

ALGEBRAIC INVARIANTS FOR BESTVINA-BRADY GROUPS

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ABSTRACT

Bestvina-Brady groups arise as kernels of length homomorphisms $G_\Gamma \rightarrow \mathbb{Z}$ from right-angled Artin groups to the integers. Under some connectivity assumptions on the flag complex Δ_Γ , we compute several algebraic invariants of such a group N_Γ , directly from the underlying graph Γ . As an application, we give examples of finitely presented Bestvina-Brady groups which are not isomorphic to any Artin group or arrangement group.

1. Introduction and statement of results

1.1. Bestvina-Brady groups

Given a finite simple graph $\Gamma = (\mathbf{V}, \mathbf{E})$, the corresponding right-angled Artin group G_Γ has presentation with a generator v for each vertex $v \in \mathbf{V}$, and a commutator relation $vw = wv$ for each edge $\{v, w\} \in \mathbf{E}$. The Bestvina-Brady group (or, Artin kernel) associated to Γ , denoted N_Γ , is the kernel of the “length” homomorphism to the additive group of integers, $\nu: G_\Gamma \rightarrow \mathbb{Z}$, which sends each generator $v \in \mathbf{V}$ to 1.

As shown by Bestvina and Brady in their seminal paper [1], the geometric and homological finiteness properties of the group N_Γ are intimately connected to the topology of the flag complex Δ_Γ . For example, N_Γ is finitely generated if and only if the graph Γ is connected; and N_Γ is finitely presented if and only if Δ_Γ is simply-connected. The groups N_Γ are complicated enough that a counterexample to either the Eilenberg-Ganea conjecture or the Whitehead asphericity conjecture can be constructed from them.

It is known that two right-angled Artin groups G_Γ and $G_{\Gamma'}$ are isomorphic if and only if the corresponding graphs, Γ and Γ' , are isomorphic; see [15], [7]. No such simple classification of the Bestvina-Brady groups is possible. Indeed, if Γ is a tree on n vertices, then $N_\Gamma = F_{n-1}$ (the free group of rank $n - 1$), as follows from [4]. Thus, for any $n \geq 4$, there exist graphs Γ and Γ' on n vertices such that $\Gamma \not\cong \Gamma'$, yet $N_\Gamma \cong N_{\Gamma'}$.

We study here a variety of algebraic invariants of a group N_Γ (mainly derived from the lower central series and the cohomology ring), showing how to compute these invariants directly from the graph Γ , provided some connectivity assumptions

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on Δ_Γ are satisfied. In turn, such invariants can be used to distinguish Bestvina-Brady groups, both among themselves, and from other, related classes of groups, such as Artin groups, or arrangement groups.

1.2. LCS quotients and Chen groups

We start by studying invariants derived from the lower central series. For a group G , this series is defined by $\gamma_1 G = G$ and $\gamma_{k+1} G = (\gamma_k G, G)$, where $(x, y) = xyx^{-1}y^{-1}$. The direct sum of the successive quotients, $\text{gr}(G) = \bigoplus_{k \geq 1} \gamma_k G / \gamma_{k+1} G$, is the *associated graded Lie algebra* of G . The Lie bracket, induced from the group commutator, is compatible with the grading. By construction, the Lie algebra $\text{gr}(G)$ is generated by $\text{gr}_1(G)$. Consequently, the derived Lie subalgebra, $\text{gr}'(G)$, coincides with $\bigoplus_{k \geq 2} \text{gr}_k(G)$.

THEOREM 1.1. *Let $\Gamma = (\mathbf{V}, \mathbf{E})$ be a connected graph, and let N_Γ be the corresponding Bestvina-Brady group. The associated graded Lie algebra $\text{gr}(N_\Gamma)$ is torsion-free, with graded ranks $\phi_k = \text{rank } \text{gr}_k(N_\Gamma)$ given by*

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \frac{P_\Gamma(-t)}{1 - t},$$

where $P_\Gamma(t) = \sum_{k \geq 0} f_k(\Gamma) t^k$ is the clique polynomial of Γ , with $f_k(\Gamma)$ equal to the number of k -cliques of Γ . Moreover, $\text{gr}'(N_\Gamma)$ is isomorphic (as a graded Lie algebra) to the derived Lie algebra of $\mathfrak{H}_\Gamma = \text{Lie}(\mathbf{V}) / ([v, w] = 0 \text{ if } \{v, w\} \in \mathbf{E})$.

For a group G , let $G' = \gamma_1 G$ be the derived group, and $G'' = (G')'$ the second derived group. Note that $H_1(G) = G/G'$ is the maximal abelian quotient of G , whereas G/G'' is the maximal metabelian quotient. Define the *Chen Lie algebra* of G to be $\text{gr}(G/G'')$.

THEOREM 1.2. *Let $\Gamma = (\mathbf{V}, \mathbf{E})$ be a connected graph, and let N_Γ be the corresponding Bestvina-Brady group. The Chen Lie algebra $\text{gr}(N_\Gamma/N_\Gamma'')$ is torsion-free, with graded ranks θ_k given by $\theta_1 = |\mathbf{V}| - 1$ and*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j \geq 2} \left(\sum_{W \subset \mathbf{V}: |W|=j} \tilde{b}_0(\Gamma_W) \right) t^j$ is the cut polynomial of Γ . Moreover, $\text{gr}'(N_\Gamma/N_\Gamma'')$ is isomorphic to the derived Lie algebra of $\mathfrak{H}_\Gamma/\mathfrak{H}_\Gamma''$.

These two theorems rely on the analogous computations for right-angled Artin groups, done in [22]. The proofs involve a homological analysis of the extension $1 \rightarrow N_\Gamma \rightarrow G_\Gamma \rightarrow \mathbb{Z} \rightarrow 0$, based on the Salvetti complex for G_Γ . This analysis shows that \mathbb{Z} acts trivially on $H_1(N_\Gamma)$, thereby allowing us to invoke the Falk-Randell lemma [11].

