TOPOLOGY
OF
HYPERPLANE ARRANGEMENTS

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Hyperplane arrangements

- *Arrangement*: A collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of hyperplanes, $H_i = \ker \alpha_i$, in $\mathbb{C}^\ell$.
- *Defining polynomial*: $Q_{\mathcal{A}} = \alpha_1 \cdots \alpha_n$.
- *Intersection lattice*: $L(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\}$.
  ordered by reverse inclusion, ranked by codimension
- *Complement*: $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.

Example. Boolean arrangement $\mathcal{D}_n$ in $\mathbb{C}^n$.

- $Q_{\mathcal{D}_n} = z_1 \cdots z_n$
- $L(\mathcal{D}_n)$: subsets of $[n] := \{1, \dots, n\}$
- $X(\mathcal{D}_n) = (\mathbb{C}^*)^n$ complex n-torus
- $\pi_1(X(\mathcal{D}_n)) = \mathbb{Z}^n$

Example. Braid arrangement $\mathcal{B}_\ell$ in $\mathbb{C}^\ell$.

- $Q_{\mathcal{B}_\ell} = \prod_{1 \le i < j \le \ell} (z_i - z_j)$
- $L(\mathcal{B}_\ell)$: partition lattice of $[\ell]$
- $X(\mathcal{B}_\ell) = F(\ell, \mathbb{C})$ configuration space of ℓ ordered points in $\mathbb{C}$
- $\pi_1(X(\mathcal{B}_\ell)) = P_\ell$ pure braid group on ℓ strings;
  in fact: $X(\mathcal{B}_\ell) \simeq K(P_\ell, 1)$

Types of arrangements

- **Complexified**
  $\mathcal{A} = \mathcal{A}_{\mathbb{R}} \otimes \mathbb{C}$ (i.e., $Q_{\mathcal{A}}$ has real coefficients)
- **Reflection**
  Complexification of reflecting hyperplanes of a Coxeter group. E.g., $\mathcal{B}_\ell$ of type $\mathrm{A}_{\ell\text{-}1}$.
- **Simplicial**
  Complexification of real arrangement, all of whose complementary regions are open simplices.
  Deligne: $\mathcal{A}$ simplicial $\implies X(\mathcal{A}) \simeq K(\pi, 1)$.
- **Fiber-type** (Falk-Randell)
  There is a tower of *linear* fibrations

$$X(\mathcal{A}) \xrightarrow{p_\ell} X(\mathcal{A}_{\ell-1}) \xrightarrow{p_{\ell-1}} \cdots \to X(\mathcal{A}_2) \xrightarrow{p_2} X(\mathcal{A}_1) = \mathbb{C}^*$$

with $\operatorname{fiber}(p_i) = \mathbb{C} \setminus \{d_i \text{ points}\}$.

Thus, $X \simeq K(G,1)$, with $G = F_{d_\ell} \rtimes_{\rho_\ell} \cdots \rtimes_{\rho_2} F_{d_1}$, where $\rho_i : \pi_1(X(\mathcal{A}_{i-1})) \to P_{d_i} \subset \operatorname{Aut}(F_{d_i})$.

Terao: $\mathcal{A}$ fiber-type $\iff L(\mathcal{A})$ supersolvable

Cohen-S.: Explicit $\mathbb{Z}G$-resolution $C_\bullet(\widetilde{X}) \to \mathbb{Z}$
(using the Fox Jacobians of ρ_i)

- **Generic**
  $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{C}^\ell$ $(2 < \ell < n)$, s.t. $\operatorname{codim} \bigcap_{H \in \mathcal{B}} H = |\mathcal{B}|$, for all $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| \le \ell$.

  E.g.: Boolean

  Hattori: $X(\mathcal{A}) \simeq S^1 \times (T^{n-1})^{(\ell-1)}$. Thus:

  $\pi_1(X) = \mathbb{Z}^n, \quad \pi_i(X) = 0 \quad (\text{if } 1 < i < \ell - 1)$

  $\pi_{\ell-1}(X) \neq 0.$

- **Hypersolvable** (Jambu-Papadima)
  Combinatorial condition, generalization of both supersolvable and generic. Hattori's result generalizes in this context (Papadima-S.)

- **Graphic**
  Each simple graph $\mathcal{G}$ with vertices $\{1, \ldots, m\}$ defines an arrangement in $\mathbb{C}^m$:

  $$\mathcal{A}_{\mathcal{G}} = \{\ker(z_i - z_j) \mid \{i, j\} \text{ edge in } \mathcal{G}\}$$

  ◇ $\mathcal{G}$ diagram of type A $\Longrightarrow$ $\mathcal{A}_{\mathcal{G}}$ Boolean

  ◇ $\mathcal{G} = K_m \Longrightarrow \mathcal{A}_{\mathcal{G}} = \mathcal{B}_m$

  ◇ $\mathcal{G}$ polygon $\Longrightarrow$ $\mathcal{A}_{\mathcal{G}}$ generic

  ◇ every cycle in $\mathcal{G}$ has a chord $\Longleftrightarrow$ $\mathcal{A}_{\mathcal{G}}$ fiber-type

  (Stanley, Fulkerson-Gross)

Cohomology ring

- $H^*(X(\mathcal{B}_\ell)) = H^*(P_\ell)$: Arnol'd
- $H^*(X(\mathcal{A}))$: Brieskorn, Orlik-Solomon
- $H^*(F(\ell, \mathbb{R}^k))$: F. Cohen
- $H^*(F(\ell, S^k))$: Feichtner-Ziegler, Xicoténcatl
- $H^*(X(\text{subspace arrangement}))$: Goresky-MacPherson, Björner-Ziegler, De Concini-Procesi, Yuzvinsky, de Longueville

Theorem. *(Orlik-Solomon) Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement. Then $L(\mathcal{A})$ determines the cohomology ring of $X = X(\mathcal{A})$:*

$$\boxed{H^*(X) = \bigwedge^* \mathbb{Z}^n \Big/ \Big(\partial e_{\mathcal{B}} \mid \operatorname{codim} \bigcap_{H \in \mathcal{B} \subseteq \mathcal{A}} H < |\mathcal{B}| \Big)}$$

where $\bigwedge^* \mathbb{Z}^n$ = exterior algebra over $\mathbb{Z}$ on generators e_i (dual to meridian of H_i) in degree 1, and, for $\mathcal{B} = \{H_{i_1}, \dots, H_{i_r}\}$, $e_{\mathcal{B}} = e_{i_1} \cdots e_{i_r}$ and $\partial e_{\mathcal{B}} = \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$.

$H^*(X)$ is torsion-free; basis: *no broken circuits* (**nbc**).

Poincaré polynomial: $\boxed{P(X,t) = \sum_{Y \in L(\mathcal{A})} \mu(Y)(-t)^{\operatorname{codim} Y}}$

where $\mu : L(\mathcal{A}) \to \mathbb{Z}$ Möbius function.

Resonance varieties

Let $A^* = H^*(X(\mathcal{A}), \mathbb{C})$ be the Orlik-Solomon algebra of $\mathcal{A}$. For $\lambda \in \mathbb{C}^n$, let $e_\lambda = \sum_{i=1}^n \lambda_i e_i \in A^1 \cong \mathbb{C}^n$.

Aomoto complex:

$$0 \to A^0 \xrightarrow{e_\lambda} A^1 \xrightarrow{e_\lambda} \cdots \to A^2 \xrightarrow{e_\lambda} A^\ell \to 0$$

The *resonance varieties* of $\mathcal{A}$ were defined by Falk as:

$$\boxed{R^k_d(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \dim_{\mathbb{C}} H^k(A, e_\lambda) \geq d\}}$$

They are actually subsets of $\Delta_n := \{\lambda \mid \sum_{i=1}^n \lambda_i = 0\}$, and depend only on $\mathcal{A}$ (up to linear iso of $\mathbb{C}^n$).

Theorem. *Each component of $R^k_d(\mathcal{A})$ is a linear subspace in $\mathbb{C}^n$.*

Conjectured by Falk. Proved by Cohen-S., Libgober ($k = 1$); Cohen-Orlik, Libgober (all k).

The varieties $R_d(\mathcal{A}) = R^1_d(\mathcal{A})$ admit a purely combinatorial description—started by Falk, completed by Libgober-Yuzvinsky, using Vinberg's classification of affine Kac-Moody algebras.

