

I. Genericity Conditions

(ESV + STV)

 $\mathcal{R} = \lambda$ affine arr. in \mathbb{C}^2
essential

$$\lambda \in \mathbb{C}^n \iff t = \exp(2\pi i \lambda) \in (\mathbb{C}^*)^n$$

$$a_\lambda = \sum_{i=1}^n \lambda_i a_i$$

 $\mathcal{L}_t =$ rank 1 local system
on M determined by t

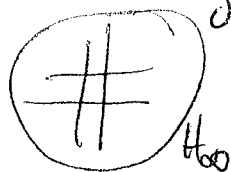
$$x \in \mathcal{L} \longrightarrow \lambda_x = \sum_{H \ni x} \lambda_i$$

(ESV + STV) iff $\lambda_x \notin \mathbb{Z}_{>0}$ for every irreducible

$$x \in \mathcal{L}(A_\infty) \text{ then } H^*(M, \mathcal{L}_t) = H^*(A, a_\lambda)$$

*0 (ie. $x \in \mathcal{L}(A), x \neq 1$)

$$\mathcal{R}_\infty = \mathcal{R} \cup \{H_\infty\}$$



$$\lambda_\infty := -\sum_{i=1}^n \lambda_i$$

(Yaz) If $\lambda_x \neq 0 \forall x \in L(\mathcal{R})$ then

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$$H^k(A, a_x) = \begin{cases} 0 & k \neq 1 \\ \mathbb{C}^{\beta(\mathcal{R})} & k = 1 \end{cases}$$

$\beta(\mathcal{R}) =$ beta invariant of $\mathcal{C}\mathcal{R}$

$$= \left| \frac{\text{Poin}(L(\mathcal{C}\mathcal{R}), t)}{(1+t)} \right|_{t=-1}$$

$$= |\chi(M(\mathcal{R}))| \stackrel{\text{Thm}}{=} \# \text{ Bdd} \\ \text{Zaslavsky Chambers of } M(\mathcal{R}) \cap \mathbb{R}^2 \\ \text{if } \mathcal{R} \text{ is simplified real}$$

II. "Gauge Transformations"

① $\lambda, \lambda' \in \mathbb{C}^n$ and $\lambda - \lambda' \in \mathbb{Z}^n$

$$\Rightarrow \mathcal{L}_\lambda = \mathcal{L}_{\lambda'} \Rightarrow H^*(M, \mathcal{L}_\lambda) = H^*(M, \mathcal{L}_{\lambda'})$$

But $H^*(A, a_\lambda) \& H^*(A, a_{\lambda'})$ are not necessarily ~~iso~~ equal.

② If $\lambda' = c\lambda$, $c \in \mathbb{C}^*$ then $H^*(A, a_\lambda) \cong H^*(A, a_{\lambda'})$

but $H^*(M, \mathcal{L}_\lambda) \& H^*(M, \mathcal{L}_{\lambda'})$ are not necessarily ~~iso~~ equal.

Note: (1) each H_i is irreducible

(2) $rk(X) = 2 \implies X$ is irreducible
iff $|R_X| \geq 3$.

Modifications of λ :

Assume $\lambda_i \notin \mathbb{Z} \forall i$ and $\sum_{i=1}^n \lambda_i \notin \mathbb{Z}$

Replace λ_i with $\lambda_i - N$, $N \gg 0, N \in \mathbb{Z}$
for $i=1, \dots, n$

Then $\lambda'_x \notin \mathbb{Z}_{>0}$ for any $x \notin H_\infty$

$$\lambda'_\infty = \left(-\sum_{i=1}^n \lambda_i\right) + nN$$

If no $X \subseteq H_\infty, X \neq H_\infty$, is irr., then
ESV/STV cond. is satisfied.

$$H^*(M, \mathcal{L}_+) \cong H^*(A, a_X)$$

Degree One Resonance Varieties

Assume $\lambda \in \mathbb{Z}$
For what λ is $H^1(A, a_\lambda) \neq 0$?

$$R_1^1(\mathbb{C}) = \{\lambda \in \mathbb{C}^n \mid H^1(A, a_\lambda) \neq 0\} \quad K \text{ any field}$$

$$R_d^1(\mathbb{C}) = \{\lambda \in \mathbb{C}^n \mid \dim_K H^1(A, a_\lambda) \geq d\}$$

$$\sigma \rightarrow A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \dots$$

\downarrow
 \mathbb{K}

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$a_\mu \in A_1$ represents a non-trivial element of $H^1(A, a_1)$
 iff $a_1 \wedge a_\mu = 0$ and $a_\mu \notin \mathbb{K}a_1$
 (i.e. $\mu \notin \mathbb{K}\lambda$)

Lifting to E :

$$0 \neq e_\lambda \wedge e_\mu \in I^2 \Leftrightarrow 0 \neq [a_\mu] \in H^1(A, a_1)$$

Recall: Assume \mathcal{A} is central (of rank 3)

$A_2 = \bigoplus_{x \in L} A_x^2$ Write $a_\lambda^x = \sum_{H_i \geq x} \lambda_i a_i$
 $\text{rk}(x) = 2$

Then $a_\lambda \wedge a_\mu = 0$ iff $a_\lambda^x \wedge a_\mu^x = 0$
 $\forall x \in L, \text{rk}(x) = 2$

Problem 1: Spns \mathcal{A} has rk two. Show

$$a_\lambda \wedge a_\mu = 0 \text{ iff } \mu \in \mathbb{K}\lambda \text{ or } \underbrace{\sum_{i=1}^n \lambda_i}_{n \geq 3} = 0 = \sum_{i=1}^n \mu_i$$

(Hint: Recall $de_{jk} = (e_i - e_j) \wedge (e_j - e_k)$)

Thm: $\lambda \in \mathbb{R}^1(A, \mathbb{K})$ iff $\exists \mu \in \mathbb{K}^n$ st. $\forall x \in L(\mathcal{A}), \text{rk}(x) = 2$

either (i) $\mu^x \parallel \lambda^x$ and $\lambda \neq \mu$

or (ii) $|L_x| \geq 3$ and $\lambda_x = \mu_x = 0$.

$$\lambda^x = (\lambda_i \mid H_i \in \mathcal{P}_x) \in \mathbb{K}^{|dx|}$$

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Assume λ & μ satisfy theorem then
we say (λ, μ) is a resonant pair.