1.3. Cohomology ring and formality

Next, we turn to cohomological invariants. If G is a group, with Eilenberg-MacLane space $K(G, 1)$, then the cohomology of G with coefficients in a commutative ring R is defined as $H^*(G, R) := H^*(K(G, 1), R)$, with ring structure given

by the cup product. The group G is said to be 1-formal if its Malcev Lie algebra is quadratically presented, cf. [24]. In this case, the rational associated graded Lie algebra $\text{gr}(G) \otimes \mathbb{Q}$ is isomorphic to the rational holonomy Lie algebra, $\mathfrak{H}_{\mathbb{Q}}(G)$, which in turn is determined by the cohomology ring in low degrees, $H^{\leq 2}(G, \mathbb{Q})$.

For a right-angled Artin group G_{Γ} , the cohomology ring can be identified with the exterior Stanley-Reisner ring of the flag complex: $H^*(G_{\Gamma})$ is the quotient of the exterior algebra on generators v^* in degree 1, indexed by the vertices $v \in V$, modulo the ideal generated by the monomials v^*w^* for which $\{v, w\}$ is not an edge of Γ ; see [14]. Furthermore, the group G_{Γ} is 1-formal; see [13].

Denote by $\iota: N_{\Gamma} \rightarrow G_{\Gamma}$ the inclusion map of the kernel, and view the homomorphism $\nu: G_{\Gamma} \rightarrow \mathbb{Z} \hookrightarrow \mathbb{Q}$ as an element in $H^1(G_{\Gamma}, \mathbb{Q})$. The next theorem determines the rational cohomology ring of N_{Γ} , in low degrees. A more general result has been independently obtained by Leary and Saadetoğlu [16].

THEOREM 1.3. *Suppose $\pi_1(\Delta_{\Gamma}) = 0$. Then $\iota^*: H^*(G_{\Gamma}, \mathbb{Q}) \rightarrow H^*(N_{\Gamma}, \mathbb{Q})$ induces a ring homomorphism $\iota^*: H^*(G_{\Gamma}, \mathbb{Q})/(\nu \cdot H^*(G_{\Gamma}, \mathbb{Q})) \rightarrow H^*(N_{\Gamma}, \mathbb{Q})$, which is an isomorphism in degrees $* \leq 2$.*

When $\pi_1(\Delta_{\Gamma}) = 0$, an explicit finite presentation for N_{Γ} was given by Dicks and Leary [4]. We use this presentation to show that the Bestvina-Brady group N_{Γ} is 1-formal.

1.4. Cohomology jumping loci

Let G be a finitely presented group, with character torus $\mathbb{T}_G = \text{Hom}(G, \mathbb{C}^*)$. Identifying the point $\rho \in \mathbb{T}_G$ with a rank one local system ${}_{\rho}\mathbb{C}$ on an Eilenberg-MacLane space $K(G, 1)$, we may define

$$\mathcal{V}_1(G) = \{\rho \in \mathbb{T}_G \mid H^1(G, {}_{\rho}\mathbb{C}) \neq 0\}.$$

The set $\mathcal{V}_1(G)$ is an algebraic subvariety of \mathbb{T}_G , called the *(first) characteristic variety* of G . Away from the origin, this variety coincides with the zero set of the annihilator of the Alexander invariant, $B(G) \otimes \mathbb{C}$.

Denote by $A = H^*(G, \mathbb{C})$ the cohomology algebra of G . For each $a \in A^1$, we have $a^2 = 0$, and so right-multiplication by a defines a cochain complex $(A, a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2$. Let $\mathcal{R}_1(G)$ be the set of points $a \in A^1$ where this complex fails to be exact,

$$\mathcal{R}_1(G) = \{a \in A^1 \mid H^1(A, a) \neq 0\}.$$

The set $\mathcal{R}_1(G)$ is a homogeneous algebraic variety in the affine space $A^1 = H^1(G, \mathbb{C})$, called the *(first) resonance variety* of G . Away from the origin, this variety coincides with the zero set of the annihilator of the infinitesimal Alexander invariant, $\mathfrak{B}(G) \otimes \mathbb{C}$.

In previous work [22], [5], we determined the resonance and characteristic varieties of right-angled Artin groups. Here, we determine these varieties for the finitely presented Bestvina-Brady groups.

Let $\mathbb{T}_V = (\mathbb{C}^*)^V$ be the character torus of G_{Γ} (of dimension $|V|$). For a subset $W \subset V$, let \mathbb{T}_W be the coordinate subtorus supported on W . Similarly, let $H_V = \mathbb{C}^V$ be the Lie algebra of \mathbb{T}_V , identified with $A^1 = H^1(G_{\Gamma}, \mathbb{C})$, and let H_W be the coordinate subspace supported on W . The inclusion $\iota: N_{\Gamma} \rightarrow G_{\Gamma}$ induces homomorphisms

$\iota^*: \text{Hom}(G_\Gamma, \mathbb{C}^*) \rightarrow \text{Hom}(N_\Gamma, \mathbb{C}^*)$ and $\iota^*: H^1(G_\Gamma, \mathbb{C}) \rightarrow H^1(N_\Gamma, \mathbb{C})$. Note that if $|\mathcal{V}| = 1$, then $N_\Gamma = \{1\}$ and thus both $\mathcal{V}_1(N_\Gamma)$ and $\mathcal{R}_1(N_\Gamma)$ are empty.

For a graph Γ on vertex set \mathcal{V} , define the connectivity $\kappa(\Gamma)$ to be the maximum integer r so that, for any set of vertices \mathcal{W} of size less than r , the full subgraph of Γ on vertex set $\mathcal{V} \setminus \mathcal{W}$ is connected.