A partition $\mathsf{P} = (\mathsf{p}_1 \mid \cdots \mid \mathsf{p}_q)$ of $\mathcal{A}$ is *neighborly* if

$$(|\,\mathsf{p}_j \cap I| \geq |\,I| - 1) \implies I \subset \mathsf{p}_j, \ \forall I \in L_2(\mathcal{A})$$

It defines a linear subspace of $\mathbb{C}^n$:

$$L_{\mathsf{P}} = \Delta_n \cap \bigcap_{\{I \in \mathcal{L}_2(\mathcal{A}) \mid I \not\subset \mathsf{p}_j, \forall j\}} \{\lambda \mid \sum_{i \in I} \lambda_i = 0\}.$$

Theorem. *(Libgober-Yuzvinsky)*

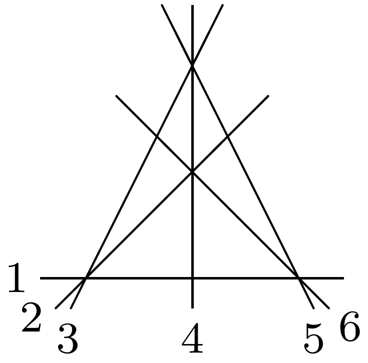
- *All components L_i of $R_1(\mathcal{A})$ arise from neighborly partitions of sub-arrangements $\mathcal{A}' \subset \mathcal{A}$.*
- $\dim L_i \geq 2$*.*
- $L_i \cap L_j = \{\mathbf{0}\}$ *for* $i \neq j$*.*
- $R_d(\mathcal{A}) = \{\mathbf{0}\} \cup \bigcup_{\dim L_i \geq d+1} L_i$*.*

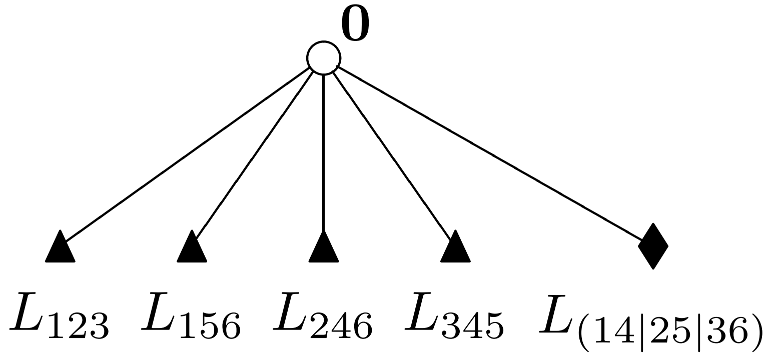
E.g., for each $I \in \mathcal{L}_2(\mathcal{A})$ with $|\,I| \geq 3$, there is a *local* component

$$L_I = \{\lambda \mid \sum_{i=1}^{n} \lambda_i = 0 \text{ and } \lambda_i = 0 \text{ for } i \notin I\}$$

corresponding to the partition (I) of $\mathcal{A}_I = \{H_i \mid i \in I\}$. Note that $\dim L_I = |\,I| - 1$, and thus $L_I \subset R_{|\,I|-2}(\mathcal{A})$.

Example. (Braid arrangement $\mathcal{B} = \mathcal{B}_4$)





The OS-algebra $A^* = H^*(X(\mathcal{B}), \mathbb{C})$ has generators $e_1, \dots, e_6$, and relations $e_i^2 = e_ie_j + e_je_i = 0$ and

$$e_1e_2 - e_1e_3 + e_2e_3 = 0, \qquad e_1e_5 - e_1e_6 + e_5e_6 = 0,$$

$$e_2e_4 - e_2e_6 + e_4e_6 = 0, \qquad e_3e_4 - e_3e_5 + e_4e_5 = 0.$$

The resonance variety $R_1(\mathcal{B}) \subset \mathbb{C}^6$ has 5 components (4 local, and 1 non-local), all 2-dimensional:

$$L_{123} = \{\lambda \mid \lambda_1 + \lambda_2 + \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0\}$$

$$L_{156} = \{\lambda \mid \lambda_1 + \lambda_5 + \lambda_6 = \lambda_2 = \lambda_3 = \lambda_4 = 0\}$$

$$L_{246} = \{\lambda \mid \lambda_2 + \lambda_4 + \lambda_6 = \lambda_1 = \lambda_3 = \lambda_5 = 0\}$$

$$L_{345} = \{\lambda \mid \lambda_3 + \lambda_4 + \lambda_5 = \lambda_1 = \lambda_2 = \lambda_6 = 0\}$$

$$L_{(14|25|36)} = \{\lambda \mid \lambda_1 - \lambda_4 = \lambda_2 - \lambda_5 = \lambda_3 - \lambda_6 = \sum \lambda_i = 0\}$$

Characteristic varieties

X finite CW-complex, $G = \pi_1(X)$.
Assume $G_{\mathrm{ab}} \cong \mathbb{Z}^n$ (with basis $t_1, \dots, t_n$).

Character variety:

$$\boxed{\mathrm{Hom}(G, \mathbb{C}^*) \cong (\mathbb{C}^*)^n}$$

algebraic torus, with coordinate ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$

Characteristic varieties:

$$\boxed{V_d^k(X) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \dim_{\mathbb{C}} H^k(X, \mathbb{C}_{\mathbf{t}}) \geq d\}}$$

where $\mathbb{C}_{\mathbf{t}}$ is the G-module $\mathbb{C}$ with action given by the representation $\mathbf{t} : G \to \mathbb{C}^*$.

$V_d^k(X)$ depends only on the homotopy type of X (up to a monomial isomorphism of $(\mathbb{C}^*)^n$).

Theorem. *(Arapura) Suppose X is the complement of a normal-crossing divisor in a compact Kähler manifold with $b_1 = 0$. Then the components of $V_d^k(X)$ are subtori of $(\mathbb{C}^*)^n$, possibly translated by roots of* $\mathbf{1}$.

Uses Deligne's mixed Hodge structures. Generalizes results of Green-Lazarsfeld, Simpson.

Characteristic varieties of $\mathcal{A} = \{H_1, \dots, H_n\}$:

$$V_d^k(\mathcal{A}) := V_d^k(X(\mathcal{A})) \subset (\mathbb{C}^*)^n$$

Recall the resonance varieties $R_d^k(\mathcal{A}) \subset \mathbb{C}^n$.

Theorem. *(Cohen-S., Cohen-Orlik, Libgober)*

$$\boxed{\mathrm{TC}_{\mathbf{1}}(V_d^k(\mathcal{A})) = R_d^k(\mathcal{A})}$$

As a consequence, the components of $V_d^k(\mathcal{A})$ passing through $\mathbf{1}$ are combinatorially determined (by $L(\mathcal{A})$).

In general, though, there do exist components that do not pass through $\mathbf{1}$ (i.e., translated subtori).

Question. Are such components combinatorially determined?

Remark. $R_d^k(X)$ may be defined for arbitrary X. Then (Libgober):

$$\mathrm{TC}_{\mathbf{1}}(V_d^k(X)) \subseteq R_d^k(X).$$

But the inclusion is *strict* in general, e.g., for link complements (Matei), and even for complements of *real* subspace arrangements (M.-S.).

Fundamental groups of arrangements

$\mathcal{A} = \{H_1, \dots, H_n\}$ hyperplane arrangement, with complement X, and fundamental group $G = \pi_1(X)$.

$\mathcal{A}' = \{l_1, \dots, l_n\}$ generic 2-section. By Lefschetz-type theorem of Hamm and Lê : $\pi_1(X) \cong \pi_1(X')$.

So reduce to the case where $\mathcal{A}$ is an arrangement of affine lines in $\mathbb{C}^2$. Let $v_1, \dots, v_s$ be the intersection points of the lines.

$$v_q = l_{i_1} \cap \dots \cap l_{i_r} \longrightarrow I_q := \{i_1, \dots, i_r\}$$

Lattice: $L_1(\mathcal{A}) = [n]$, $L_2(\mathcal{A}) = \{I_1, \dots, I_s\}$.

Group: $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$.

Question. Is $G(\mathcal{A})$ combinatorially determined? I.e.:

$$L(\mathcal{A}_1) \cong L(\mathcal{A}_2) \implies G(\mathcal{A}_1) \cong G(\mathcal{A}_2)?$$

According to Rybnikov, the answer is no.

