3}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \qquad \text{or} \qquad \overset{4}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet},$$

respectively. Then $G_3^p = [k, 4]^p$, so the facets are maps of type $\{6, 4\}$ or $\{p, 4\}$, respectively; see [17, §5.8-5.9] for details. Of course, the vertex-figures are cubes $\{4, 3\}$. Since $\mathrm{disc}(V) \sim -15$ or $-2$, respectively, $V$ is singular if $k = 6$ and $p = 3$ or 5.

Suppose $k = 6$. For $p = 3$, the group $G^3$ has order 2592 and appeared earlier in (13) with different generators. In this new guise, $G^3$ is the group of the universal polytope of type

$$\{\{6, 4\}_4\,, \{4, 3\}\}\ . \tag{16}$$

Indeed, for $p = 3$, each basic system gives just this polytope, whose facets are isomorphic to the map $\{6, 4\}_4$ (the Petrial of the toroidal map $\{4, 4\}_{(3,3)}$). (See [13, 7B2] for a description of $\mathcal{Q}^\pi$, the *Petrial* or *Petrie dual* of a map $\mathcal{Q}$.)

For $p = 5$, the facets are Coxeter-Petrie polyhedra $\{6, 4\,|\,3\}$ (see [6]). Since $G^5$ has order 30000, the corresponding polytope has 125 of these facets and 625 vertices.

Otherwise, if $G = [6, 4, 3]$ and $p > 5$, then we have

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4, p, 1)\ , & \text{if } p \equiv 1, 23, 47, 49 \pmod{120} \\ O_1(4, p, -1)\ , & \text{if } p \equiv 71, 73, 97, 119 \pmod{120} \\ O(4, p, 1)\ , & \text{if } p \equiv 17, 19, 31, 53, 61, 77, 79, 83, 91, 107, 109, 113 \pmod{120} \\ O(4, p, -1)\ , & \text{if } p \equiv 7, 11, 13, 29, 37, 41, 43, 59, 67, 89, 101, 103 \pmod{120}. \end{cases}$$

Finally, when $G = [\infty, 4, 3]$, the polytopes are of type $\{p, 4, 3\}$ and their groups are those described in (15) above (now allowing $p = 3$), with new generators of course. For $p = 3$ we obtain the 24-cell $\{3, 4, 3\}$. When $p = 5$, the polytope has facets isomorphic to Gordan's map $\{5, 4\}_6$ of genus 4 (see [6]).

### 4.2. Groups with a Euclidean subgroup $[l, m]$

Next we consider the crystallographic groups $G = [k, l, m]$ with a Euclidean subgroup $[l, m]$. The groups with a spherical subgroup $[k, l]$ have already been discussed in the dual setting, so we may further assume that $[k, l]$ is Euclidean or hyperbolic. By Theorems 3.4 and 3.5, $G^p$ is a C-group for all $p \geq 3$, with these exceptions: for $G = [6, 3, 6]$ only $p = 3$ is acceptible, and for $G = [\infty, 3, 6]$, only $p = 3$ and $p \equiv \pm 5 \pmod{12}$. The corresponding regular 4-polytopes $\mathcal{P}$ all have toroidal vertex-figures.

For the group $G = [k, 4, 4]$, with $k = 4$, 6 or $\infty$, we take the diagram

$$\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}, \qquad \overset{3}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \qquad \text{or} \qquad \overset{4}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet},$$

respectively. Their polytopes $\mathcal{P}$ have toroidal vertex-figures $\{4, 4\}_{(p,0)}$ and facets isomorphic to the maps of type $\{k, 4\}$ described in [17, §5.5,5.8,5.9]. Now $\mathrm{disc}(V) \sim -1$ in each case, except when $k = 6$ and $p = 3$; in this latter case, $V$ is singular (for each admissible diagram).

When $G = [4, 4, 4]$, the polytope is self-dual and its facets are also maps $\{4, 4\}_{(p,0)}$. The group $\Gamma(\mathcal{P})$ is the same as in (6); that is, we have

$$[4, 4, 4]^p \cong [4, 4, 3]^p$$

for each $p \geq 3$. In fact, the Coxeter group $[4,4,4]$ is known to be a subgroup of index 3 in $[4,4,3]$, and under the modular reduction this index collapses to 1; see, for example, [13, §10E], which also explains the corresponding relationship between the polytopes. In particular, if $[4,4,3] = \langle r_0, \ldots, r_3 \rangle$ (say), then we can identify $[4,4,4]$ with the subgroup $\langle r_1, r_0, r_2r_1r_2, r_3 \rangle$, with the generators taken in this order; modulo $p$, this subgroup is the group itself.

For $p = 3$ we obtain the universal regular polytope

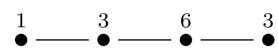
$$\{\{4,4\}_{(3,0)}, \{4,4\}_{(3,0)}\}$$

with group $S_6 \times C_2$, which is related to the polytope in (8) (see [13, 10E6 and Thm. 10C12]).

For $G = [6,4,4]$ and $p > 3$, we have

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4,p,1)\ , & \text{if } p \equiv 1 \pmod{24} \\ O_1(4,p,-1)\ , & \text{if } p \equiv 23 \pmod{24} \\ O(4,p,1)\ , & \text{if } p \equiv 5, 13, 17 \pmod{24} \\ O(4,p,-1)\ , & \text{if } p \equiv 7, 11, 19 \pmod{24}. \end{cases}$$

When $p = 5$ the facets of $\mathcal{P}$ are Coxeter-Petrie polyhedra $\{6,4\,|\,3\}$ (see [6]). For $p = 3$, the facets are maps $\{6,4\}_4$, the vertex-figures are maps of type $\{4,4\}_{(3,0)}$, and the diagram given above yields a group $G^3$ with order 1296. However, the alternate diagram




results instead in a group $G^3$ of order 3888. We thus obtain different polytopes with the same local structure.

When $G = [\infty,4,4]$, the polytopes are of type $\{p,4,4\}$, and although differently generated, the group $\Gamma(\mathcal{P})$ is again described by (6) when $p \geq 3$. For $p = 3$ we obtain the universal regular polytope

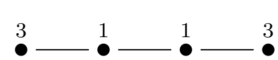
$$\{\{3,4\}, \{4,4\}_{(3,0)}\},$$

the dual of (8), with group $S_6 \times C_2$. For $p = 5$ the polytope $\mathcal{P}$ has facets isomorphic to Gordan's map $\{5,4\}_6$ of genus 4.