THEOREM 1.4. *Let Γ be a graph. Suppose $\pi_1(\Delta_\Gamma) = 0$ and $|\mathcal{V}| > 1$.*

- (i) *If $\kappa(\Gamma) = 1$, then $\mathcal{V}_1(N_\Gamma) = \text{Hom}(N_\Gamma, \mathbb{C}^*)$ and $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$.*
- (ii) *If $\kappa(\Gamma) > 1$, then the irreducible components of $\mathcal{V}_1(N_\Gamma)$, respectively $\mathcal{R}_1(N_\Gamma)$, are the subtori $\mathbb{T}'_{\mathcal{W}} = \iota^*(\mathbb{T}_{\mathcal{W}})$, respectively the subspaces $H'_{\mathcal{W}} = \iota^*(H_{\mathcal{W}})$, of dimension $|\mathcal{W}|$, one for each subset $\mathcal{W} \subset \mathcal{V}$, maximal among those for which the induced subgraph $\Gamma_{\mathcal{W}}$ is disconnected.*

1.5. Comparison with other classes of groups

It turns out that Bestvina-Brady groups share many common features with other, much-studied classes of groups: finite-type Artin groups and fundamental groups of complements of complex hyperplane arrangements. We catalogue here some of these common features, and indicate certain overlaps between the various classes. As a counterpoint, and as an application of our methods, we give examples of finitely presented Bestvina-Brady groups which are not isomorphic to any group from those two other classes.

THEOREM 1.5. *There exists an infinite family of graphs $\{\Gamma_i\}_{i \in \mathbb{N}}$ such that the Bestvina-Brady group N_{Γ_i} is finitely presented, yet not isomorphic to either an Artin group, or an arrangement group.*

These graphs are obtained as the 1-skeleta of certain ‘extra-special’ triangulations of the 2-disk.

Using some of the machinery developed here, a complete classification of the Bestvina-Brady groups which can be realized as fundamental groups of quasi-projective varieties is given in [6].

1.6. Organization of the paper

We start in Section 2 with a review of the Bestvina-Brady groups, and a discussion of the Dicks-Leary presentation.

In Section 3 we recall the Salvetti complex for G_Γ , and use it to analyze the induced homomorphism $\iota_*: H_1(N_\Gamma) \rightarrow H_1(G_\Gamma)$.

In Section 4 we give presentations for the Alexander invariants $B(G_\Gamma)$ and $B(N_\Gamma)$.

In Section 5 we relate the graded Lie algebras attached to G_Γ and N_Γ , and prove Theorems 1.1 and 1.2.

In Section 6 we show that finitely presented Bestvina-Brady groups are 1-formal.

In Section 7 we compute the homology groups $H_*(N_\Gamma, \mathbb{k})$, and prove Theorem 1.3.

In Section 8 we relate the characteristic and resonance varieties of G_Γ to those of N_Γ , and prove Theorem 1.4.

Finally, in Section 9, we compare finitely presented Bestvina-Brady groups with Artin groups and arrangement groups, and prove Theorem 1.5.

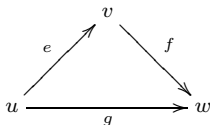


FIGURE 1. A directed triangle

2. Bestvina-Brady groups

Let Γ be a finite graph without loops or multiple edges, with vertex set \mathbf{V} and edge set $\mathbf{E} \subset \binom{\mathbf{V}}{2}$. The flag complex of Γ , denoted Δ_Γ , is the maximal simplicial complex with 1-skeleton equal to Γ : the k -simplices of Δ_Γ correspond to the $(k+1)$ -cliques of Γ .

To the graph Γ , there is associated a *right-angled Artin group*, G_Γ , with a generator v for each vertex in \mathbf{V} , and with a commutator relation for each edge in \mathbf{E} :

$$G_\Gamma = \langle v \in \mathbf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathbf{E} \rangle. \quad (2.1)$$

For example, if Γ is the empty (or null) graph on n vertices, then $G_\Gamma = F_n$ (the free group of rank n), whereas if Γ is the complete graph K_n , then $G_\Gamma = \mathbb{Z}^n$.

DEFINITION 2.1. The *Bestvina-Brady group* associated to the graph $\Gamma = (\mathbf{V}, \mathbf{E})$, denoted N_Γ , is the kernel of the “length” homomorphism $\nu: G_\Gamma \rightarrow \mathbb{Z}$ which sends each generator $v \in \mathbf{V}$ to 1.

If $\iota: N_\Gamma \rightarrow G_\Gamma$ denotes the inclusion map, we have an exact sequence

$$1 \longrightarrow N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\nu} \mathbb{Z} \longrightarrow 0. \quad (2.2)$$

The group N_Γ need not be finitely generated. For example, if Γ is the empty graph on $n > 1$ vertices, then $N_\Gamma = \ker(\nu: F_n \twoheadrightarrow \mathbb{Z})$ is a free group of countably infinite rank. More generally, it was shown by Meier–VanWyk [18] and Bestvina–Brady [1] that the group N_Γ is finitely generated if and only if the graph Γ is connected.

Even if the graph Γ is connected, the group N_Γ may not have a finite presentation. For example, if Γ is a 4-cycle, then $G_\Gamma = F_2 \times F_2$; as noted by Stallings [26], $H_2(N_\Gamma)$ is not finitely generated, and so N_Γ is not finitely presented. Much more generally, Bestvina and Brady [1] showed that N_Γ is finitely presented if and only if the flag complex Δ_Γ is simply-connected. In this case, an explicit finite presentation was given by Dicks and Leary [4].

Fix a linear order on the vertices, and orient the edges increasingly. A triple of edges (e, f, g) forms a directed triangle if $e = \{u, v\}$, $f = \{v, w\}$, $g = \{u, w\}$, and $u < v < w$; see Figure 1.

THEOREM 2.2 (Dicks–Leary [4]). *Suppose the flag complex Δ_Γ is simply connected. Then N_Γ has presentation*

$$N_\Gamma = \langle e \in \mathbf{E} \mid ef = fe, ef = g \text{ if } (e, f, g) \text{ is a directed triangle} \rangle. \quad (2.3)$$

Moreover, the inclusion $\iota: N_\Gamma \rightarrow G_\Gamma$ is given by $\iota(e) = uv^{-1}$, for $e = \{u, v\}$ as above.