Presentation for $G = G(\mathcal{A})$

(Van Kampen, Artin/Randell, Salvetti, Arvola/Moishezon, Libgober, E. Hironaka, Cordovil-Fachada, Cohen-S., …):

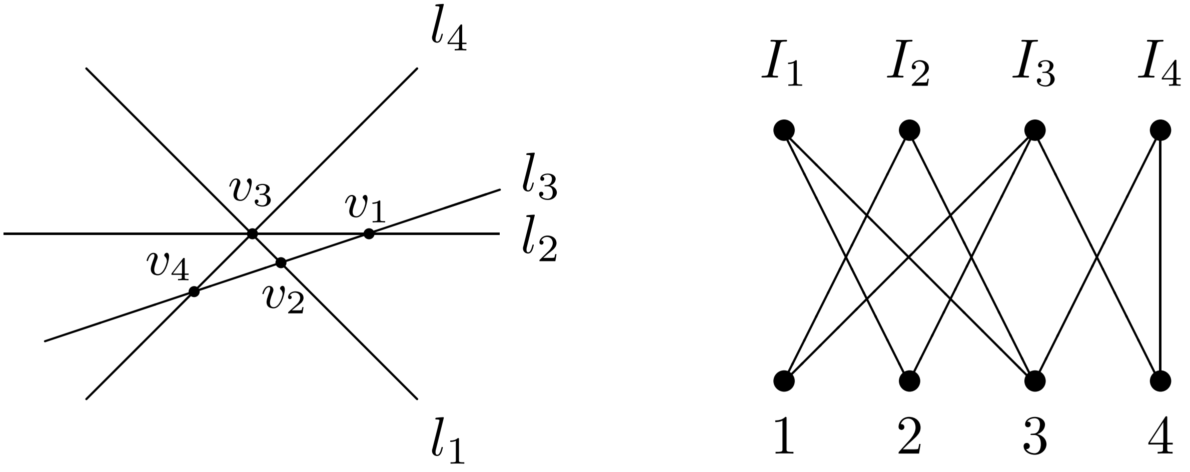
$$\boxed{G = \langle x_1, \ldots, x_n \mid \alpha_1(x_i) = x_i, \ldots, \alpha_s(x_i) = x_i \ (1 \le i \le n) \rangle}$$

where

$$\boxed{\alpha_q = A_{I_q}^{\delta_q} \in P_n}$$

acting on $F_n = \langle x_1, \ldots x_n \rangle$ via the Artin representation (A_I = full twist on I-strands, and $\delta_q \in B_n$ can be read from a "braided wiring diagram").

Example.



$$\alpha_1 = A_{23}, \quad \alpha_2 = A_{13}^{A_{23}}, \quad \alpha_3 = A_{124}, \quad \alpha_4 = A_{34}.$$

$$G = \left\langle \begin{array}{c} x_1, x_2, \\ x_3, x_4 \end{array} \middle| \begin{array}{l} x_1 x_2 x_4 = x_4 x_1 x_2 = x_2 x_4 x_1, \\ [x_1, x_3] = [x_2, x_3] = [x_4, x_3] = 1 \end{array} \right\rangle \cong F_2 \times \mathbb{Z}^2$$

Characteristic varieties of G (over field $\mathbb{K}$):

$$\boxed{V_d(G,\mathbb{K}) = \{\mathbf{t} \in \operatorname{Hom}(G,\mathbb{K}^*) \mid \dim_{\mathbb{K}} H^1(G,\mathbb{K}_{\mathbf{t}}) \geq d\}}$$

For $d < n$, we have (E. Hironaka):

$$\boxed{V_d(G,\mathbb{K}) = \{\mathbf{t} \in (\mathbb{K}^*)^n \mid \operatorname{rank}_{\mathbb{K}} A_G(\mathbf{t}) < n-d\}}$$

where $A_G = J_G^{\mathrm{ab}}$ is the *Alexander matrix* of G, obtained by abelianizing the Fox Jacobian $J_G = \left(\frac{\partial r_i}{\partial x_j}\right)$.

Resonance varieties of G (over $\mathbb{K}$):

$$\boxed{R_d(G,\mathbb{K}) = \left\{ \lambda \in H^1(G,\mathbb{K}) \;\middle|\; \begin{array}{l} \exists \text{ subspace } W \subset H^1(G,\mathbb{K}), \\ \dim W = d+1,\ \lambda \cdot W = 0 \end{array} \right\}}$$

Then (Matei-S.):

$$\boxed{R_d(G,\mathbb{K}) = \{\lambda \in \mathbb{K}^n \mid \operatorname{rank}_{\mathbb{K}} A_G^{(1)}(\lambda) < n-d\}}$$

where $A_G^{(1)}(\lambda) = (A_G|_{t_i=1-\lambda_i})^{\text{linear terms}}$ is the *linearized Alexander matrix* of G.

Remark. If $G = \pi_1(X(\mathcal{A}))$, then $R_d(\mathcal{A}) = R_d(G,\mathbb{C})$.

But: $R_d(G,\mathbb{C}) \bmod p \neq R_d(G,\mathbb{F}_p)$,

$\mathrm{TC}_{\mathbf{1}}(V_d(G,\mathbb{F}_p)) \neq R_d(G,\mathbb{F}_p)$.

Homology of finite covers

Theorem. *(Libgober, Sakuma) G f.p. group, $G_{\mathrm{ab}} = \mathbb{Z}^n$, $1 \to K \to G \xrightarrow{\gamma} \Gamma \to 1$. If Γ finite abelian, then:*

$$\boxed{b_1(K) = n + \sum_{\mathbf{1} \neq \rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)} (\operatorname{corank} J_G^{\rho\circ\gamma} - 1)}$$

More generally, let

$$b_1^{(q)}(G) := \dim_{\mathbb{K}} H_1(G, \mathbb{K})$$

where $\mathbb{K}$ is a field of characteristic q.

Theorem. *(Matei-S.) If Γ finite and $q \nmid |\Gamma|$, then:*

$$\boxed{b_1^{(q)}(K) = b_1^{(q)}(G) + \sum_{\rho \neq \mathbf{1}} n_\rho (\operatorname{corank} J_G^{\rho\circ\gamma} - n_\rho)}$$

(sum over all non-trivial irreps $\rho : \Gamma \to \mathrm{GL}(n_\rho, \mathbb{K})$,
$\mathbb{K}$ field of char. q containing all roots of 1 of order $\exp \Gamma$).

Corollary. *(M.-S.) Let* $K = \ker(\gamma : G \twoheadrightarrow \mathbb{Z}_N)$*. Then:*

$$\boxed{b_1^{(q)}(K) = b_1^{(q)}(G) + \sum_{1 \neq k \mid N} \phi(k)\, \mathrm{depth}_{\mathbb{K}}(\gamma^{N/k})}$$

where $\mathrm{depth}_{\mathbb{K}}(\mathbf{t}) := \max\{d \mid \mathbf{t} \in V_d(G, \mathbb{K})\}$.

Application to homology of Milnor fiber of a (central) arrangement $\mathcal{A}$. Milnor fibration:

$$\boxed{F(\mathcal{A}) \to X(\mathcal{A}) \xrightarrow{Q_{\mathcal{A}}} \mathbb{C}^*}$$

> $H_*(F(\mathcal{A}))$ studied by Randell, Orlik-Randell, Artal Bartolo, Cohen-S., Denham, Cohen-Orlik, . . .

$F(\mathcal{A})$ is the n-fold cyclic cover of $X(\overline{\mathcal{A}})$, given by

$$\gamma : \overline{G} \to \mathbb{Z}_n, \quad \gamma(x_i) = 1$$

Thus:

$$b_1^{(q)}(F) = n - 1 + \sum_{1 \neq k \mid n} \phi(k)\, \mathrm{depth}_{\mathbb{K}}(\gamma^{n/k})$$

Question. Is $H_*(F(\mathcal{A}))$ combinatorially determined?

Question. Is $H_*(F(\mathcal{A}))$ torsion-free? In particular, is $b_1^{(q)}(F) = b_1(F)$, for all $q \nmid n$?