Next we investigate the groups $[k,3,6]$. By Theorem 3.5, when $k = 6$ we need only consider $p = 3$. The diagrams

(17)

give isomorphic groups $G^3$ of order 1944, yet *distinct* polytopes of type $\{\{6,3\}_{(3,0)}, \{3,6\}_{(3,0)}\}$ (with the same type of facets and vertex-figures, the latter type being the dual of the former). However, only the first of these two 4-polytopes is in itself self-dual. The other essentially distinct diagram



(18)

yields a group $G^3$ of order 216 and the universal self-dual polytope of type

$$\{\{6,3\}_{(1,1)},\{3,6\}_{(1,1)}\} \tag{19}$$

(see [13, 11C8]).

Let us now turn to the case $k=\infty$, with diagram

$$\overset{4}{\bullet}\text{---}\overset{1}{\bullet}\text{---}\overset{1}{\bullet}\text{---}\overset{3}{\bullet}\,. \tag{20}$$

Now $\mathrm{disc}(V)\sim -3$, so $V$ is singular if $p=3$. When $p>3$, the vertex-figures of $\mathcal{P}$ are toroidal maps $\{3,6\}_{(p,0)}$ (see [17, §5.6]), and the facets are the maps of type $\{p,3\}$ described in [17, §5.7]. Keeping in mind Theorem 3.5, we have just two possibilities:

$$\Gamma(\mathcal{P})=\begin{cases} O(4,p,1)\,, & \text{if } p\equiv 7 \pmod{12} \\ O(4,p,-1)\,, & \text{if } p\equiv 5 \pmod{12}. \end{cases} \tag{21}$$

For $p=5$ the facets of $\mathcal{P}$ are dodecahedra $\{5,3\}$, and for $p=7$ they are isomorphic to Klein's map $\{7,3\}_8$ of genus 3. When $p=3$, both the diagram in (20) and its alternative yield the universal regular polytope

$$\mathcal{P}=\{\{3,3\},\{3,6\}_{(3,0)}\},$$

namely the dual of (11). Again the group is $[1\,1\,2]^3\rtimes C_2$ of order 1296.

It remains to study the groups $G=[k,6,3]$ with $k=3$, 4, 6 or $\infty$, where we take the diagrams

$$\overset{1}{\bullet}\text{---}\overset{1}{\bullet}\text{---}\overset{3}{\bullet}\text{---}\overset{3}{\bullet},\quad \overset{2}{\bullet}\text{---}\overset{1}{\bullet}\text{---}\overset{3}{\bullet}\text{---}\overset{3}{\bullet},\quad \overset{3}{\bullet}\text{---}\overset{1}{\bullet}\text{---}\overset{3}{\bullet}\text{---}\overset{3}{\bullet}\quad\text{or}\quad \overset{4}{\bullet}\text{---}\overset{1}{\bullet}\text{---}\overset{3}{\bullet}\text{---}\overset{3}{\bullet},$$

respectively. When $p>3$ the vertex-figures of $\mathcal{P}$ are toroidal maps $\{6,3\}_{(p,0)}$, and the facets are the maps of type $\{k,6\}$ or $\{p,6\}$, for $k=3,4,6$ or $k=\infty$, as described in [17, §5.6,5.8,5.10,5.11]. Since $\mathrm{disc}(V)\sim -3$ in each case, $V$ is singular if $p=3$.

When $G=[3,6,3]$ and $p>3$, the polytope $\mathcal{P}$ is self-dual and its facets are toroidal maps $\{3,6\}_{(p,0)}$. The group $\Gamma(\mathcal{P})$ is the same as in (10); that is, we have

$$[3,6,3]^p\cong[6,3,3]^p \tag{22}$$

for $p>3$. In fact, the Coxeter group $[3,6,3]$ is a subgroup of index 4 in $[6,3,3]$, and under reduction modulo $p$ this index becomes 1, again so long as $p>3$; see, for example, [13, Sect. 11G,11H], which also describes the relationship between the polytopes. In particular, if $[6,3,3]=\langle r_0,\ldots,r_3\rangle$ (say), then $[3,6,3]$ can be identified with the subgroup $\langle r_0,r_1r_0r_1,r_2,r_3\rangle$, with the generators taken in this order; modulo $p$, this is the whole group.

When $p=3$, we obtain the (combinatorially flat) universal regular polytope

$$\mathcal{P}=\{\{3,6\}_{(1,1)},\{6,3\}_{(3,0)}\} \tag{23}$$

with group

$$\Gamma(\mathcal{P})\cong[1\,1\,1]^3\rtimes S_3,$$

of order 324, where $[1\,1\,1]^3$ denotes a certain unitary reflection group in $\mathbb{C}^3$ (see [13, Thm. 11E7]). Of course, $\mathcal{P}$ is not self-dual, although the alternative diagram

$$\overset{3}{\bullet}\!-\!\!-\!\!-\overset{3}{\bullet}\!-\!\!-\!\!-\overset{1}{\bullet}\!-\!\!-\!\!-\overset{1}{\bullet}$$

does yield the dual. Note also that the isomorphism in (22) must fail for $p = 3$, regardless of choice of diagrams. Indeed, $[3, 6, 3]^3$ does have index 4 in $[6, 3, 3]^3$.

Incidentally, we also have

$$[6, 3, 6]^p \cong [6, 3, 3]^p$$

for $p > 3$; but then, as we have seen, $[6, 3, 6]^p$ is not a C-group with its natural generators. (In this context, note that [13, Thm. 11H10] is incorrect for the parameter vectors $(s, 0)$ with $s$ not divisible by 3, so does not yield any polytopes of type $\{6, 3, 6\}$.)

If $G = [4, 6, 3]$ and $p > 3$, then $\Gamma(\mathcal{P})$ is the group described earlier in (12), though again with new generators. For $p = 5$ the facets of $\mathcal{P}$ are Coxeter-Petrie polyhedra $\{4, 6\,|\,3\}$ (see [6]). For $p = 3$, the facets are maps $\{4, 6\}_4$; but the diagrams

$$\overset{2}{\bullet}\!-\!\!-\!\!-\overset{1}{\bullet}\!-\!\!-\!\!-\overset{3}{\bullet}\!-\!\!-\!\!-\overset{3}{\bullet}\;,\quad \overset{6}{\bullet}\!-\!\!-\!\!-\overset{3}{\bullet}\!-\!\!-\!\!-\overset{1}{\bullet}\!-\!\!-\!\!-\overset{1}{\bullet}$$

lead respectively to vertex-figures $\{6, 3\}_{(3,0)}$ and group order 3888, or to vertex-figures $\{6, 3\}_{(1,1)}$ and group order 432.