Define $\text{supp}(\lambda, \mu) = \{i \in [n] \mid d_i \neq 0 \text{ or } m_i \neq 0\}$

Assume $\text{supp}(\lambda, \mu) = [n]$

Define an equivalence relation on $[n]$ by $i \sim j$

$$\text{iff } \begin{vmatrix} d_i & m_i \\ d_j & m_j \end{vmatrix} = 0.$$

The Theorem now says:

$\forall X \in \mathcal{L}, \text{rk}(X) = 2$ either

$$(i) i \sim j \quad \forall i, j \in X$$

or (ii) $|X| \geq 3$ and $\lambda_X = 0 = \mu_X$

Lemma: $\text{Sp}_3 \mathcal{A}$ has rk two.

Then A_2 has basis

$$\{a_i, \mu_i \mid 2 \leq i \leq n\}$$

Pf: Exercise

Thm! Sp's $X = \{1, \dots, P\}$ and $\text{rk} X = 2$

$$i \sim j \quad \forall \quad i, j \geq 2$$

Then $i \sim j \quad \forall \quad i, j \geq 1$

Pf:

$$0 = \left(\sum_{i=1}^P \lambda_i a_i \right) \wedge \left(\sum_{j=1}^P \mu_j a_j \right) = \sum_j \begin{vmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{vmatrix} a_i \wedge a_j = 0$$

$$= \sum_j \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_j & \mu_j \end{vmatrix} a_1 \wedge a_j$$

By lemma $\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_j & \mu_j \end{vmatrix} = 0 \quad \forall j.$

Def: A partition \mathcal{P} of $[n]$ is neighborly (w.r.t. \mathbb{R})

iff for every block B of \mathcal{P} and
 every flat X of $\text{rk} X = 2$
 $|X \cap B| \geq |X| - 1$ then $X \subseteq B$

Cor. If (λ, μ) is a resonant pair,
 then the induced partition of $[n]$
 is neighborly.

Cor. If (λ, μ) is a resonant pair and X
 is a flat of $\text{rk} X = 2$ with $X \not\subseteq$ any block
 of the induced partition then $\lambda_x = 0 = \mu_x.$

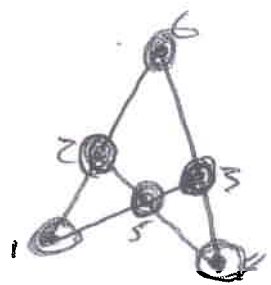
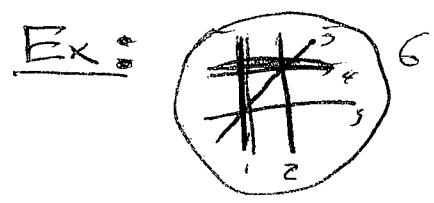
So, we have:

If $\mathcal{A} \in \mathcal{R}^1$ supports a resonant pair of weights (d, n) then \mathcal{A} supports a neighborly partition \mathcal{P} and

\exists two independent vectors in the kernel of the incidence matrix

$x \notin B \rightarrow$
 "multi-colored flats"
 \rightarrow
 \rightarrow

$\left[\begin{matrix} 1 & \dots & n \\ \vdots & & \vdots \end{matrix} \right]$



$P = 14 | 23 | 56$

★ Find examples of arr.'s that support neighborly partitions.

Cordune

IF $|X \cap B| \geq |X| - 1 \Rightarrow X \subseteq B$

IF $|X| = 2$ then $X \cap B \neq \emptyset \Rightarrow X \subseteq B$

126	}	1	2	3	4	5	6	has multi: ≥ 2
135		1	1	0	0	0	1	
245		1	0	1	0	1	0	
246		0	1	0	1	1	0	
346		0	0	1	1	0	1	

⇒ λ = (1, 0, 0, 1, -1, -1) = red - green

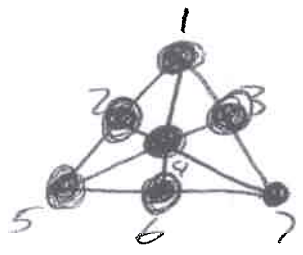
μ = (0, 1, 1, 0, -1, -1) = blue - green

Conclusion: (a₁ + a₄ - a₅ - a₆) ∧ (a₂ + a₃ - a₅ - a₆) = 0

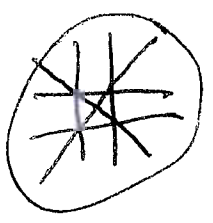
a₁ ∧ a₂ = 0

⇒ [Ka₁ + Ka₂ ⊆ R'(A)]

Ex:



P = 1 | 2 3 6 | 4 | 5 | 7



$$I = \begin{matrix} 125 \\ 137 \\ 247 \\ \vdots \\ 6 \end{matrix} \left[\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 \times 7 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & \end{array} \right]$$

nullity_ℝ(I) = 1

nullity_{ℤ₂}(I) = 3

nullspace has basis: _____

over ℤ₂, there is a vector in R'_{ℤ₂}(A)