Corollary. *(M.-S.)* $K = \ker(\gamma : G \twoheadrightarrow \mathbb{Z}_p)$. *Then:*

$$\boxed{b_1^{(q)}(K) = b_1(G) + (p-1)\,\mathrm{depth}_{\mathbb{K}}(\gamma)}$$

where $\mathbb{K} = \mathbb{C}$ *if* $q = 0$, *or* $\mathbb{K} = \mathbb{F}_{q^s}$ *if* q *prime,* $q \neq p$.

$s = \mathrm{ord}_p(q) =$ *smallest positive integer s.t.* $p | q^s - 1$

$\mathbb{F}_q^s = \mathbb{F}_q(\zeta)$, *where* ζ *is a primitive* p*-th root of* 1

Define:

$$\beta_{p,d}^{(q)}(G) = \sharp\left\{ K \lhd G \;\middle|\; \begin{array}{l} [G : K] = p \ \text{ and} \\ b_1^{(q)}(K) = b_1^{(q)}(G) + (p-1)d \end{array} \right\}$$

Then:

$$\boxed{\beta_{p,d}^{(q)}(G) = \frac{\sharp\,\mathrm{Tors}_p(V_d(G,\mathbb{K}) \setminus V_{d+1}(G,\mathbb{K}))}{p-1}}$$

where

$$\mathrm{Tors}_N(V) = \{\mathbf{t} \in V \mid \mathbf{t}^N = \mathbf{1} \text{ and } \mathbf{t} \neq \mathbf{1}\}$$

for $V \subset \mathbb{K}^{*n}$.

Representations onto finite groups

G f.g. group, Γ finite group.

$$\begin{aligned}\delta_\Gamma(G) :&= |\operatorname{Epi}(G,\Gamma)/\operatorname{Aut}(\Gamma)| \\ &= \sharp\{\text{factor groups of } G \text{ that are isomorphic to } \Gamma\}\end{aligned}$$

May compute $\delta_\Gamma(G)$ when Γ abelian, or a semidirect product of (certain) abelian groups (Matei-S.)

- **Γ abelian**

Write $\Gamma = \prod_{p \mid |\Gamma|} \Gamma_p$, where Γ_p is a finite abelian p-group. Clearly, $\delta_\Gamma(G) = \prod_{p \mid |\Gamma|} \delta_{\Gamma_p}(G)$. Write

$$\Gamma_p = \mathbb{Z}_{p^{\nu_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\nu_k}}, \ \nu = (\nu_1 \geq \cdots \geq \nu_k).$$

If $G_{\mathrm{ab}} = \mathbb{Z}^n$, then:

$$\boxed{\delta_{\Gamma_p}(G) = \frac{p^{|\nu|(n-1)-2\langle\nu\rangle}\varphi_n(p^{-1})}{\varphi_{n-k}(p^{-1})\prod_{r\geq 1}\varphi_{m_r(\nu)}(p^{-1})}}$$

where: $|\nu| = \sum_{i=1}^k \nu_i, \quad \langle\nu\rangle = \sum_{i=1}^k (i-1)\nu_i,$
$m_r(\nu) = \sharp\{j \mid \nu_j = r\}, \quad \varphi_m(t) = \prod_{i=1}^m (1-t^i).$

- $\Gamma = \mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p$

  ($p \neq q$ primes, $s = \mathrm{ord}_p(q)$, $\sigma \in \mathrm{Aut}(\mathbb{Z}_q^s)$ of order p)

$$\boxed{\delta_{\mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p}(G) = \tfrac{p-1}{s(q^s-1)} \sum_{d=1}^{n} \beta_{p,d}^{(q)}(G)(q^{sd}-1)}$$

Thus, we may compute $\delta_{\mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p}(G)$ from $V_d(G, \mathbb{F}_{q^s})$. E.g., for $S_3 = \mathbb{Z}_3 \rtimes_{(-1)} \mathbb{Z}_2$ and $A_4 = \mathbb{Z}_2^2 \rtimes_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)} \mathbb{Z}_3$:

$$\delta_{S_3}(G) = \tfrac{1}{2} \sum_{d \geq 1} \sharp\, \mathrm{Tors}_2(V_d(G, \mathbb{F}_3) \setminus V_{d+1}(G, \mathbb{F}_3))(3^d - 1)$$

$$\delta_{A_4}(G) = \tfrac{1}{3} \sum_{d \geq 1} \sharp\, \mathrm{Tors}_3(V_d(G, \mathbb{F}_4) \setminus V_{d+1}(G, \mathbb{F}_4))(4^d - 1)$$

This gives info about $a_k(G) = \sharp\{\text{index } k \text{ subgroups of } G\}$, $a_k^{\triangleleft}(G) = \sharp\{\text{index } k \text{ normal subgroups of } G\}$, e.g.:

$$a_3 = \tfrac{1}{2}(3^n - 1) + 3\delta_{S_3}$$

$$a_4 = \tfrac{1}{3}(2^{n+1} - 1)(2^n - 1) + 4(\delta_{D_8} + \delta_{A_4} + \delta_{S_4})$$

$$a_3^{\triangleleft} = \tfrac{1}{2}(3^n - 1)$$

$$a_4^{\triangleleft} = \tfrac{1}{3}(2^{n+1} - 1)(2^n - 1)$$

$$a_6^{\triangleleft} = \tfrac{1}{2}(3^n - 1)(2^n - 1) + \delta_{S_3}$$

$$a_8^{\triangleleft} = \tfrac{1}{21}(2^{n+2} - 1)(2^{n+1} - 1)(2^n - 1) + \delta_{D_8} + \delta_{Q_8}$$

Congruence covers

X finite cell complex, with $H_1(X) \cong \mathbb{Z}^n$.

X_N is the regular $(\mathbb{Z}_N)^n$-cover of X determined by

$$\pi_1(X) \xrightarrow{\mathrm{ab}} H_1(X, \mathbb{Z}) \xrightarrow{\mathrm{mod}\, N} H_1(X; \mathbb{Z}_N).$$

By Libgober and Sakuma:

$$\boxed{b_1(X_N) = n + \sum_{\mathbf{t} \in \mathrm{Tors}_N(\mathbb{C}^*)^n} \mathrm{depth}_{\mathbb{C}}(\mathbf{t}).}$$

Theorem. *(Sarnak-Adams, Sakuma) The sequence* $\{b_1(X_N)\}_{N\in\mathbb{N}}$ *is* polynomially periodic, *i.e., there is* $T \geq 1$, *and polynomials* $P_1(x), \dots, P_T(x)$, *such that*

$$b_1(X_N) = P_i(N), \quad \textit{if } N \equiv i \mod T.$$

Follows from above formula, and

Theorem. *(Laurent) If* V *is a subvariety of* $(\mathbb{C}^*)^n$, *then* $\mathrm{Tors}_N(V) = \bigcup_{i=1}^{v} \mathrm{Tors}_N(S_i)$, *where* S_i *are subtori of* $(\mathbb{C}^*)^n$, *possibly translated by roots of unity.*

For $n \leq 6$: $b_1(X_N(\mathcal{A})) = P_{\mathcal{A}}(N)$.

For $n = 7$: $\exists\ \mathcal{A}$ s.t. $b_1(X_N(\mathcal{A}))$ has period $T = 2$.

Hirzebruch covering surfaces

$\mathcal{A}$ arrangement of n planes in $\mathbb{C}^3$. $\overline{\mathcal{A}}$ projectivized arrangement of lines in $\mathbb{CP}^2$.

- $X_N(\overline{\mathcal{A}})$ congruence cover of $X(\overline{\mathcal{A}})$.
- $\widehat{X}_N(\overline{\mathcal{A}})$ associated branched cover $\mathbb{CP}^2$.
- $M_N(\mathcal{A})$ minimal desingularization of $\widehat{X}_N(\overline{\mathcal{A}})$.

Hirzebruch computed the Chern numbers of $M_N(\mathcal{A})$:

$$c_1^2 = \big((3b_2 - s - 5n + 9)N^2 - 4(b_2 - n)N + (b_2 + n + m_2)\big)N^{n-3}$$
$$c_2 = \big((b_2 - 2n + 3)N^2 - 2(b_2 - n)N + (b_2 + s - m_2)\big)N^{n-3}$$

where $m_r = \sharp\{I \in L_2(\mathcal{A}) \mid |I| = r\}$,
$s = \sum m_r, \quad b_2 = \sum_r m_r(r-1)$

Sakuma computed the first Betti number of $M_N(\mathcal{A})$:

$$\boxed{b_1(M_N(\mathcal{A})) = \sum_{\mathbf{t} \in \mathrm{Tors}_N(\mathbb{C}^*)^n} \mathrm{depth}_{\mathbb{C}}(\mathbf{t}|_{\mathcal{A}_{\mathbf{t}}})}$$

where $\mathcal{A}_{\mathbf{t}} = \{H_i \in \mathcal{A} \mid \mathbf{t}(x_i) \neq 1\}$.