When $G = [6, 6, 3]$ and $p > 3$, the facets of $\mathcal{P}$ are self-dual maps of type $\{6, 6\}$, with group $O_1(3, p, 0)$ or $O(3, p, 0)$ according as $p \equiv \pm 1 \bmod 12$ or $p \not\equiv \pm 1 \bmod 12$. In fact, $\Gamma(\mathcal{P})$ is a newly generated version of the group described in (10). When $p = 3$, we find that the type of facets depends on the diagram chosen; they are isomorphic to the Petrials of two maps first described by Sherk in [18] : $\{6, 6\}_{(1,1)}$, with group of order 72 and genus 4, and $\{6, 6\}_{(3,0)}$, with group 216 and genus 10. Note that $\{6, 6\}_{(1,1)} \cong \{6, 6 \mid 2\}$ (see [6, § 8.5]). The various diagrams yield four distinct universal polytopes, summarized in the chart below:

| $\Delta(G)$ | $\lvert G^3 \rvert$ | The Universal Polytope |
|---|---|---|
| $\overset{1}{\bullet}-\overset{3}{\bullet}-\overset{1}{\bullet}-\overset{1}{\bullet}$ | 216 | $\{\{6, 6\}^{\pi}_{(1,1)}\,, \{6, 3\}_{(1,1)}\}$ |
| $\overset{9}{\bullet}-\overset{3}{\bullet}-\overset{1}{\bullet}-\overset{1}{\bullet}$ | 648 | $\{\{6, 6\}^{\pi}_{(3,0)}\,, \{6, 3\}_{(1,1)}\}$ |
| $\overset{3}{\bullet}-\overset{1}{\bullet}-\overset{3}{\bullet}-\overset{3}{\bullet}$ | 648 | $\{\{6, 6\}^{\pi}_{(1,1)}\,, \{6, 3\}_{(3,0)}\}$ |
| $\overset{1}{\bullet}-\overset{3}{\bullet}-\overset{9}{\bullet}-\overset{9}{\bullet}$ | 5832 | $\{\{6, 6\}^{\pi}_{(3,0)}\,, \{6, 3\}_{(3,0)}\}$ |

(24)

(The two groups of order 648 are isomorphic, though differently generated.)

Finally, if $G = [\infty, 6, 3]$ and $p > 3$, we have the same group as in (10), now yielding a polytope $\mathcal{P}$ of type $\{p, 6, 3\}$. In particular, if $p = 5$, the facets are maps $\{5, 6\}_4$, with group $S_5 \times C_2$. When $p = 3$, we again obtain the universal polytope (23), or its dual, depending on choice of diagram.

## 5 Groups $[k, l, m]$ with a hyperbolic subgroup $[l, m]$

We now consider the crystallographic groups $G = [k, l, m]$, for which $[l, m]$ is of hyperbolic type. The groups with a spherical or Euclidean subgroup $[k, l]$ have already occurred in the previous section in the dual setting, so we may assume that $[k, l]$ is also of hyperbolic type. Then, except possibly for small primes $p$, all three subspaces $V$, $V_0$ and $V_3$ are non-singular, but $V_{03}$ is non-singular only if $l \neq \infty$. By Corollary 3.2, if $l = \infty$, then $G^p$ is a C-group for $p \geq 3$. On the other hand, by Corollary 3.1, if $l \neq \infty$, then $G^p$ can only be a C-group if $p$ is small.

**5.1. The groups** $[k, \infty, m]$

We begin by discussing the groups $G = [k, l, m]$ with $l = \infty$ and with diagrams specified below. Inspection of the discriminants shows that $V$ is non-singular, except when

$$(k, m, p) \text{ or } (m, k, p) = \left\{ \begin{array}{l} (3,3,7), (4,3,5), (6,3,3), (6,3,13), (4,4,3), (6,4,3), (6,4,7), \\ (6,6,3), (6,6,5), (\infty,6,3). \end{array} \right.$$

Moreover, $V_0$ and $V_3$ are non-singular except occasionally when $p = 3$. If all three spaces $V$, $V_0$, $V_3$ are non-singular, then the vertex-figures of the polytope $\mathcal{P}$ with group $G^p$ are the maps of type $\{p, m\}$ or $\{p, p\}$ associated with $[\infty, m]^p$ (see [17, 5.7,5.9,5.11,5.12]), and the facets are duals of such maps (with $m$ replaced by $k$).

When $G = [k, \infty, 3]$, with $k = 3$, 4, 6 or $\infty$, we take the diagram

$\overset{1}{\bullet}$ — $\overset{1}{\bullet}$ — $\overset{4}{\bullet}$ — $\overset{4}{\bullet}$, $\overset{2}{\bullet}$ — $\overset{1}{\bullet}$ — $\overset{4}{\bullet}$ — $\overset{4}{\bullet}$, $\overset{3}{\bullet}$ — $\overset{1}{\bullet}$ — $\overset{4}{\bullet}$ — $\overset{4}{\bullet}$ or $\overset{4}{\bullet}$ — $\overset{1}{\bullet}$ — $\overset{4}{\bullet}$ — $\overset{4}{\bullet}$,

respectively. Then $\mathrm{disc}(V) \sim -7$, $-5$, $-39$ or $-1$, respectively, so $V$ may be singular for $p = 3$, 5, 7 or 13.

From $[3, \infty, 3]$ we obtain a self-dual regular 4-polytope $\mathcal{P}$ of type $\{3, p, 3\}$ for all $p \geq 3$. In the non-singular cases with $p \neq 3, 7$ we have

$$\Gamma(\mathcal{P}) = \left\{ \begin{array}{ll} O_1(4, p, 1)\ , & \text{if } p \equiv 1, 9, 11, 15, 23, 25 \pmod{28} \\ O_1(4, p, -1)\ , & \text{if } p \equiv 3, 5, 13, 17, 19, 27 \pmod{28}. \end{array} \right. \tag{25}$$

In particular, if $p = 5$, then $\Gamma(\mathcal{P}) = O_1(4, 5, -1)$, and $\mathcal{P}$ has 130 icosahedral facets and 130 dodecahedral vertex-figures. For $p = 7$ the $7^3$ facets are maps $\{3, 7\}_8$, and so dually the vertex-figures are maps $\{7, 3\}_8$. Of course, for $p = 3$ we get the 4-simplex with group $S_5$.