The Dicks-Leary presentation is far from being minimal (unless Γ is a tree). Indeed, there are $|\mathbf{E}|$ generators in (2.3), whereas $H_1(N_\Gamma)$ has rank $|\mathbf{V}| - 1$, as we shall see in Proposition 4.2. Nevertheless, (2.3) can be simplified via Tietze moves to a presentation where all the relations are commutators.

COROLLARY 2.3. *If $\pi_1(\Delta_\Gamma) = 0$, then N_Γ admits a commutator-relators presentation, $N_\Gamma = F/R$, with F the free group generated by the edges in a maximal tree \mathbf{T} , and R a finitely generated normal subgroup of F' .*

Proof. Fix a maximal tree \mathbf{T} for Γ . Suppose $e = \{u, v\}$ is an edge not in \mathbf{T} . Picking a path e_1, \dots, e_r in \mathbf{T} connecting u to v , we see that $\iota(e) = \iota(e_1^{\epsilon_1} \dots e_r^{\epsilon_r})$ in G_Γ , for some suitable signs ϵ_i . Thus, $e = e_1^{\epsilon_1} \dots e_r^{\epsilon_r}$ in N_Γ . This shows that N_Γ is generated by the edges of \mathbf{T} . Now note that N_Γ/N_Γ' is free abelian, of rank equal to the number of edges in \mathbf{T} . Eliminating the redundant generators from (2.3), we arrive at the desired presentation. \square

In certain situations, the Dicks-Leary presentation permits us to identify the group N_Γ in terms of better known groups.

EXAMPLE 2.4. Suppose Γ is a tree on n vertices. Then Γ has no triangles, and $\pi_1(\Delta_\Gamma) = 0$. Since Γ has $n - 1$ edges, we see that $N_\Gamma = F_{n-1}$.

EXAMPLE 2.5. Suppose Γ is the cone on Γ' . Then $G_\Gamma = G_{\Gamma'} \times \mathbb{Z}$, and so $N_\Gamma = G_{\Gamma'}$. In particular, if $\Gamma = K_n$, then $N_\Gamma = \mathbb{Z}^{n-1}$.

In general, though, N_Γ is not isomorphic to any right-angled Artin group, as we shall show later.

Noteworthy is the situation when Δ_Γ is a triangulation of the 2-disk. In this case, N_Γ admits a 2-dimensional $K(N_\Gamma, 1)$, see [3, Corollary 2.3].

DEFINITION 2.6. A triangulation of the disk is said to be *special* if it is obtained from a triangle by adding one triangle at a time, along a unique boundary edge.

LEMMA 2.7. *Let Δ be a special triangulation of D^2 , with 1-skeleton $\Gamma = (\mathbf{V}, \mathbf{E})$. Then:*

- (i) $2|\mathbf{V}| - |\mathbf{E}| = 3$.
- (ii) $\Delta_\Gamma = \Delta$.
- (iii) N_Γ admits a presentation with $|\mathbf{V}| - 1$ generators and $|\mathbf{V}| - 2$ commutator relators.

Proof. By induction on the number t of triangles. Evidently, all statements hold for a single triangle. Now suppose Δ is a special triangulation with t triangles, and a directed triangle (e, f, g) is added along edge e , to form Δ' . In the process, one vertex and two edges are added, and so the quantity $2|\mathbf{V}| - |\mathbf{E}|$ does not change. The only new 3-cycle in the graph Γ' is the boundary of (e, f, g) ; thus, Δ' is a flag complex. Furthermore, if \mathbf{T} is a maximal tree for Γ , we can build a maximal tree \mathbf{T}' for Γ' by adding a new edge, say, f . The Dicks-Leary relation $ef = g$ (with e expressed as a word in the edges of \mathbf{T}) may be used to eliminate the generator g . Thus, $N_{\Gamma'}$ has only one new generator, f , and only one new relation, $ef = fe$. \square

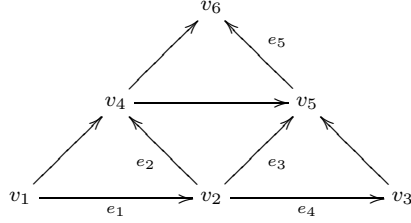


FIGURE 2. A special triangulation of the 2-disk

EXAMPLE 2.8. Let Γ be the graph in Figure 2. Choosing a maximal tree $T = \{e_1, \dots, e_5\}$ as indicated, the presentation from Lemma 2.7(iii) reads as follows:

$$N_\Gamma = \langle e_1, \dots, e_5 \mid (e_1, e_2), (e_2, e_3), (e_3, e_4), e_5 e_2^{-1} e_3 = e_2^{-1} e_3 e_5 \rangle.$$

We shall see in Proposition 9.4 that $N_\Gamma \not\cong G_{\Gamma'}$, no matter what the graph Γ' is.

3. The Salvetti complex

For a simple graph $\Gamma = (V, E)$, let K_Γ be the CW-complex obtained by joining tori in the manner prescribed by the flag complex Δ_Γ . More precisely, if $T^n = (S^1)^{\times n}$ is the torus of dimension $n = |V|$, with the usual CW-decomposition, then K_Γ is the subcomplex obtained by deleting the cells corresponding to the non-faces of Δ_Γ .

Clearly, the fundamental group of K_Γ is the right-angled Artin group G_Γ . In fact, K_Γ is an Eilenberg-MacLane space of type $K(G_\Gamma, 1)$; see [18]. Note that $H_k(K_\Gamma)$ is free abelian, of rank equal to the number of k -cliques in Γ ; thus, the Poincaré polynomial of K_Γ equals the clique polynomial of the graph, $P_\Gamma(t)$.

We use the cell structure of the space K_Γ to describe a finite, free resolution of \mathbb{Z} , viewed as a trivial module over the group ring $\mathbb{Z}G_\Gamma$. This resolution was first determined by Salvetti [25], in the more general context of Artin groups, and further extended by Charney and Davis [2]. For the benefit of the reader, we include a self-contained, direct computation of the Salvetti complex in our particular case.