Theorem. *(Hironaka, Sakuma) The sequence* $\{b_1(M_N(\mathcal{A}))\}_{N\in\mathbb{N}}$ *is polynomially periodic.*

For $n \leq 7$: $b_1(M_N(\mathcal{A})) = P_{\mathcal{A}}(N)$.

For $n = 8$: $\exists\ \mathcal{A}$ s.t. $b_1(M_N(\mathcal{A}))$ has period $T = 4$.

Lower central series quotients

G f.g. group. *Lower central series*:

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots, \quad \text{where } \gamma_{k+1} G = [\gamma_k G, G]$$

LCS quotients: $\operatorname{gr}_k G = \gamma_k G / \gamma_{k+1} G$ (f.g. abelian)

$$\boxed{\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G)}$$

Chen groups: $\operatorname{gr}_k(G/G'')$.

$$\boxed{\theta_k(G) = \operatorname{rank}(\operatorname{gr}_k G/G'')}$$

Clearly, $\phi_1 = \theta_1$, $\phi_2 = \theta_2$, $\phi_3 = \theta_3$, $\phi_k \geq \theta_k$. E.g.:

$$\phi_k(F_n) = w_k(n) := \tfrac{1}{k} \sum_{d|k} \mu(d) n^{k/d} \qquad \text{(Witt)}$$

$$\theta_k(F_n) = \tbinom{n+k-2}{k}(k-1), \quad \text{for } k \geq 2$$

(Murasugi, Massey-Traldi)

Massey: $\boxed{\operatorname{gr}_k(G/G'') = \mathfrak{I}^{k-2} B / \mathfrak{I}^{k-1} B}$

$B = G'/G''$: Alexander invariant ($\mathbb{Z}[G/G']$-module),

$\mathfrak{I} = \ker \epsilon$: augmentation ideal.

Hence, if $G/G' \cong \mathbb{Z}^n$: $\boxed{\sum_{k\geq 0} \theta_{k+2}\, t^k = \operatorname{Hilb}(\operatorname{gr} B)}$

A presentation for $\operatorname{gr} B = \bigoplus_{k\geq 0} \operatorname{gr}_k B$ (as module over $\operatorname{gr} \mathbb{Z}[G/G'] \cong \mathbb{Z}[\lambda_1, \dots, \lambda_n]$) can be obtained from a presentation for B via a Gröbner basis algorithm.

Let $\mathcal{A}$ be arrangement of n hyperplanes, $G = G(\mathcal{A})$.

Falk: $\phi_k = \phi_k(G)$ combinatorially determined.

Cohen-S.: Presentation for the Alexander invariant $B = G'/G''$. This gives algorithm for computing the Chen groups of $G(\mathcal{A})$. E.g., for $P_n = \rtimes_{i=1}^{n-1} F_i$:

$$\theta_k(P_n) = (k-1)\tbinom{n+1}{4}, \quad \text{for } k \geq 3.$$

It follows that $\theta_k(P_n) \neq \theta_k(\Pi_n)$, where $\Pi_n = \prod_{i=1}^{n-1} F_i$. On the other hand, $\phi_k(P_n) = \phi_k(\Pi_n)$, by

Theorem. *(LCS formula of Falk and Randell) If $\mathcal{A}$ fiber-type, with exponents $d_1, \dots, d_\ell$:*

$$\boxed{\prod_{k\geq 1} (1-t^k)^{\phi_k} = P(X, -t) = \prod_{i=1}^{\ell} (1 - d_i t)}$$

i.e., $\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i})$.

Kohno: First proved LCS formula for $\mathcal{A} = \mathcal{B}_\ell$.

Shelton-Yuzvinsky: Consequence of Koszul duality.

Papadima-Yuz.: Extend LCS formula to formal rational $K(G,1)$ spaces. Also, if $\mathcal{A}$ arrangement in $\mathbb{C}^3$:

$$\text{LCS formula holds} \Longleftrightarrow \mathcal{A} \text{ fiber-type.}$$

Jambu-Papadima: Extend LCS formula to hypersolvable arrangements.

Let $\mathcal{R}_1(\mathcal{A}) = \bigcup_{i=1}^{v} L_i$, with L_i linear subspaces of $\mathbb{C}^n$. Let $h_r = \sharp\{L_i \mid \dim L_i = r\}$.

Conjecture. The Chen groups of $G = G(\mathcal{A})$ are free abelian, of rank

$$\boxed{\theta_k(G) = \sum_{r\geq 2} h_r \theta_k(F_r), \quad \text{for } k \geq 4.}$$

(This would imply that the Chen groups of an arrangement are combinatorially determined.)

Conjecture. If $\phi_4 = \theta_4$, then $\operatorname{gr}_k G$ is free abelian, of rank

$$\boxed{\phi_k(G) = \sum_{r\geq 2} h_r \phi_k(F_r), \quad \text{for } k \geq 4.}$$

Resonance varieties & nilpotent quotients

Let X be a finite CW-complex, with $H_*(X)$ torsion-free, and $H^*(X)$ generated in degree 1.
E.g.: $X = X(\mathcal{A})$, or $X = X(\text{link in } S^3 \text{ with } \mathrm{lk}_{i,j} = \pm 1)$.
Let $G = \pi_1(X)$.

Theorem. *(Matei-S.) If X, X' as above, then:*

$$\boxed{H^{\le 2}(X) \cong H^{\le 2}(X') \Longleftrightarrow G/\gamma_3 G \cong G'/\gamma_3 G'.}$$

For p prime, $d \ge 0$, define:

$$\nu_{p,d}(G/\gamma_3 G) = \sharp \left\{ K \lhd G/\gamma_3 G \;\middle|\; \begin{array}{l} [G/\gamma_3 G : K] = p \;\text{ and} \\ \dim_{\mathbb{F}_p}(\operatorname{Tors} H_1(K)) \otimes \mathbb{F}_p = d \end{array} \right\}$$

Then:

$$\boxed{\nu_{p,d}(G/\gamma_3 G) = \frac{\sharp(R_{p,d}(G, \mathbb{F}_p) \setminus R_{p,d+1}(G, \mathbb{F}_p))}{p-1}}$$

One may define higher-order resonance varieties, $S_d^{(k)}(G, \mathbb{K})$, using higher-order Massey products $<x, \lambda, \dots, \lambda>$, or higher-order truncations of A_G (D. Matei, Ph.D. thesis). One may also define $\nu_{p,d}(G/\gamma_{k+2} G)$, as above. An analogue of the framed formula remains to be found.

Higher homotopy groups

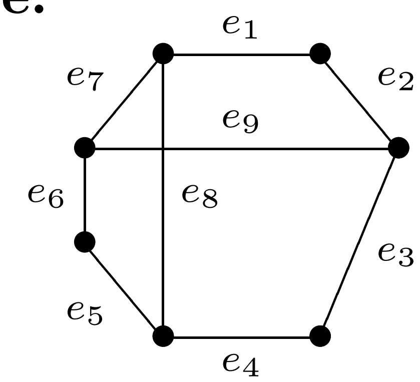
Let $\mathcal{G}$ be a graph, $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$, $X = X(\mathcal{A})$.

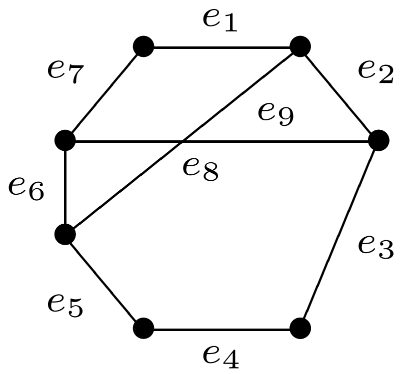
Assume $\mathcal{G}$ has no 3-cycles. Then $\pi_1(X) = \mathbb{Z}^n$, where $n = \sharp$edges. Let $\mathcal{S}$ be the set of 4-cycles.