The group $G = [4, \infty, 3]$, with $p > 5$, yields regular polytopes $\mathcal{P}$ of type $\{4, p, 3\}$ with group

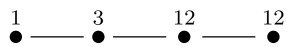
$$\Gamma(\mathcal{P}) = \left\{ \begin{array}{ll} O_1(4, p, 1)\ , & \text{if } p \equiv 1, 7, 9, 23 \pmod{40} \\ O_1(4, p, -1)\ , & \text{if } p \equiv 17, 31, 33, 39 \pmod{40} \\ O(4, p, 1)\ , & \text{if } p \equiv 3, 21, 27, 29 \pmod{40} \\ O(4, p, -1)\ , & \text{if } p \equiv 11, 13, 19, 37 \pmod{40}. \end{array} \right. \tag{26}$$

For $p = 3$ we obtain the 4-cube $\{4,3,3\}$. When $p = 7$, the vertex-figures of $\mathcal{P}$ are isomorphic to Klein's map $\{7,3\}_8$ of genus 3. Lastly, when $p = 5$ we obtain a polytope with 125 facets of type $\{4,5\}_6$ and with 250 dodecahedral vertex-figures.

When $G = [6,\infty,3]$, with $p \neq 3, 13$, we obtain regular polytopes $\mathcal{P}$ of type $\{6,p,3\}$ with group

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4,p,1)\ , & \text{if } p \equiv 1,11,25,47,49,59,61,71,83,119,121,133 \pmod{156} \\ O_1(4,p,-1)\ , & \text{if } p \equiv 23,35,37,73,85,95,97,107,109,131,145,155 \pmod{156} \\ O(4,p,1)\ , & \text{if } p \equiv 5,41,43,55,79,89,103,125,127,137,139,149 \pmod{156} \\ O(4,p,-1)\ , & \text{otherwise.} \end{cases} \tag{27}$$

For $p = 5$ we have a polytope with facets isomorphic to $\{6,5\}_4$ and with dodecahedral vertex-figures. When $p = 3$, both the original diagram and the alternative



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yield the universal polytope described earlier in (11). Finally, even though $V$ is singular for $p = 13$, we still obtain a C-group $G^{13}$ of order $13^4 \cdot 7 \cdot 4!$.

Suppose now that $G = [\infty,\infty,3]$. For $p \geq 3$, we obtain a regular polytope of type $\{p,p,3\}$, which we shall denote $\mathcal{C}_{p,p,3}$, again to suggest connections with the classical star-polytopes. To explain this, we resurrect the earlier group $[\infty,3,3]$, now with generators $\langle s_0, s_1, s_2, s_3\rangle$, to which we apply the *mixing operation*

$$(s_0, s_1, s_2, s_3) \to (s_0, s_1 s_0 s_1, s_2, s_3) =: (r_0, r_1, r_2, r_3)\ .$$

From this construction we find that the reflection group $G = [\infty,\infty,3] := \langle r_0, r_1, r_2, r_3\rangle$ is a subgroup of $[\infty,3,3]$, with index 4 in characteristic 0. (See [14, Lemma 2.1] or [15] for an equivalent geometric dissection.) Now, as explained in the proof of [13, Thm. 7D16(a)], the index collapses to 1 in characteristic $p \geq 3$. Thus $G^p$ is the group described in (14), though with the new generators.

Again for $p = 3$, $\mathcal{C}_{3,3,3}$ is the 4-simplex with group $S_5$. If $p = 5$, then

$$\Gamma(\mathcal{C}_{5,5,3}) = O_1(4,5,1) \cong H_4,$$

which is isomorphic to the symmetry group of the 120-cell $\{5,3,3\}$ (or any regular star-polytope in Euclidean 4-space associated with it). The facets of $\mathcal{C}_{5,5,3}$ are maps $\{5,5\,|\,3\}$ ( $= \mathcal{M}_{5,5}$) of genus 4, which can be metrically realized in Euclidean 3-space by the small stellated dodecahedron $\{\frac{5}{2},5\}$; the vertex-figures of $\mathcal{C}_{5,5,3}$ are dodecahedra $\{5,3\}$ ( $= \mathcal{M}_{5,3}$). We know from [13, Thm. 7D16] (or [10]) that the universal abstract polytope $\{\{5,5\,|\,3\},\{5,3\}\}$ is isomorphic to the regular star-polytope $\{\frac{5}{2},5,3\}$ in Euclidean 4-space. Therefore, since they have the same group, we must have

$$\mathcal{C}_{5,5,3} \cong \{\{5,5\,|\,3\},\{5,3\}\} \cong \{\tfrac{5}{2},5,3\}.$$

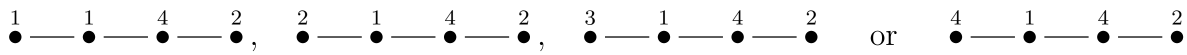
More generally, our earlier conjecture can be restated as:

**Conjecture 2**: For primes $p \geq 3$, $\mathcal{C}_{p,p,3}$ is the universal polytope of type

$$\{\,\mathcal{M}_{p,p}\,,\,\mathcal{M}_{p,3}\,\}\ .$$

Geometrically, it is useful to view this mixing of new generators for the same group as a *stellation* of the polygonal faces of $\mathcal{C}_{p,3,3}$, thereby yielding $\mathcal{C}_{p,p,3}$. In a sense, the two polytopes share the same edges, which are, however, allocated differently to form the $p$-gons in $\mathcal{C}_{p,p,3}$. For $p > 3$, the two polytopes have equal numbers of facets, polygons and edges, though rather different numbers of vertices.

For the groups $G = [k, \infty, 4]$, with $k = 3$, 4, 6 or $\infty$, our diagrams are




respectively. Now $\mathrm{disc}(V) = -5$, $-3$, $-21$ or $-2$, respectively, so $V$ may be singular for the primes $p = 3$, 5 or 7. We have already discussed the case $k = 3$ in the dual setting.

From $[4, \infty, 4]$, with $p > 3$, we obtain a self-dual regular 4-polytope $\mathcal{P}$ of type $\{4, p, 4\}$ with group

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4,p,1)\ , & \text{if } p \equiv 1, 7 \pmod{24} \\ O_1(4,p,-1)\ , & \text{if } p \equiv 17, 23 \pmod{24} \\ O(4,p,1)\ , & \text{if } p \equiv 13, 19 \pmod{24} \\ O(4,p,-1)\ , & \text{if } p \equiv 5, 11 \pmod{24}. \end{cases} \tag{28}$$

For $p = 5$, the polytope $\mathcal{P}$ has facets isomorphic to $\{4, 5\}_6$ and vertex-figures isomorphic to $\{5, 4\}_6$. When $p = 3$, the space $V$ is singular, and we obtain a regular toroid $\{4, 3, 4\}_{(3,0,0)}$ of rank 4 with automorphism group $G^3 \cong C_3^3 \rtimes [3, 4]$.

