Geometry 1, MTH G122.
Fall 2004. Professor Mikhail Shubin.

Textbook:

Foundations of Differentiable Manifolds and Lie groups, by Frank W. Warner. Springer-Verlag New York, Inc., 1983.

Office: 460 Lake Hall. **Phone:** (617)373-5676 **E-mail:** shubin@neu.edu

Homework assignment no. 5
(due October 14)

1. Let M and N be C^∞ manifolds, $\varphi : M \to N$ a C^∞ map. Given $X_p \in M_p$ (i.e. X_p is a tangent vector to M at p), define $Y_{\varphi(p)} = d\varphi(p)X_p \in N_{\varphi(p)}$ by the formula $Y_{\varphi(p)}f = X_p(f \circ \varphi)$, $f \in C^\infty(N)$. Prove that this agrees with the definition of $Y_{\varphi(p)}$ in terms of curves: if $\gamma : (-\varepsilon, +\varepsilon) \to M$ is a smooth curve in M, such that $\gamma(0) = p$ and $\gamma'(0) = X_p$, then $Y_{\varphi(p)} = (\varphi \circ \gamma)'(0)$.

2. Let X be a smooth vector field on a compact manifold M, and let $\varphi : M \to M$ be a diffeomorphism. Let $\{X_t | \, t \in \mathbf{R}\}$ be the flow on M defined by X, that is $X_t : M \to M$, $X_t(x) = \gamma(t; x)$ where $t \mapsto \gamma(t; x) \in M$ is the solution of the differential equation

$$\frac{\partial \gamma(t; x)}{\partial t} = X(\gamma(t; x)),$$

satisfying the initial condition $\gamma(0; x) = x$.

(a) Prove that the transformations X_t form a one-parametric group of diffeomorphisms of M, that is $X_t : M \to M$ is a diffeomorphism of M onto itself for every $t \in \mathbf{R}$, such that $(t, x) \mapsto X_t(x)$ is a smooth map $\mathbf{R} \times M \to M$, X_0 is the identity map, and $X_t \circ X_s = X_{t+s}$ for all $t, s \in \mathbf{R}$.

(b) Let $\varphi : M \to N$ be a diffeomorphism of manifolds, and let $Y = d\varphi(X)$ be the vector field on N, obtained by the action of φ on the vector fields. Prove that the flows of X and Y are related by the formula $Y_t = \varphi \circ X_t \circ \varphi^{-1}$.

(c) Establish an explicit relation between the actions of X_t and Y_t on C^∞ functions.