Proposition. *(Papadima-S.) If* $\mathcal{S} = \emptyset$, *then* $\pi_2(X) = 0$. *Otherwise, the* $\mathbb{Z}\pi_1$*-module* $\pi_2(X)$ *is combinatorially determined (by the graph* $\mathcal{G}$*), and* $\pi_2(X)_{\pi_1} = \mathbb{Z}[\mathcal{S}] \neq 0$.

Let $V_d(\pi_2) = V\big(\bigwedge^d \pi_2\big) \subset (\mathbb{C}^*)^n$. These varieties may distinguish π_2's with the same coinvariants.

Example.





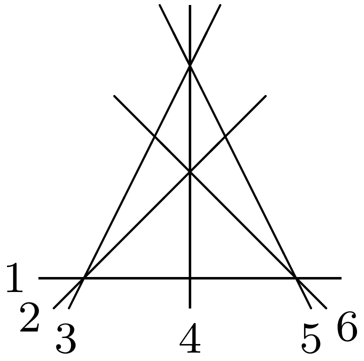
The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have no 3-cycles. Each graph has exactly two 4-cycles. The complements $X_i = X(\mathcal{A}_{\mathcal{G}_i})$ have:

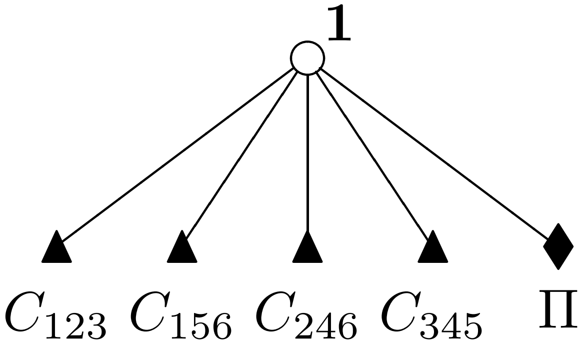
$$\pi_1 = \mathbb{Z}^9, \quad (\pi_2)_{\pi_1} = \mathbb{Z}^2, \quad b_2 = 36, \quad b_3 = 82.$$

$V_1(\pi_2(X_1))$ has 2 components, $V_1(\pi_2(X_1))$ has 3. Hence:

$$\pi_2(X_1) \not\cong \pi_2(X_2) \qquad \text{(as } \mathbb{Z}\pi_1\text{-modules)}.$$

Example (Braid arrangement).



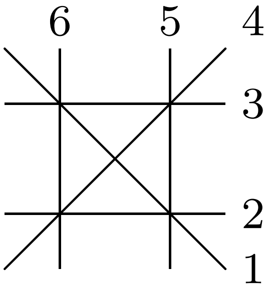


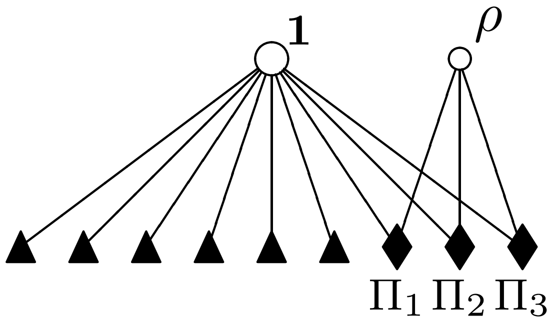
- $Q = xyz(x-y)(x-z)(y-z)$.
- $n = 6$, $s = 7$, $m_2 = 3$, $m_3 = 4$.
- $P(X,t) = (1+t)(1+2t)(1+3t)$.
- $G = P_4 = F_3 \rtimes F_2 \rtimes F_1$.
- $V_1(G,\mathbb{K}) = C_{124} \cup C_{135} \cup C_{236} \cup C_{456} \cup \Pi$, where

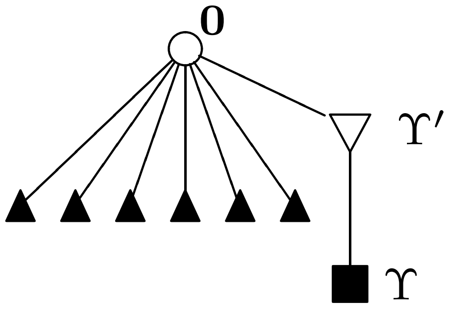
  $\Pi = C_{(14|25|36)} = \{(s,t,(st)^{-1},s,t,(st)^{-1}) \mid s,t \in \mathbb{K}^*\}$.

  $V_2(G,\mathbb{K}) = \{\mathbf{1}\}$
- $\beta^{(q)}_{p,1} = \nu_{p,1} = 5(p+1)$.
- $b_1(X_N) = 5N^2 + 1$.
- $b_1(M_N) = 5(N-1)(N-2)$
  $c_1^2(M_N) = 5N^3(N-2)^2$, $c_2(M_N) = N^3(2N^2-10N+15)$.
- $\delta_{S_3} = 15$, $\delta_{A_4} = 20$, $a_2 = 63$, $a_3^{\triangleleft} = 364$, $a_3 = 409$.
- $\phi_1 = 6$, $\phi_2 = 4$, $\phi_3 = 10$, $\phi_4 = 21$, $\phi_k = w_k(2) + w_k(3)$.
- $\theta_1 = 6$, $\theta_2 = 4$, $\theta_3 = 10$, $\theta_4 = 15$, $\theta_k = 5(k-1)$.

Example (Non-Fano plane).