3.1. A free $\mathbb{Z}G_\Gamma$ -resolution of \mathbb{Z}

For each subset $W \subset V$, let Γ_W be the induced subgraph of Γ on vertex set W . Let $G_W = G_{\Gamma_W}$ be the corresponding right-angled Artin group, and let $K_W = K_{\Gamma_W}$ be the corresponding CW-complex. The inclusion $W \subset V$ gives rise to a cellular inclusion map $j_W: K_W \rightarrow K_\Gamma$. The induced homomorphism, $(j_W)_\#: G_W \rightarrow G_\Gamma$, is a split injection, with retract $G_\Gamma \rightarrow G_W$ given on generators by $v \mapsto v$ if $v \in W$, and $v \mapsto 1$ otherwise.

Denote by $C_\bullet = C_\bullet(K_\Gamma)$ the cellular chain complex of K_Γ . A basis for C_k is given by the complete k -subgraphs of Γ : to each complete subgraph on vertex set $W \subset V$, there corresponds a cell c_W . Since C_\bullet is a sub-complex of the cellular chain complex of T^n , the boundary maps $C_k \rightarrow C_{k-1}$ are the zero maps.

Now let $\tilde{C}_\bullet = (C_\bullet(\tilde{K}_\Gamma), \partial_\bullet)$ be the equivariant chain complex of the universal cover of K_Γ . The augmentation map, $\epsilon: \tilde{C}_0 \rightarrow \mathbb{Z}$, extends to a finite, free resolution $\tilde{C}_\bullet \rightarrow \mathbb{Z}$ over the group ring $\mathbb{Z}G_\Gamma$.

PROPOSITION 3.1. *Under the identification $\tilde{C}_k = \mathbb{Z}G_\Gamma \otimes C_k$, the boundary map $\partial_k: \tilde{C}_k \rightarrow \tilde{C}_{k-1}$ is given by:*

$$\partial_k(c_W) = \sum_{r=1}^k (-1)^{r-1} (v_{i_r} - 1) c_{W \setminus \{i_r\}} \quad (3.1)$$

where $W = \{v_{i_1}, \dots, v_{i_k}\}$ is a k -clique in Γ .

Proof. Let $K_W = T^k$ be the corresponding CW-subcomplex of K_Γ . The equivariant chain complex $\tilde{C}_\bullet^W = (C_\bullet(\tilde{K}_W), \partial_\bullet^W)$ is simply the Koszul complex on the variables from W . In particular, $\partial_k^W(c_W)$ is given by the right-hand side of (3.1). The commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Z}G_W \otimes C_k(K_W) & \xrightarrow{\partial_k^W} & \mathbb{Z}G_W \otimes C_{k-1}(K_W) \\ \downarrow j_\# \otimes j_* & & \downarrow j_\# \otimes j_* \\ \mathbb{Z}G_\Gamma \otimes C_k(K_\Gamma) & \xrightarrow{\partial_k} & \mathbb{Z}G_\Gamma \otimes C_{k-1}(K_\Gamma) \end{array}$$

completes the proof. \square

3.2. An injectivity lemma

Recall that the Bestvina-Brady group associated to a graph $\Gamma = (V, E)$ is the kernel of the homomorphism $\nu: G_\Gamma \rightarrow \mathbb{Z}$ that sends each generator $v \in V$ of G_Γ to 1 in \mathbb{Z} . Let $\iota: N_\Gamma \rightarrow G_\Gamma$ be the inclusion map.

LEMMA 3.2. *If the graph Γ is connected, then the induced homomorphism $\iota_*: H_1(N_\Gamma) \rightarrow H_1(G_\Gamma)$ is injective.*

Proof. Let $\tilde{C}_\bullet = (C_\bullet(\tilde{K}_\Gamma), \partial_\bullet)$ be the Salvetti complex, with boundary maps given by (3.1). Identify the group ring $\mathbb{Z}\mathbb{Z}$ with the ring of Laurent polynomials $\mathbb{Z}[\tau, \tau^{-1}]$. By Shapiro's Lemma, $H_*(N_\Gamma) = H_*(\mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} \tilde{C}_\bullet)$, where $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[\tau, \tau^{-1}]$ is viewed as a right G_Γ -module via the map $v \mapsto \tau$.

After identifying $\mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} \tilde{C}_k$ with $\mathbb{Z}\mathbb{Z} \otimes C_k$, the boundary map ∂_k takes the form $(\tau - 1) \otimes d_k: \mathbb{Z}\mathbb{Z} \otimes C_k \rightarrow \mathbb{Z}\mathbb{Z} \otimes C_{k-1}$, where d_k is the simplicial boundary map on the $(k-1)$ -simplices of Δ_Γ . With these identifications, the chain map $\iota_\#: \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} C_\bullet(\tilde{K}_\Gamma) \rightarrow C_\bullet(K_\Gamma)$ takes the form:

$$\begin{array}{ccccc} \cdots & \longrightarrow & \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z}^E & \xrightarrow{\partial_2 = (\tau-1) \otimes d_2} & \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z}^V & \xrightarrow{\partial_1 = (\tau-1) \otimes d_1} & \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z} \\ & & \downarrow \iota_2 = \epsilon \otimes \text{id} & & \downarrow \iota_1 = \epsilon \otimes \text{id} & & \downarrow \iota_0 = \epsilon \otimes \text{id} \\ \cdots & \longrightarrow & \mathbb{Z} \otimes \mathbb{Z}^E & \xrightarrow{0} & \mathbb{Z} \otimes \mathbb{Z}^V & \xrightarrow{0} & \mathbb{Z} \otimes \mathbb{Z} \end{array}$$

The map $\iota_*: H_1(N_\Gamma) \rightarrow H_1(G_\Gamma)$ is the homomorphism induced on H_1 by the middle down arrow. To show ι_* is injective, we need to prove: $\ker \partial_1 \cap \ker \iota_1 \subset \text{im } \partial_2$.