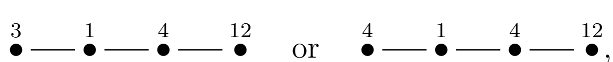
The group $G = [6, \infty, 4]$, with $p \neq 3, 7$, yields regular polytopes $\mathcal{P}$ of type $\{6, p, 4\}$ with group

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4,p,1)\ , & \text{if } p \equiv 1, 23, 25, 71, 95, 121 \pmod{168} \\ O_1(4,p,-1)\ , & \text{if } p \equiv 47, 73, 97, 143, 145, 167 \pmod{168} \\ O(4,p,1)\ , & \text{if } p \equiv \begin{cases} 5, 11, 17, 19, 31, 37, 41, 55, 85, 89, 101, 103, \\ 107, 109, 115, 125, 139, 155 \pmod{168} \end{cases} \\ O(4,p,-1)\ , & \text{otherwise.} \end{cases} \tag{29}$$

For $p = 5$ we have a polytope with facets isomorphic to $\{6, 5\}_4$ and vertex-figures isomorphic to $\{5, 4\}_6$. When $p = 3$ both the given diagram and its alternate yield groups $G^3$ which are similar in $GL_4(\mathbb{Z}_3)$ to the group of order 2592 defined by the diagrams in (13). We get the same non-universal polytope of type $\{\{6, 3\}_{(3,0)}, \{3, 4\}\}$. Lastly, in the singular case with $p = 7$, we still obtain a C-group of order $2^5 \cdot 3 \cdot 7^4$.

When $G = [\infty, \infty, 4]$, with $p > 3$, the corresponding polytopes $\mathcal{P}$ are of type $\{p, p, 4\}$ and their groups are once more newly generated versions of those described in (15). For $p = 3$ we obtain the cross-polytope $\{3, 3, 4\}$, whose group has index 3 in $O(4, 3, 1) \cong F_4$. For $p = 5$, the polytope $\mathcal{P}$ has facets isomorphic to $\{5, 5\,|\,3\} \cong \{\frac{5}{2}, 5\}$ and vertex-figures isomorphic to $\{5, 4\}_6$.

We now turn to the groups $G = [k, \infty, 6]$, with $k = 3$, 4, 6 or $\infty$. We have already investigated the cases $k = 3$ or 4 in the dual setting, so we need only consider the diagrams

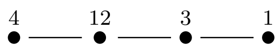



for which $\mathrm{disc}(V) \sim -15$ or $-3$, respectively.

From $G = [6, \infty, 6]$, with $p \neq 3, 5$, we obtain self-dual regular polytopes of type $\{6, p, 6\}$ with group

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(4,p,1)\ , & \text{if } p \equiv 1, 23, 47, 49 \pmod{60} \\ O_1(4,p,-1)\ , & \text{if } p \equiv 11, 13, 37, 59 \pmod{60} \\ O(4,p,1)\ , & \text{if } p \equiv 17, 19, 31, 53 \pmod{60} \\ O(4,p,-1)\ , & \text{otherwise.} \end{cases} \tag{30}$$

For $p = 3$ or 5, each possible diagram $\Delta(G)$ gives a singular space $V$, though we still obtain polytopes. When $p = 5$, $\mathcal{P}$ is self-dual with facets $\{6, 5\}_4$ and vertex-figures $\{5, 6\}_4$ and a group of order $2^4 \cdot 3 \cdot 5^4$. For $p = 3$, the essentially distinct diagrams



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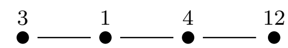
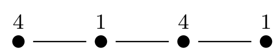
and



describe exactly the same groups $G^3$ as the diagrams displayed in (17) and (18). Thus only the first and last diagrams yield self-dual polytopes.

When $G = [\infty, \infty, 6]$ and $p > 3$, the polytope $\mathcal{P}$ is of type $\{p, p, 6\}$ and its group is given by (10). For $p = 5$, $\mathcal{P}$ has facets $\{5, 5 \,|\, 3\} \cong \{\frac{5}{2}, 5\}$ and vertex-figures $\{5, 6\}_4$, and $\Gamma(\mathcal{P}) = O(4, 5, -1)$. For $p = 3$ each admissible diagram yields the universal polytope $\{\{3, 3\}, \{3, 6\}_{(3,0)}\}$ (dual to the polytope described in (11)).

It remains to investigate the group $G = [\infty, \infty, \infty]$, with diagram



.

Then $\mathrm{disc}(V) \sim -1$, independent of $p$. In each case we obtain a regular polytope $\mathcal{C}_{p,p,p}$ of type $\{p, p, p\}$, again related to a classical star-polytope. Note that $\mathcal{C}_{p,p,p}$, along with its facets and vertex-figures, is self-dual. Taking an approach similar to that for $\mathcal{C}_{p,p,3}$, we begin with the group $[\infty, 3, 3] = \langle s_0, s_1, s_2, s_3 \rangle$, to which we apply the mixing operation

$$(s_0, s_1, s_2, s_3) \to (s_1, s_0, s_2 s_1 s_0 s_1 s_2, s_3) =: (r_0, r_1, r_2, r_3)\ .$$

Then in characteristic 0, $G = [\infty, \infty, \infty] := \langle r_0, r_1, r_2, r_3 \rangle$ is a subgroup of index 6 in $[\infty, 3, 3]$ (see [15, part (6)]). Again following the proof of [13, Thm. 7D16(c)], we find that the index collapses to 1 in characteristic $p \geq 3$. Once more $G^p$ is the group described in (14), though with the new generators.