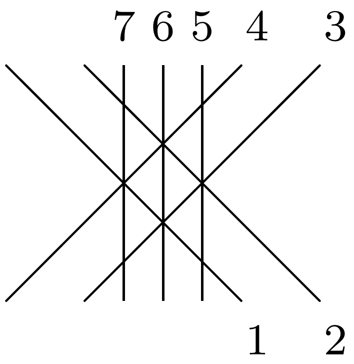


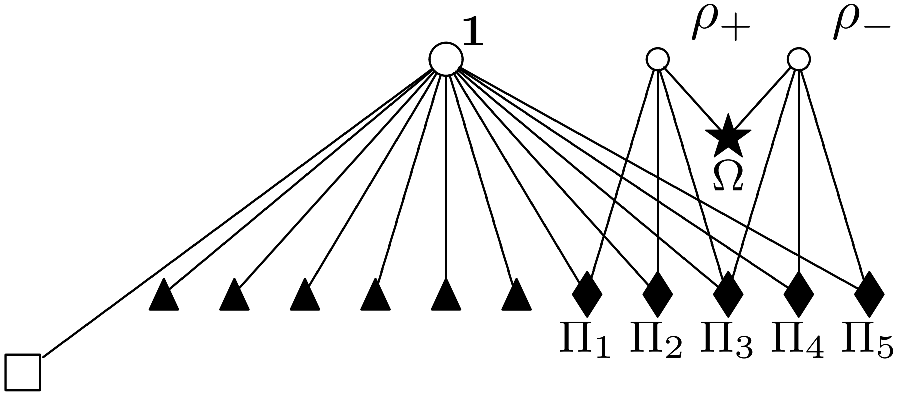


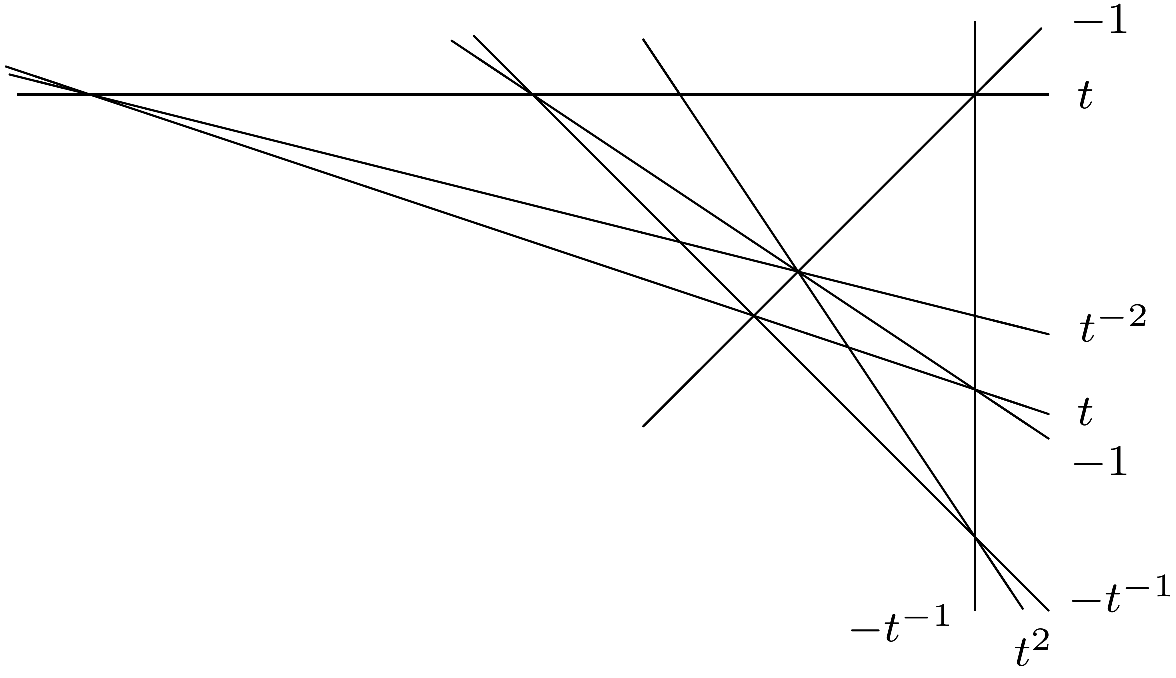


- $Q = xyz(x-y)(x-z)(y-z)(x+y-z)$.
- $n = 7$, $s = 9$, $m_2 = 3$, $m_3 = 6$.
- $P(X,t) = (1+t)(1+3t)^2$.
- $V_1(G,\mathbb{K})$ has nine 2-dim components.
  $V_2(G,\mathbb{K}) = \Pi_1 \cap \Pi_2 \cap \Pi_3 = \{\mathbf{1}, \rho\}$, where $\rho^2 = \mathbf{1}$
- $\mathcal{R}_1(G,\mathbb{F}_2)$ has 3-dim component $\Upsilon \not\subset \mathrm{TC}_{\mathbf{1}}(V_1(G,\mathbb{F}_2))$.
  $\mathcal{R}_2(G,\mathbb{F}_2)$ has 1-dim component $\Upsilon' \not\subset \mathrm{TC}_{\mathbf{1}}(V_2(G,\mathbb{F}_2))$.
- $\beta^{(q)}_{p,1} = \nu_{p,1} = 9(p+1)$, except for:
  $$\beta^{(q)}_{2,1} = \nu_{2,1} = 24,\ \beta^{(q)}_{2,2} = \nu_{2,2} = 1.$$
- $b_1(X_N) = \begin{cases} 9N^2 - 3 & \text{if } N \text{ even,} \\ 9N^2 - 2 & \text{if } N \text{ odd.} \end{cases}$
- $b_1(M_N) = 9(N-1)(N-2)$.
- $\delta_{S_3} = 28$, $\delta_{A_4} = 36$, $a_2 = 127$, $a_3^{\triangleleft} = 1,093$, $a_3 = 1,177$.
- $\phi_1 = 7$, $\phi_2 = 6$, $\phi_3 = 17$, $\phi_4 = 42$, $\phi_5 = 123$, $\phi_6 = 341$, $\phi_7 = 1,041$.
- $\theta_1 = 7$, $\theta_2 = 6$, $\theta_3 = 17$, $\theta_4 = 27$, $\theta_k = 9(k-1)$.

Example (Deleted B_3 arrangement).







- $Q = xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z)$.
- $n = 8$, $s = 11$, $m_2 = 4$, $m_3 = 6$, $m_4 = 1$.
- $V_1(G, \mathbb{K})$ has a 1-dim component which does *not* pass through $\mathbf{1}$ (unless $\operatorname{char}\mathbb{K} = 2$):

$$\Omega = \{(t, -t^{-1}, -t^{-1}, t, t^2, -1, t^{-2}, -1) \mid t \in \mathbb{K}^*\}$$

- $\nu_{p,1} = 11(p+1)$, $\nu_{p,2} = \frac{p^3-1}{p-1}$, $\beta^{(q)}_{p,d} = \nu_{p,d}$, except:

$$\beta^{(q)}_{2,1} = 27,\ \beta^{(q)}_{2,2} = 9,\ \beta^{(2)}_{3,1} = 45.$$

Distribution of index 3, normal subgroups in G:

K_{ab}	$\mathbb{Z}^8$	$\mathbb{Z}^8 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$
$\sharp K$	$1,035$	$1^{(*)}$	44	13

$^{(*)}$ $K = \ker(\gamma : G \twoheadrightarrow \mathbb{Z}_3)$, $\gamma = (\omega, \omega^2, \omega^2, \omega, \omega^2, 1, \omega, 1) \in (\mathbb{Z}_3)^8$.
Since $\gamma \notin V_1(G, \mathbb{C})$, but $\gamma \in \Omega \subset V_1(G, \mathbb{F}_2(\omega))$:

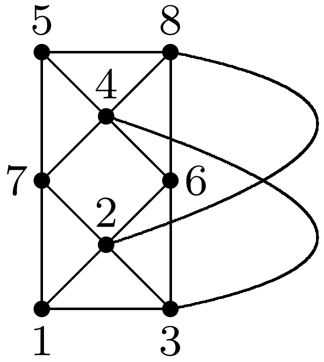
$$b_1(K) = 8, \quad b_1^{(2)}(K) = 8 + (3-1)\cdot 1 = 10.$$

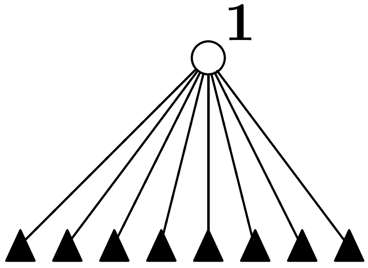
- $b_1(X_N) = \begin{cases} 2N^3 + 11N^2 + N - 9 & \text{if } N \text{ even,} \\ 2N^3 + 11N^2 - 5 & \text{if } N \text{ odd,} \end{cases}$
- $b_1(M_N) =$

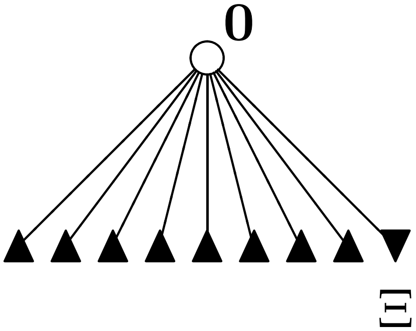
$$\begin{cases} (N-1)(2N^2 + 9N - 24) + N - 2 & \text{if } N \equiv 0 \mod 4 \\ (N-1)(2N^2 + 9N - 24) + \frac{1}{2}(N-2) & \text{if } N \equiv 2 \mod 4 \\ (N-1)(2N^2 + 9N - 24) & \text{if } N \text{ odd.} \end{cases}$$

- $\delta_{S_3} = 63$, $\delta_{A_4} = 110$, $a_2 = 255$, $a_3^{\triangleleft} = 3,280$, $a_3 = 3,469$.
- $\phi_1 = 8$, $\phi_2 = 9$, $\phi_3 = 28$, $\phi_4 = 78$, $\phi_k = w_k(3) + w_k(4)$.
- $\theta_1 = 8$, $\theta_2 = 9$, $\theta_3 = 28$, $\theta_4 = 48$, $\theta_k = (k+12)(k-1)$.

Example (MacLane arrangement).