Let $z = \sum_{v \in V} p_v \otimes v \in \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z}^V$. Suppose z belongs to $\ker \partial_1$. Since $\partial_1(z) = (\tau - 1)(\sum_{v \in V} p_v)$, and since the ring $\mathbb{Z}\mathbb{Z}$ has no zero-divisors, we must have $\sum_{v \in V} p_v = 0$. Now suppose z belongs to $\ker \iota_1$. Then $\sum_{v \in V} \epsilon(p_v) \otimes v = 0$, which can only happen if $\epsilon(p_v) = 0$, for all $v \in V$. Thus, for each $v \in V$, there is $q_v \in \mathbb{Z}\mathbb{Z}$ such that

$p_v = (\tau - 1)q_v$. We conclude that $\ker \partial_1 \cap \ker \iota_1$ is generated by elements of the form $(\tau - 1)q \otimes (v - u)$.

Let e_1, \dots, e_s be a path in Γ with $e_i = (u_{i-1}, u_i)$, joining $u_0 = u$ to $u_s = v$. Then:

$$\partial_2 \left(\sum_{i=1}^s q \otimes e_i \right) = (\tau - 1) \sum_{i=1}^s q \otimes (u_i - u_{i-1}) = (\tau - 1)q \otimes (v - u).$$

This finishes the proof. \square

4. Alexander invariants

Let G be a group, with abelianization $H_1(G) = G/G'$. The *Alexander invariant* of G is the quotient group $B(G) = G'/G''$, endowed with the $\mathbb{Z}H_1(G)$ -module structure induced by conjugation in G/G'' , via the exact sequence $0 \rightarrow G'/G'' \rightarrow G/G'' \rightarrow G/G' \rightarrow 0$. Alternatively, if K is a connected CW-complex with $\pi_1(K) = G$, and K^{ab} is the universal abelian cover of K , then $B(G) = H_1(K^{\text{ab}})$, with module structure coming from the action of $H_1(G)$ by deck transformations.

In this Section, we determine the Alexander invariants of right-angled Artin groups and of finitely-generated Bestvina-Brady groups. We start by giving a finite presentation for $B(G_\Gamma)$, viewed as a module over $\mathbb{Z}H_1(G_\Gamma)$, for an arbitrary finite graph $\Gamma = (\mathbb{V}, \mathbb{E})$.

After fixing a total ordering on \mathbb{V} , we may identify $\mathbb{Z}H_1(G_\Gamma)$ with the ring of Laurent polynomials in variables labeled by the vertices, $\Lambda = \mathbb{Z}[\mathbb{V}^{\pm 1}]$. Let t be a triple of vertices, and let e be a 2-element subset of t . We denote by v_e the third vertex of t , and by ϵ_e the sign of the permutation (v_e, u, w) where $e = \{u, w\}$, with $u < w$.

THEOREM 4.1. *The Alexander invariant $B(G_\Gamma)$ is the $\mathbb{Z}[\mathbb{V}^{\pm 1}]$ -module generated by the non-edges $e \in \mathbb{E}_\Gamma$, and with relators*

$$\sum_{e \subset t, e \in \mathbb{E}_\Gamma} \epsilon_e (v_e - 1) \otimes e,$$

indexed by the triples of vertices $t \in \binom{\mathbb{V}}{3}$ which are not 3-cliques of Γ .

Proof. As before, let $\tilde{C}_\bullet = (C_\bullet(\tilde{K}_\Gamma), \partial_\bullet)$ be the equivariant chain complex of K_Γ . By Shapiro's Lemma, $B(G_\Gamma) = H_1(\mathbb{Z}H_1(G_\Gamma) \otimes_{\mathbb{Z}G_\Gamma} \tilde{C}_\bullet)$, as Λ -modules.

Recall K_Γ is a CW-subcomplex of the torus T^n , where $n = |\mathbb{V}|$. Since $T^n = K(\mathbb{Z}^n, 1)$, the equivariant chain complex $(C_\bullet(\tilde{T}^n), \delta_\bullet)$ gives a free Λ -resolution of \mathbb{Z} . Using (3.1), it is readily seen that $\Lambda \otimes_{\mathbb{Z}G_\Gamma} \tilde{C}_\bullet = \Lambda \otimes C_\bullet$ is a Λ -subcomplex of $C_\bullet(\tilde{T}^n) = \Lambda^{\binom{\mathbb{V}}{\bullet}}$, and that these two complexes coincide up to degree $\bullet = 1$. A diagram chase yields

$$B(G_\Gamma) = \text{coker} \left(\delta_3 + \text{incl}: \Lambda^{\binom{\mathbb{V}}{3}} \oplus \Lambda^{\mathbb{E}} \rightarrow \Lambda^{\binom{\mathbb{V}}{2}} \right). \quad (4.1)$$

Row-reducing the above presentation matrix finishes the proof. \square

A homomorphism $\iota: N \rightarrow G$ induces in a natural way a change of rings map $\iota_*: \mathbb{Z}H_1(N) \rightarrow \mathbb{Z}H_1(G)$ and a $\mathbb{Z}H_1(N)$ -linear map $B(\iota): B(N) \rightarrow B(G)$.

PROPOSITION 4.2. *Let Γ be a connected graph, and let $\iota: N_\Gamma \rightarrow G_\Gamma$ be the inclusion map. Then, the following hold.*

- (i) *There is a split exact sequence $0 \rightarrow H_1(N_\Gamma) \xrightarrow{\iota_*} H_1(G_\Gamma) \xrightarrow{\nu_*} \mathbb{Z} \rightarrow 0$.*
- (ii) *The inclusion ι restricts to an equality $N'_\Gamma = G'_\Gamma$.*
- (iii) *The induced map $B(\iota): B(N_\Gamma) \rightarrow B(G_\Gamma)$ is a $\mathbb{Z}H_1(N_\Gamma)$ -linear isomorphism.*

Proof. ((i)) This is a direct consequence of Lemma 3.2.