For $p = 3$ we again get the regular simplex $\{3, 3, 3\}$. If $p = 5$, the facets and vertex-figures are maps $\{5, 5 | 3\}$ $(= \mathcal{M}_{p,p})$ of genus 4, and

$$\Gamma(\mathcal{C}_{5,5,5}) = O_1(4, 5, 1) \cong H_4.$$

We know from [13, Thm. 7D16] (or [10]) that the universal abstract regular 4-polytope $\{\{5, 5 | 3\}, \{5, 5 | 3\}\}$ is isomorphic to the regular star-polytope $\{\frac{5}{2}, 5, \frac{5}{2}\}$ in Euclidean 4-space, which also has group $H_4$. Then it follows that

$$\mathcal{C}_{5,5,5} \cong \{\{5, 5 | 3\}, \{5, 5 | 3\}\} \cong \{5, \tfrac{5}{2}, 5\}.$$

This suggests a final variant of our earlier conjecture:

**Conjecture 3**: For primes $p \geq 3$, $\mathcal{C}_{p,p,p}$ is the universal polytope of type

$$\{ \mathcal{M}_{p,p} \, , \, \mathcal{M}_{p,p} \} \ .$$

### 5.2. The groups $[k, l, m]$ with $l = 3$, 4 or 6

By Corollary 3.1, the remaining groups $G = [k, l, m]$ with $l = 3$, 4 or 6 do not generally yield a C-group $G^p$. In particular, if $V$, $V_0$, $V_3$, $V_{0,3}$ are non-singular and $G_0^p$, $G_3^p$ are of orthogonal type, then $G^p$ fails to be a C-group if $p > 13$. Hence only small primes need to be considered.

Every group $G = [k, 3, m]$, except $[\infty, 3, \infty]$, has a spherical or Euclidean subgroup $[k, 3]$ or $[3, m]$, so $G$ has already been investigated in the previous section.

When $G = [\infty, 3, \infty]$, with diagram


the three subspaces $V$, $V_0$, $V_3$ have discriminant $-1$, and $\mathrm{disc}(V_{0,3}) \sim 3$, so all four subspaces are certainly non-singular for $p > 3$. Moreover, $G_0^p$, $G_3^p$ are of orthogonal type for $p > 3$, and $\epsilon(V_{0,3}) = 1$ if and only if $p \equiv 1, 7 \bmod 12$. (Recall that $\epsilon(V_{0,3}) = 1$ or $-1$ according as $\mathrm{disc}(V_{0,3}) \sim -1$ or $-\gamma$, with $\gamma$ a non-square.) Now Corollary 3.1(a) shows that $G^p$ can only be a C-group if $p \leq 7$. Computations with GAP confirm that $G^p$ is indeed a C-group for $p = 3, 5$ or 7. The corresponding polytope has type $\{p, 3, p\}$, and we shall denote it by $\mathcal{C}_{p,3,p}$, again to suggest a connection with the classical cases. Even though there are just three polytopes in this family, the group $G^p$ is still described by (14) for any prime $p \geq 3$.

Clearly, when $p = 3$ we reacquire the simplex $\{3, 3, 3\}$. For $p = 5$ we obtain a self-dual polytope $\mathcal{C}_{5,3,5}$ with 120 dodecahedral facets $\{5, 3\}$ and 120 icosahedral vertex-figures $\{3, 5\}$, and with

$$\Gamma(\mathcal{C}_{5,3,5}) = O_1(4, 5, 1) \cong H_4.$$

We know from [13, Thm. 7D16] that the regular star-polytope $\{5, 3, \frac{5}{2}\}$ in $\mathbb{E}^4$ is isomorphic to the abstract regular polytope $\{5, 3, 5 \,|\, 3\}$, which is the quotient of the hyperbolic tessellation $\{5, 3, 5\}$ obtained by imposing the extra relation

$$(\rho_0 \rho_1 \rho_2 \rho_3 \rho_2 \rho_1)^3 = 1$$

on its group $\langle \rho_0, \ldots, \rho_3 \rangle$ (the “3” in the symbol for the quotient signifies this relation). It is straightforward to check that the corresponding element $r_0 r_1 r_2 r_3 r_2 r_1$ of $G^5$ indeed has order 3, so we also have

$$\mathcal{C}_{5,3,5} \cong \{5, 3, \tfrac{5}{2}\} \cong \{5, 3, 5 \,|\, 3\}.$$

(In fact, this isomorphism proves directly that $G^5$ is a C-group.)

When $p = 7$, the self-dual polytope $\mathcal{C}_{7,3,7}$ has 350 facets isomorphic to $\{7, 3\}_8$ and 350 vertex-figures isomorphic to $\{3, 7\}_8$, and $\Gamma(\mathcal{C}_{7,3,7}) = O_1(4, 7, -1)$, of order $7^2(7^4 - 1)$. Thus Klein's dual pair of maps $\{7, 3\}_8$ and $\{3, 7\}_8$ of genus 3 occur as facets and vertex-figures of a self-dual regular 4-polytope. (In contrast to the case $p = 5$, we have no interesting presentation for $G^7$.)

All but three groups $[k, 4, m]$ have a spherical or Euclidean subgroup $[k, 4]$ or $[4, m]$, so their polytopes have already been discussed in the previous section. The three exceptions are $G = [6, 4, 6]$, $[\infty, 4, 6]$ (or $[6, 4, \infty]$) and $[\infty, 4, \infty]$, with respective diagrams

$$\overset{3}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{2}{\bullet}\!-\!\!-\!\overset{6}{\bullet}, \quad \overset{4}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{2}{\bullet}\!-\!\!-\!\overset{6}{\bullet} \qquad \text{and} \qquad \overset{4}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{2}{\bullet}\!-\!\!-\!\overset{8}{\bullet}\,.$$

When $G = [6, 4, 6]$, the four subspaces $V$, $V_0$, $V_3$ and $V_{0,3}$ have discriminants $\sim -7$, $-6$, $-6$ and 1, respectively, and hence are non-singular for $p \neq 3, 7$. Moreover, $G_0^p$, $G_3^p$ are of orthogonal type for $p > 3$, and $\epsilon(V_{0,3}) = 1$ if and only if $p \equiv 1 \bmod 4$. Then Corollary 3.1(a) implies that $G^p$ can only be a C-group if $p \leq 7$. In fact, $G^7$ also fails to be a C-group.

However, $G^5$ is a C-group by Corollary 3.1(b), because $G_0^5$ and $G_3^5$ are full orthogonal groups. The corresponding regular polytope $\mathcal{P}$ is self-dual of type $\{6, 4, 6\}$. A dual pair of Petrie-Coxeter polyhedra $\{6, 4\,|\,3\}$ and $\{4, 6\,|\,3\}$ occur as the types of the 130 facets and 130 vertex-figures of $\mathcal{P}$, respectively; and $\Gamma(\mathcal{P}) = O(4, 5, -1)$.