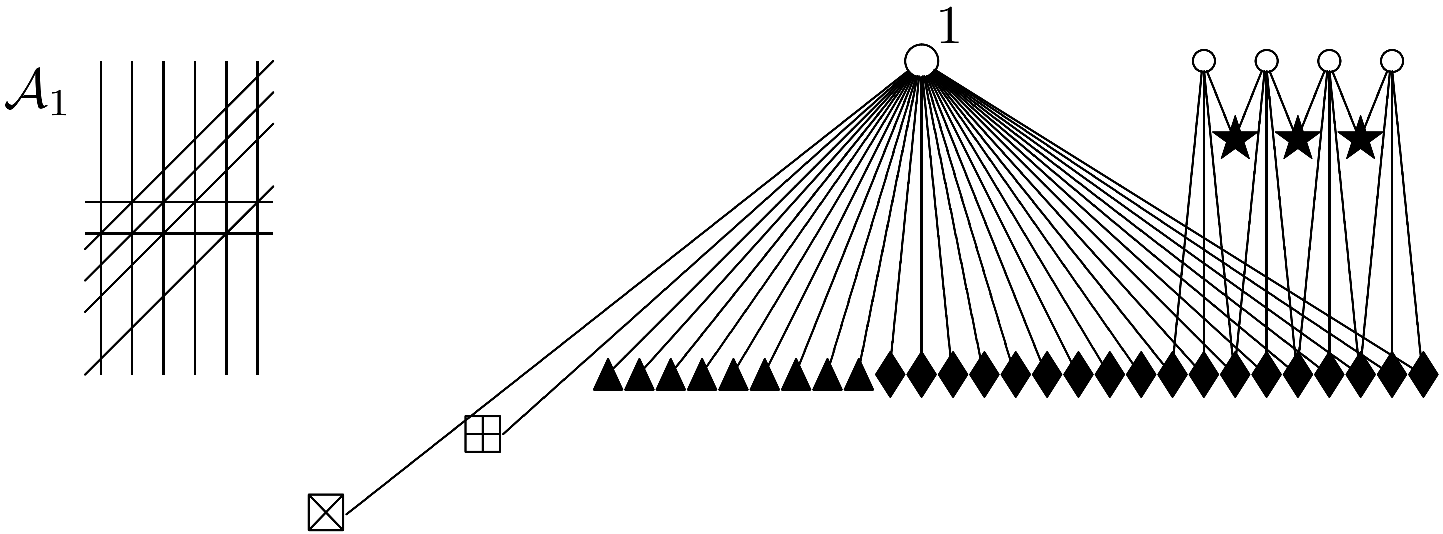


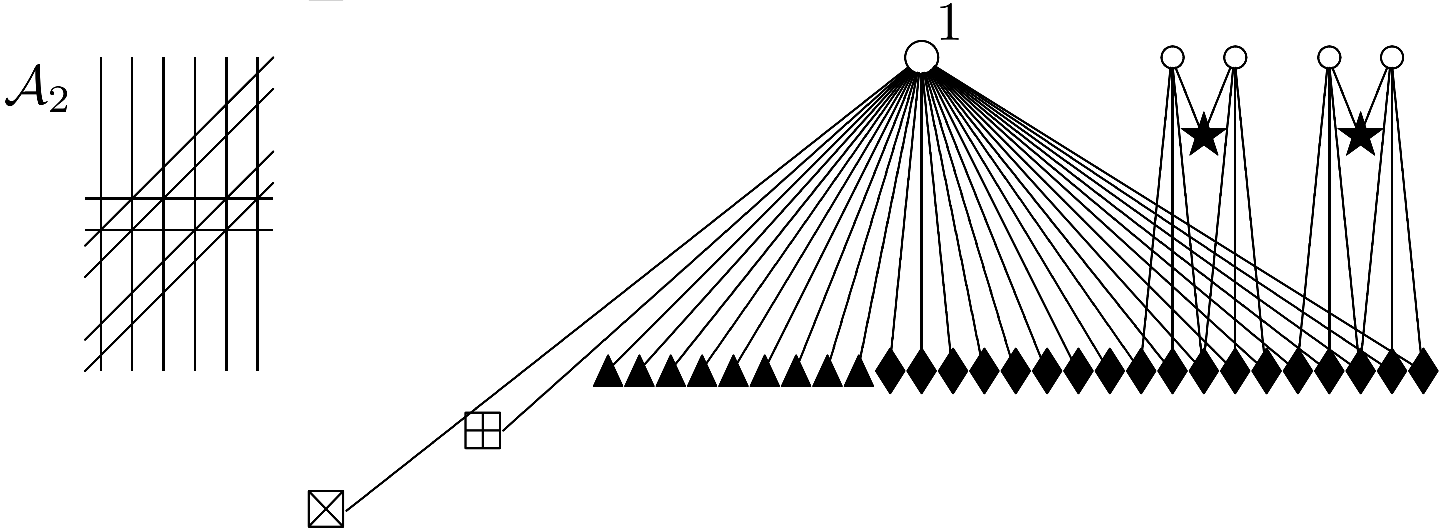
- $Q = xyz(y-x)(z-x)(z+\omega y)(z+\omega^2 x+\omega y)(z-x-\omega^2 y)$.
- $n = 8$, $s = 12$, $m_2 = 4$, $m_3 = 8$.
- $P(X,t) = (1+t)(1+7t+13t^2)$.
- $V_d(G,\mathbb{K})$ has only local components
- $\mathcal{R}_1(G,\mathbb{F}_3)$ has one non-local, 2-dim component:
  $\Xi = \{(\lambda-\mu, \lambda, \mu, -\lambda, -\mu, \lambda+\mu, -\lambda-\mu, \mu-\lambda) \mid \lambda, \mu \in \mathbb{F}_3\}$
- $\beta_{p,1}^{(q)} = 8(p+1)$ and $\nu_{p,1} = 8(p+1)$, except for $\nu_{3,1} = 36$.
- $b_1(X_N) = 8N^2$.
- $b_1(M_N) = 8(N-1)(N-2)$.
- $\delta_{S_3} = 24$, $\delta_{A_4} = 32$, $a_2 = 255$, $a_3^{\triangleleft} = 3,280$, $a_3 = 3,352$.
- $\phi_1 = 8$, $\phi_2 = 8$, $\phi_3 = 21$, $\phi_4 = 42$, $\phi_5 = 87$, $\phi_6 = 105$.
- $\theta_1 = 8$, $\theta_2 = 8$, $\theta_3 = 21$, $\theta_4 = 24$, $\theta_k = 8(k-1)$.

Note: There is torsion in the LCS quotients of G. E.g.:

$$\operatorname{gr}_5 G = \mathbb{Z}^{87} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_3$$

Example (Ziegler arrangements).





- $n = 13$, $s = 31$, $m_2 = 20$, $m_3 = 9$, $m_5 = 1$, $m_7 = 1$.
- $P(X_i, t) = (1 + t)(1 + 6t)^2$.
- $\phi_1 = 13$, $\phi_2 = 30$, $\phi_3 = 140$, $\phi_k = 2w_k(6)$.
- $\theta_k = \frac{(k-1)(k^4+10k^3+47k^2+86k+696)}{24}$.
- $R_d(G_1, \mathbb{K})$ and $R_d(G_2, \mathbb{K})$ are (abstractly) isomorphic. Hence, $\nu_{p,d}(G_1) = \nu_{p,d}(G_2)$.
- $a_2 = 8,191$, $a_3^{\triangleleft} = 797,161$, $a_3 = 820,870$.

- $V_1(G_1, \mathbb{K})$ has $\tau_1 = 3$ translated subtori
  $V_1(G_2, \mathbb{K})$ has $\tau_2 = 2$ translated subtori
- $\beta_2^{(q)} = (69, 4, 15, 0, 63)$
  $\beta_p^{(q)} = (27(p+1), 0, \frac{p^4-1}{p-1}, 0, \frac{p^6-1}{p-1})$
  except if $p = 3$, $q = 2$, $d = 1$

  Thus: $\delta_\Gamma(G_1) = \delta_\Gamma(G_2)$, if $\Gamma = \mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p \not\cong A_4$.
  But:

$$\boxed{\beta_{3,1}^{(2)}(G_1) = 111, \quad \beta_{3,1}^{(2)}(G_2) = 110.}$$

Hence:

$$\boxed{\delta_{A_4}(G_1) = 124,435, \quad \delta_{A_4}(G_2) = 124,434.}$$

- $b_1(X_N(\mathcal{A}_i)) =$
$$\begin{cases} 5N^6 + 3N^4 + 27N^2 + \tau_i(N-2) - 26 & \text{if } N \text{ even} \\ 5N^6 + 3N^4 + 27N^2 - 22 & \text{if } N \text{ odd} \end{cases}$$
- $b_1(M_N(\mathcal{A}_i)) = \begin{cases} f(N) + \tau_i(N-2) & \text{if } N \equiv 0 \mod 4 \\ f(N) + \frac{\tau_i}{2}(N-2) & \text{if } N \equiv 2 \mod 4 \\ f(N) & \text{if } N \text{ odd,} \end{cases}$
  where
  $f(N) = (N-1)(5N^5 - 2N^4 + N^3 - 4N^2 + 23N - 58)$