((ii)) Clearly, $N'_\Gamma \subset G'_\Gamma$. Now let $g \in G'_\Gamma$; then $\nu(g) = 0$, and so $g = \iota(n)$, for some $n \in N_\Gamma$. Since $\iota_*: H_1(N_\Gamma) \rightarrow H_1(G_\Gamma)$ is injective, we must have $n \in N'_\Gamma$.

((iii)) By the above, $N''_\Gamma = G''_\Gamma$, and so $N'_\Gamma/N''_\Gamma = G'_\Gamma/G''_\Gamma$. \square

COROLLARY 4.3. *Let Γ be a connected graph, with vertex set $\mathbf{V} = \{1, \dots, n\}$. The Alexander invariant $B(N_\Gamma)$ is isomorphic to the restriction of the $\mathbb{Z}\mathbb{Z}^n$ -module $B(G_\Gamma)$, with presentation given in Theorem 4.1, via the change of rings $\iota_*: \mathbb{Z}\mathbb{Z}^{n-1} \rightarrow \mathbb{Z}\mathbb{Z}^n$.*

Let G be a finitely presented group, with torsion-free abelianization. The *holonomy Lie algebra* of G , denoted $\mathfrak{h}(G)$, is the quotient of the free Lie algebra on $H_1(G)$ by the ideal generated by the image of the comultiplication map $\nabla_G: H_2(G) \rightarrow H_1(G) \wedge H_1(G)$.

The *infinitesimal Alexander invariant* of G is $\mathfrak{B}(G) = \mathfrak{h}(G)'/\mathfrak{h}(G)''$, with module structure over the symmetric algebra $\text{Sym}(H_1(G))$ coming from the exact sequence

$$0 \rightarrow \mathfrak{h}(G)'/\mathfrak{h}(G)'' \rightarrow \mathfrak{h}(G)/\mathfrak{h}(G)'' \rightarrow \mathfrak{h}(G)/\mathfrak{h}(G)' \rightarrow 0.$$

This module is isomorphic to the ‘‘linearization’’ of the classical Alexander invariant of the group G , see [21].

A homomorphism $\iota: N \rightarrow G$ induces a change of rings map $\iota_*: \text{Sym}(H_1(N)) \rightarrow \text{Sym}(H_1(G))$ and a $\text{Sym}(H_1(N))$ -linear map $\mathfrak{B}(\iota): \mathfrak{B}(N) \rightarrow \mathfrak{B}(G)$.

Now let $\Gamma = (\mathbf{V}, \mathbf{E})$ be a finite graph. After fixing a total ordering on \mathbf{V} , we may identify $\text{Sym}(H_1(G_\Gamma))$ with the polynomial ring $S = \mathbb{Z}[\mathbf{V}]$. Using [21, Theorem 6.2], we obtain the following infinitesimal analogue of Theorem 4.1.

PROPOSITION 4.4. *The infinitesimal Alexander invariant of a right-angled Artin group, $\mathfrak{B}(G_\Gamma) = \mathfrak{h}(G_\Gamma)'/\mathfrak{h}(G_\Gamma)''$, is the S -module generated by the non-edges $e \in \mathbf{E}_{\overline{\Gamma}}$, and with relators*

$$\sum_{e \subset t, e \in \mathbf{E}_{\overline{\Gamma}}} \epsilon_e v_e \otimes e,$$

indexed by the triples $t \in \binom{\mathbf{V}}{3}$ which are not 3-cliques of Γ .

5. Lower central series

In this section, we determine the associated graded Lie algebra and the Chen Lie algebra of the Bestvina-Brady group corresponding to a finite, connected graph, thus proving Theorems 1.1 and 1.2 from the Introduction.

5.1. Lie algebras associated to right-angled Artin groups

A graph $\Gamma = (\mathbf{V}, \mathbf{E})$ determines in a natural way a graded, finitely-presented Lie algebra \mathfrak{H}_Γ , as follows:

$$\mathfrak{H}_\Gamma = \text{Lie}(\mathbf{V}) / ([v, w] = 0 \text{ if } \{v, w\} \in \mathbf{E}), \quad (5.1)$$

where $\text{Lie}(\mathbf{V})$ is the free Lie algebra on the vertex set \mathbf{V} .

For each $k \geq 1$, let $f_k(\Gamma)$ be the number of complete k -subgraphs of Γ , and set $f_0(\Gamma) = 1$.

THEOREM 5.1 ([8], [9], [22]). *Let Γ be a finite graph, and let G_Γ be the corresponding right-angled Artin group. Then $\text{gr}(G_\Gamma) \cong \mathfrak{H}_\Gamma$, as graded Lie algebras. Moreover, the graded pieces of $\text{gr}(G_\Gamma)$ are torsion-free, with ranks $\phi_k = \text{rank}(\text{gr}_k(G_\Gamma))$ given by:*

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t),$$

where $P_\Gamma(t) = \sum_{k \geq 0} f_k(\Gamma)t^k$ is the clique polynomial of Γ .

For each $j \geq 1$, let $c_j(\Gamma) = \sum_{\mathbf{W} \subset \mathbf{V}: |\mathbf{W}|=j} \tilde{b}_0(\Gamma_{\mathbf{W}})$, where $\tilde{b}_0(\Gamma) = \text{rank } \tilde{H}_0(\Gamma)$ is the number of components of Γ minus 1. Note that $c_1(\Gamma) = 0$, and also $c_j(\Gamma) = 0$, if $j > |\mathbf{V}| - \kappa(\Gamma)$, where $\kappa(\Gamma)$ is the connectivity of Γ .