When $p = 3$, each possible diagram yields a polytope with facets isomorphic to $\{6, 4\}_4$ and vertex-figures isomorphic to $\{4, 6\}_4$. Both the diagram given above for this case and the alternate

$$\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\overset{6}{\bullet}\!-\!\!-\!\overset{2}{\bullet}\,,$$

yield isomorphic, self-dual polytopes with group order 2592. (This is a *new* group of that order, not isomorphic for example to the group defined for $p = 3$ by the diagrams in (13).) For the remaining diagram

$$\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\overset{6}{\bullet}\!-\!\!-\!\overset{18}{\bullet}\,,$$

we find that $G^3$ has order 7776 and is the automorphism group of the universal self-dual polytope

$$\{\{6, 4\}_4\,, \{4, 6\}_4\}\ . \qquad (31)$$

For $G = [\infty, 4, 6]$, the discriminants of $V$, $V_0$, $V_3$ and $V_{0,3}$ are $\sim -6$, $-3$, $-1$ and 1, respectively. Hence, if $p > 3$, then all four subspaces are non-singular, $G_0^p$, $G_3^p$ are of orthogonal type, and $\epsilon(V_{0,3}) = 1$ if and only if $p \equiv 1 \bmod 4$. Then Corollary 3.1(a) shows again that $G^p$ can only be a C-group if $p \leq 7$. Indeed, for $p = 7$ we do obtain a C-group of order $2^9 \cdot 3^2 \cdot 7^2$.

We see also that $G^5$ is a C-group by Corollary 3.1(b). The corresponding polytope $\mathcal{P}$ has 120 facets isomorphic to $\{5, 4\}_6$ and 120 vertex-figures isomorphic to $\{4, 6\,|\,3\}$, and

$$\Gamma(\mathcal{P}) = O(4, 5, 1) \cong H_4 \rtimes C_2.$$

When $p = 3$, both the given diagram and its alternate produce the dual of the universal polytope (16), with group order 2592.

For $G = [\infty, 4, \infty]$, the discriminants of $V$, $V_0$, $V_3$ and $V_{0,3}$ are $\sim -2$, $-2$, $-1$ and 1, respectively, so these spaces are non-singular for each $p$. Corollary 3.1(a) once again implies that $G^p$ can only be a C-group if $p \leq 7$, and Corollary 3.1(b) confirms that $G^5$ actually is a C-group. Moreover, computation with GAP shows that $G^3$ and $G^7$ also are C-groups. The corresponding regular polytopes $\mathcal{P}$ are self-dual and are of type $\{p,4,p\}$ in each case. In particular, for $p=3$ we have

$$\Gamma(\mathcal{P}) \cong O(4,3,1) \cong F_4,$$

so $\mathcal{P}$ is the 24-cell $\{3,4,3\}$. When $p=5$, the polytope $\mathcal{P}$ has 130 facets $\{5,4\}_6$ and 130 vertex-figures $\{4,5\}_6$, and $\Gamma(\mathcal{P}) \cong O(4,5,-1)$. Finally, if $p=7$, then $\mathcal{P}$ has 350 vertices and 350 facets, and $\Gamma(\mathcal{P}) = O_1(4,7,-1)$; the vertex-figures are isomorphic to the map R10.9 of [3].

For the groups $[k,6,m]$ we may assume that $k,m \neq 3$, as the other groups have already been studied previously. Now $\mathrm{disc}(V_{0,3}) \sim 3$ in each case, so $\epsilon(V_{0,3}) = 1$ if and only if $p \equiv 1,7 \bmod 12$.

When $G=[k,6,4]$, with $k=4$, 6 or $\infty$, we take the diagram

$$\overset{2}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\overset{6}{\bullet}, \quad \overset{3}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\overset{6}{\bullet} \quad \text{or} \quad \overset{4}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\overset{6}{\bullet}, \tag{32}$$

respectively. Then $\mathrm{disc}(V) \sim -2$, $-15$ or $-6$, and $\mathrm{disc}(V_3) \sim -6$, $-2$ or $-1$, respectively. Moreover, $\mathrm{disc}(V_0) \sim -2$ in each case. It follows that the subspaces $V$, $V_0$, $V_3$ and $V_{0,3}$ certainly are non-singular for $p>3$, except when $k=6$ and $p=5$; in particular, $G_0^p$ and $G_3^p$ are then of orthogonal type. Now Corollary 3.1(a) eliminates all primes $p>13$.

For $G=[4,6,4]$, the subgroups $G_0^p$ and $G_3^p$ are full orthogonal for all primes $p \leq 13$, so Corollary 3.1(b) shows (for $p>3$) that $G^p$ is a C-group only when $p=5,7$. For $p=5$, we obtain a self-dual regular polytope $\mathcal{P}$ with 130 facets isomorphic to $\{4,6\,|\,3\}$ and 130 vertex-figures isomorphic to $\{6,4\,|\,3\}$, and with group $\Gamma(\mathcal{P}) = O(4,5,-1)$. Thus $\mathcal{P}$ has a dual pair of Petrie-Coxeter polyhedra as its facets and vertex-figures. For $p=7$, we have another self-dual polytope $\mathcal{P}$ of type $\{4,6,4\}$, now with 350 vertices, 350 facets, and automorphism group $O(4,7,-1)$. Finally, when $p=3$, $G^3$ has order 5184 and is the automorphism group of the universal, self-dual polytope

$$\{\{4,6\}_4\,,\,\{6,4\}_4\}\ , \tag{33}$$

with 36 facets (and vertices).

Let $G=[6,6,4]$. Now Corollary 3.1(b) applies only to $p=7$ among primes $p \leq 13$, so in particular $G^7$ is a C-group. The corresponding regular polytope $\mathcal{P}$ is of type $\{6,6,4\}$, has 350 vertices and 350 facets, and has $O(4,7,-1)$ as its group. Using GAP we find that $G^p$ is also a C-group for the remaining primes $p \leq 13$, thereby giving a few regular polytopes $\mathcal{P}$ of type $\{6,6,4\}$.

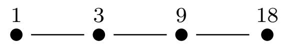
For $p=3$, the group $G^3$ depends on the diagram used for reduction. Both the middle diagram in (32) and

$$\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{2}{\bullet}$$

give the group of order 864 for the universal polytope

$$\{\{6,6\}^{\pi}_{(1,1)}\,,\,\{6,4\}_4\}$$

whose 12 facets are isomorphic to the Petrial of Sherk's map $\{6,6\}_{(1,1)}$, and with 6 vertex-figures isomorphic to $\{6,4\}_4$. On the other hand, both



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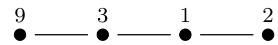



yield the group of order 7776 for the universal polytope

$$\{\{6,6\}^{\pi}_{(3,0)}\,,\,\{6,4\}_4\}\ ,$$

whose 36 facets isomorphic to the Petrial of $\{6,6\}_{(3,0)}$. (This is *not* the group for the polytope in (31).)

