of fiber type

Cor. $\bigcap_{n\geq 1} G_n = 1$

i.e. G is residually nilpotent.

Sucin

LCS: $G = G_1 \supset G_2 \supset \cdots \supset G_r \supset \cdots$

rank $G_r = \varphi_r$

LCS Quotients G_r/G_{r+1}

Nilpotent quotients G/G_r

$G/G_2 = G^{ab} = \mathbb{Z}^n$

$\to G_2/G_3 \to G/G_3 \longrightarrow G/G_2 \longrightarrow 1$

$\vdots$

$1 \to G_r/G_{r+1} \to G/G_{r+1} \to G/G_r \to 1$

ex: $G = F_2 = \langle x, y \rangle$

$$\begin{array}{ccccccc} \to G_2/G_3 & \to & G/G_3 & \xrightarrow{ab} & G/G_2 & \longrightarrow & 1 \\ \| & & & & \| & & \\ [x,y] & & & & \mathbb{Z}^2 & & \\ & & & & x, y & & \end{array}$$

$$G/G_3 = \langle x, y \mid [x,[x,y]] = [y,[x,y]] = 1 \rangle$$

$$\text{Heisenberg grp} = \left\{ \begin{array}{cccc} & x & & \\ \left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} a \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} c \\ b \\ 1 \end{matrix}\right) & \begin{matrix} \\ y \\ \\ \end{matrix} \end{array} \;\middle|\; a, b, c \in \mathbb{Z} \right\}$$

<u>Thm:</u> $G = \pi_1(M)$
$K = \pi_1(N)$

$$G/G_3 \cong K/K_3 \iff H^{\leq 2}(M) \cong H^{\leq 2}(N)$$

(in particular G/G_3 is combinatorially determined)

<u>Warning</u> G/G_4 is <u>not</u> comb. determined.

Let
$$\nu_{p,d}(G) := \#\left\{ K \triangleleft G/G_3 \;\middle|\; \begin{array}{l} [G/G_3 : K] = p \\ \dim_{\mathbb{Z}_p} H^1(K, \mathbb{Z}_p) = n+d \end{array} \right\}$$
p is prime, $d \geq 0$

<u>Thm:</u> $$\nu_{p,d}(G) = \frac{1}{p-1} \left| \mathcal{R}_d(G, \mathbb{Z}_p) \setminus \mathcal{R}_{d+1}(G, \mathbb{Z}_p) \right|$$

Associated graded Lie alg

$$gr(G) = \bigoplus_{r=1}^{\infty} \underbrace{G_r / G_{r+1}}_{gr_r(G)}$$

$$[\ ,\ ] : gr_k(G) \times gr_s(G) \longrightarrow gr_{k+s}(G)$$

given by commutator $[x,y] = xyx^{-1}y^{-1}$

$[x,y] = -[y,x]$ and Jacobi identity

Ex: $G = F_n$ free grp

$gr(G) = L_n$ free Lie alg.

$U(L_n) = T_n$ = tensor alg.

univer. enveloping alg. $\qquad \phi_r = \operatorname{rank} gr_r(G)$

P.B.W. Thm

$$\prod_{r=1}^{\infty} (1-t^r)^{-\phi_r} = \mathrm{Hilb}(U(gr(G)))$$

Ex: $G = F_n$ $\qquad \prod_{r=1}^{\infty} (1-t^r)^{-\phi_r(F_n)} = \mathrm{Hilb}(T_n) = \frac{1}{1-nt}$

Chen Lie alg (~1950)

$\underline{gr}(G/G'')$ $\qquad \theta_r = \text{rank}\, \underline{gr}_r(G/G'')$

this is a quotient of $\underline{gr}(G)$, so

$\phi_r \geq \theta_r$ also $\phi_i = \theta_i$ for $i=1,2,3$

Thm: (Massey)

$$\theta_r = \dim_{\mathbb{C}} gr_{r-2} B \otimes \mathbb{C}$$

$$(B = G'/G'' \text{ Alex inv.})$$

filtered by I^r

$$\underline{gr}_r B = I^r B / I^{r+1} B$$

in particular

$$\phi_2 = \theta_2 = \dim B/IB$$

$$\phi_3 = \theta_3 = \dim IB/I^2B$$

Theorem (PSO)

$$\theta_r = \dim \underline{gr}_{r-2} B^{lin} \otimes \mathbb{C}$$

combinatorially determined

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Application

Chen • $\Theta_r(F_n) = (r-1)\binom{n+r-2}{r}$ $\quad r \geq 2$

C.S. 94 $\quad \Theta_r(P_n) = (r-1)\binom{n+1}{4}$ for $r \geq 2$

$\quad \#$

$\quad \Theta_r(F_{n-1} \times \cdots \times F_1)$ for $n \geq 4$

Digression

Recall

$$R_d(G) = V(\operatorname{ann} B^{\dim})$$

Question: Can we write Θ_r (and maybe ϕ_r) in terms of $R_d(dR)$?

Resonance LCS Conjectures (2000)

$R_1 = R^1_1 = \bigcup_{i=1}^{s} L_i$, L_i linear sbspce of $\mathbb{C}^n$

$h_r := \#\{L_i \mid \dim L_i = r\}$ $\quad$ [= for non-resonant arrs] [= for graph arr's] [$\geq$ true (SS 04)]

Then :

- $\Theta_k = \sum_{r \geq 2} h_r \cdot \Theta_k(F_r)$ for $k \gg 0$.
- $\phi_k = \sum_{r \geq 2} h_r \phi_k(F_r)$ if $\Theta_4 = \phi_4$ for $k \gg 0$

<u>Thm</u>: If $\Phi_3 = \sum_{x \in L_2(\delta\ell)} \Phi_3(F_{\mu(x)})$ then

$$\prod_{r=1}^{\infty} (1 - t^r)^{\Phi_r(\delta\ell)} = (1-t)^{|\delta\ell| - \sum_{x \in L_2(\delta\ell)} \mu(x)} \prod_{x \in L_2(\delta\ell)} (1 - \mu(x) t)$$

THE END