LCS Formula [Falk]

$\mathcal{A}$ central arr. $G = \pi_1(M)$

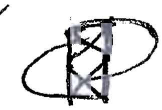
$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$ L.C.S.

$G_i = [G_{i-1}, G]$

$\phi_i(\mathcal{A}) = \operatorname{rank} G_i / G_{i+1}$

G_i/G_{i+1} abelian, f.g. not necessarily torsion free

$P_M(t) = \operatorname{Poin}(\mathcal{A}, t)$

()

LCS formula

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k(\mathcal{A})} = P_M(-t)$$

First ex: 

$\mathcal{A} = \{H\}$ $G = \mathbb{Z}$

$\phi_1 = 1$ $\phi_2 = 0 = \phi_k$ $k \geq 2$

$(1-t) = 1 - t$

First Example

Kohno: $\mathcal{A} = \mathcal{A}_\ell$ - Braid arr.

$$\Longrightarrow \quad \prod_{k \geq 1} (1-t^k)^{\phi_k} = (1-t)(1-2t)(1-3t)\cdots(1-(\ell-1)t)$$

<u>pf</u>: rational homotopy theory / holonomy Lie alg.

$\mathcal{A}_\ell$ is supersolvable / fiber-type w/ exponents $1, 2, \dots, \ell-1$

$$\begin{tikzcd}
 & & & \mathbb{C}^* \arrow[d, phantom, "\shortparallel"] \\
M(\mathcal{A}_\ell) \arrow[r] & M(\mathcal{A}_{\ell-1}) \arrow[r] & M(\mathcal{A}_{\ell-2}) \arrow[r] & \cdots \arrow[r] & M(\mathcal{A} \\
\mathbb{C} \setminus (\ell-1 \text{ pts}) \arrow[u] \arrow[r] & \mathbb{C} \setminus (\ell-2 \text{ pts}) \arrow[u] \arrow[r] & \cdots \arrow[u]
\end{tikzcd}$$

$$\pi_1(\mathbb{C} \setminus (d \text{ pts})) \cong F_d$$

<u>Witt's formula</u>

$$\prod_{k \geq 1} (1-t^k)^{\phi_k(F_d)} = 1 - dt$$

$$\phi_k(F_d) = \sum_{r \mid k} \frac{1}{k} \mu\left(\frac{k}{r}\right) d^r$$

$$\begin{array}{ccc} \mathbb{C}\setminus(d \text{ pts}) & \longrightarrow & M(\mathcal{F}) \\ & \text{fiber bundle} & \Big\downarrow \text{projection} \\ & & M(\mathcal{B}) \end{array}$$

f.b.– type $\Downarrow$ ($M(\mathcal{B})$ is aspherical)

long exact htpy seq.

$$\pi_2(\mathcal{B}) \to \pi_1(\mathbb{C}\text{-}d\text{pts}) \to \pi_1(M(\mathcal{F})) \to \pi_1(M(\mathcal{B})) \to 1$$

$\pi_2(\mathcal{B}) = 1$

So we have

$$1 \to \underset{A}{F_d} \to \underset{B}{\pi_1(M(\mathcal{F}))} \underset{\sigma}{\rightleftarrows} \underset{C}{\pi_1(M(\mathcal{B}))} \to 1$$

Ingredients: (i) $\exists$ a section

(ii) action of $\pi_1(M(\mathcal{B}))$ on F_d is trivial on $F_d/[F_d, F_d]$

$$1 \to A \to B \rightleftarrows C \to 1$$

$\overset{\text{hard work}}{\Longrightarrow}$ $1 \to A_n \to B_n \rightleftarrows C_n \to 1$

$$\Longrightarrow 1 \to A_n/A_{n+1} \to B_n/B_{n+1} \to C_n/C_{n+1} \to 1$$

from free gp

Conclusion (by induction)

B_n/B_{n+1} is free abelian and

$$\mathrm{rk}\left(B_n/B_{n+1}\right) = \mathrm{rk}\left(C_n/C_{n+1}\right) + \mathrm{rk}\left(A_n/A_{n+1}\right)$$

for $\mathcal{A}$ fiber type w/ ~~exponents~~ fibers $\mathbb{C} - d_i$ pts $i=1,\dots,\ell$

$\Longrightarrow$ ① $P_M(t) = \prod_{i=1}^{\ell}(1+d_i t)$

② $\varphi_k(G) = \sum_{i=1}^{\ell} \varphi_k(F_{d_i})$ $\Bigg\rangle \prod_{k\geq 1}(1-t^k)^{\varphi_k(G)}$

③ G_n/G_{n+1} is free Abelian

$$\prod_{i=1}^{\ell}\prod_{k\geq 1}(1-t^k)^{\varphi_k(F_{d_i})} = \prod_{k\geq 1}(1-t^k)^{\sum \varphi_k(F_{d_i})}$$

$\overset{\text{Witt}}{\|}$

$$\prod_{i=1}^{\ell}(1-d_i t) = P_M(-t)$$

<u>Thm</u> If $\mathcal{A}$ is fiber type then this LCS formula holds.