When $p=5$, the underlying space $V$ is singular and $G^5$ is of order 30000; the corresponding polytope $\mathcal{P}$ has 125 facets isomorphic to the map R11.5 of [3], and 125 vertex-figures isomorphic to $\{6,4\,|\,3\}$. Finally, we have the groups $G^{11}=O(4,11,-1)$ and $G^{13}=O(4,13,-1)$, each contributing a regular polytope of type $\{6,6,4\}$, with 2684 or 4420 facets, respectively, and with half as many vertices.

Next let $G=[\infty,6,4]$. The primes 5 and 7 are easily seen to yield C-groups by Corollary 3.1(b); in fact, $G^p$ is a C-group for the remaining primes $p\leq 13$. In particular, when $p=5$, we have a polytope $\mathcal{P}$ with 120 facets $\{5,6\}_4$ and 120 vertex-figures $\{6,4\,|\,3\}$, and with

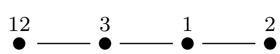
$$\Gamma(\mathcal{P})=O(4,5,1)\cong H_4\rtimes C_2.$$

When $p=7$, $\mathcal{P}$ is of type $\{7,6,4\}$ and has 336 facets and 336 vertices, and $\Gamma(\mathcal{P})=O(4,7,1)$. Similarly, we find that $G^{11}=O(4,11,+1)$ and $G^{13}=O(4,13,-1)$. These groups yield regular polytopes of type $\{11,6,4\}$ or $\{13,6,4\}$, respectively, with 2640 or 4420 facets and half as many vertices.

When $p=3$, the right-most diagram in (32) yields the group $G^3$ of order 432 for the universal regular polytope

$$\{\{3,6\}_{(1,1)}\,,\,\{6,4\}_4\}\ .$$

This polytope has with 12 toroidal facets $\{3,6\}_{(1,1)}$ and 3 vertex-figures $\{6,4\}_4$. However, the alternate diagram

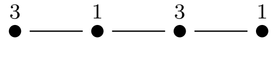



gives the group of order 3888 for the universal regular polytope

$$\{\{3,6\}_{(3,0)}\,,\,\{6,4\}_4\}\ ,$$

now with 36 toroidal facets $\{3,6\}_{(3,0)}$ and 27 vertex-figures.

At last then, we are left to consider only the groups $G=[6,6,6]$, $[\infty,6,6]$ and $[\infty,6,\infty]$, with diagrams



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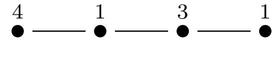

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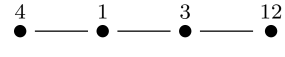

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and $\mathrm{disc}(V)\sim -11$, $-1$ and $-3$, respectively. Again the subspaces $V$, $V_0$, $V_3$ and $V_{0,3}$ are non-singular for $p>3$ (excluding $p=11$ for the left diagram); and then $G^p_0$ and $G^p_3$ are of orthogonal type. By Corollary 3.1(a) we exclude all $p>13$. For each diagram, $\epsilon(V_{0,3})=1$ if and only if $p\equiv 1\,,\,7 \bmod 12$. Thus, by Corollary 3.1(b), we do get a C-group for $p=5,7$.

In fact, from GAP we find for each diagram that $G^p$ is a C-group for $p = 3, 11, 13$, too. We describe some features of the corresponding polytopes in the more notable cases only.

Let us first consider $G = [6, 6, 6]$. The resulting polytope $\mathcal{P}$ is self-dual for $p = 5, 7, 11, 13$. When $p = 5$, the facets (and vertex-figures) are copies of the map R11.5 listed in [3]. For $p = 11$, $V$ is singular and $G^{11}$ has order $11^4 \cdot 5!$. For $p = 3$, the facets and vertex-figures are the Petrials of Sherk's maps, as described earlier. The diagrams

$$\overset{3}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{1}{\bullet} \quad \text{and} \quad \overset{1}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{9}{\bullet} \text{---} \overset{27}{\bullet}$$

yield, respectively, the groups of orders 432 and 11664 for the self-dual, universal regular polytopes

$$\{\{6,6\}^{\pi}_{(1,1)}\,,\,\{6,6\}^{\pi}_{(1,1)}\} \quad \text{and} \quad \{\{6,6\}^{\pi}_{(3,0)}\,,\,\{6,6\}^{\pi}_{(3,0)}\}\ .$$

The remaining pertinent diagrams

$$\overset{1}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{9}{\bullet} \text{---} \overset{3}{\bullet} \quad \text{and} \quad \overset{3}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{9}{\bullet}$$

yield the group of order 1296 for the universal regular polytope

$$\{\{6,6\}^{\pi}_{(3,0)}\,,\,\{6,6\}^{\pi}_{(1,1)}\}\ ,$$

and its dual, respectively.

Next suppose $G = [\infty, 6, 6]$. For $p = 5$ we reobtain $\Gamma(\mathcal{P}) = O(4, 5, 1) \cong H_4 \rtimes C_2$ as the group of order 28800 for a polytope $\mathcal{P}$ whose 120 facets are isomorphic to $\{5, 6\}_4$ and whose 120 vertex-figures are copies of map R11.5 in [3].

For $p = 3$, the situation is quite analogous to that in the previous case. The diagrams

$$\overset{4}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{1}{\bullet} \quad \text{and} \quad \overset{36}{\bullet} \text{---} \overset{9}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{1}{\bullet}$$

yield, respectively, the groups of orders 216 and 5832 for the duals of the first and last of the universal regular polytopes displayed in (24). The two other pertinent diagrams

$$\overset{12}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{3}{\bullet} \quad \text{and} \quad \overset{4}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{3}{\bullet} \text{---} \overset{9}{\bullet}$$

yield isomorphic groups of order 648. The given generators then provide the (non-isomorphic) duals of the second and third of the universal regular polytopes in (24).

Finally, consider $G = [\infty, 6, \infty]$. When $p = 3$ we again get the universal polytope described in (23). For $p = 5, 7, 11, 13$, we obtain self-dual polytopes of type $\{p, 6, p\}$, with groups $O(4, 5, -1)$, $O(4, 7, +1)$, $O_1(4, 11, -1)$ and $O_1(4, 13, +1)$, respectively. In particular, for $p = 5$, we get the universal self-dual regular polytope

$$\{\{5,6\}_4\,,\,\{6,5\}_4\}\ .$$

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