Geometry 1, MTH G122.
Fall 2004. Professor Mikhail Shubin.

Textbook:

Foundations of Differentiable Manifolds and Lie groups, by Frank W. Warner. Springer-Verlag New York, Inc., 1983.

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Homework assignment no. 4

(due October 7)

1. Let M be a compact manifold, $\dim M = d$, and we are given a C^∞ map $f : M \to \mathbb{R}^d$, $f(x) = (f_1(x), \ldots, f_d(x))$, $x \in M$, where the functions $f_j \in C^\infty(M)$, $j = 1, \ldots, d$, define the components of the vector function f. Let $p \in M$ be a point where f_1 attains its maximum. Prove that p is a degeneracy point of the map f, i.e. the differential

$$(df)_p : M_p \to \mathbb{R}^d_{f(p)}$$

is not injective (or, equivalently, not surjective).

2. Let us consider the system of differential equations

$$\begin{cases} \frac{\partial u}{\partial x} = f_1(x, y, u, v) \\ \frac{\partial u}{\partial y} = f_2(x, y, u, v) \\ \frac{\partial v}{\partial x} = g_1(x, y, u, v) \\ \frac{\partial v}{\partial y} = g_2(x, y, u, v) \end{cases}$$

with the initial conditions

$$\begin{cases} u(x_0, y_0) = u_0 \\ v(x_0, y_0) = v_0 \end{cases}$$

where f_1, f_2, g_1, g_2 are C^∞ functions with respect to $(x, y, u, v) \in \mathcal{U}$, where $\mathcal{U}$ is an open set in $\mathbb{R}^4$, $(x_0, y_0, u_0, v_0) \in \mathcal{U}$, $u = u(x, y)$ and $v = v(x, y)$ are C^∞ functions defined in a neighborhood $\mathcal{V}$ of (x_0, y_0) in $\mathbb{R}^2$, such that $(x, y, u(x, y), v(x, y)) \in \mathcal{U}$ for all $(x, y) \in \mathcal{V}$.

(a) Using the Frobenius theorem, establish necessary and sufficient conditions for the following existence statement to be true: for every point $(x_0, y_0, u_0, v_0) \in \mathcal{U}$ there exists a solution (u, v) defined in a neighborhood $\mathcal{V}$ of (x_0, y_0) in $\mathbb{R}^2$ such that $(x, y, u(x, y), v(x, y)) \in \mathcal{U}$ for all $(x, y) \in \mathcal{V}$.

(b) Prove that such a solution is unique: if two solutions with the same initial conditions are defined in a connected neighborhood $\mathcal{V}$ of (x_0, y_0), then they coincide in $\mathcal{V}$.