THEOREM 5.2 [22]. *Let Γ be a finite graph. Then $\text{gr}(G_\Gamma/G_\Gamma'') \cong \mathfrak{H}_\Gamma/\mathfrak{H}_\Gamma''$, as graded Lie algebras. Moreover, the graded pieces of $\text{gr}(G_\Gamma/G_\Gamma'')$ are torsion-free, with ranks $\theta_k = \text{rank}(\text{gr}_k(G_\Gamma/G_\Gamma''))$ given by:*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j=2}^{|\mathbf{V}|-\kappa(\Gamma)} c_j(\Gamma)t^j$ is the cut polynomial of Γ .

5.2. Monodromy action

Let N_Γ be the Bestvina-Brady group associated to the graph Γ . Recall we have an exact sequence

$$1 \longrightarrow N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\nu} \mathbb{Z} \longrightarrow 0. \quad (5.2)$$

This sequence admits a splitting $s: \mathbb{Z} \rightarrow G_\Gamma$, given by $s(1) = v$, for some fixed generator $v \in \mathbf{V}$.

PROPOSITION 5.3. *Let Γ be a connected graph. Then, in the split extension (5.2), the group \mathbb{Z} acts trivially on the abelianization $H_1(N_\Gamma)$.*

Proof. The monodromy of the semidirect product $G_\Gamma = N_\Gamma \rtimes_\sigma \mathbb{Z}$ is given by $\sigma(1)(x) = vxv^{-1}$, for some fixed generator $v \in \mathbf{V}$. Conjugation by any element of G_Γ acts trivially on $H_1(G_\Gamma)$. On the other hand, we know from Lemma 3.2 that $H_1(N_\Gamma)$ injects into $H_1(G_\Gamma)$. Hence, conjugation by an element of G_Γ also acts trivially on $H_1(N_\Gamma)$. \square

For a homomorphism $\alpha: G \rightarrow H$, let $\bar{\alpha}: G/G'' \rightarrow H/H''$ be the induced homomorphism on maximal metabelian quotients.

PROPOSITION 5.4. *Let Γ be a connected graph. Then, the sequence*

$$1 \longrightarrow N_\Gamma/N_\Gamma'' \xrightarrow{\bar{\iota}} G_\Gamma/G_\Gamma'' \xrightarrow{\bar{\nu}} \mathbb{Z} \longrightarrow 0, \quad (5.3)$$

is split exact, with trivial monodromy action on $H_1(N_\Gamma/N_\Gamma'') = H_1(N_\Gamma)$.

Proof. From the proof of Proposition 4.2, we know that $N_\Gamma'' = G_\Gamma''$. All claimed properties of sequence (5.3) follow from the corresponding properties of (5.2). \square

5.3. Lie algebras associated to Bestvina-Brady groups

Recall now the following well-known result of Falk and Randell.

THEOREM 5.5 ([11]). *Let $1 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 1$ be a split exact sequence of groups. Suppose C acts trivially on $H_1(A)$. Then*

$$0 \longrightarrow \text{gr}(A) \xrightarrow{\text{gr}(\alpha)} \text{gr}(B) \xrightarrow{\text{gr}(\beta)} \text{gr}(C) \longrightarrow 0$$

is a split exact sequence of graded Lie algebras.

This result permits us to reduce the computation of the LCS and Chen quotients of a Bestvina-Brady group to the computation of the LCS and Chen quotients of the corresponding right-angled Artin group.

THEOREM 5.6. *Let Γ be a finite, connected graph. Then, the inclusion map $\iota: N_\Gamma \rightarrow G_\Gamma$ induces isomorphisms of graded Lie algebras*

- (i) $\text{gr}'(\iota): \text{gr}'(N_\Gamma) \xrightarrow{\cong} \text{gr}'(G_\Gamma)$.
- (ii) $\text{gr}'(\bar{\iota}): \text{gr}'(N_\Gamma/N_\Gamma'') \xrightarrow{\cong} \text{gr}'(G_\Gamma/G_\Gamma'')$.

Proof. Using Propositions 5.3 and 5.4, we may apply Theorem 5.5 to the exact sequences (5.2) and (5.3). Noting that $\text{gr}(\mathbb{Z}) = \mathbb{Z}$ (concentrated in degree 1), yields isomorphisms (i) and (ii), respectively. \square

Combining Theorem 5.6 with Theorems 5.1 and 5.2 finishes the proof of Theorems 1.1 and 1.2 from the Introduction.

EXAMPLE 5.7. Suppose Γ is a tree on n vertices. Then recall N_Γ is a free group of rank $n - 1$. By Theorem 5.6, $\phi_k(G_\Gamma) = \phi_k(F_{n-1})$ and $\theta_k(G_\Gamma) = \theta_k(F_{n-1})$, for all $k \geq 2$, which recovers the computations from [22, §6.2].

REMARK 5.8. Suppose $\Gamma = K_{n_1, \dots, n_r}$ is a complete multi-partite graph. Then $G_\Gamma = F_{n_1} \times \dots \times F_{n_r}$, and so, by Theorem 5.6, $\text{gr}'(N_\Gamma) = \text{gr}'(F_{n_1} \times \dots \times F_{n_r})$. Even though, from the point of view of the lower central series quotients, N_Γ looks like a product of free groups, it is not of this type, except when some $n_i = 1$. Indeed, if all $n_i > 1$, then $H_{r-1}(\Delta_\Gamma)$ is a free abelian group of rank $\prod_{i=1}^r (n_i - 1) > 0$, and so it follows from [1] that N_Γ does not have a finite $K(N_\Gamma, 1)$.

6. Holonomy Lie algebra and 1-formality

Let G be a finitely presented group, with torsion-free abelianization. Recall that the holonomy Lie algebra $\mathfrak{h}(G)$ is the quotient of the free Lie algebra $\text{Lie}(H_1(G))$ by the ideal generated by the image of the comultiplication map $\nabla_G: H_2(G) \rightarrow H_1(G) \wedge H_1(G)$. Note that $\mathfrak{h}(G)$ inherits a natural grading from the free Lie algebra, compatible with the Lie bracket. By construction, $\mathfrak{h}(G)$ is generated by $\mathfrak{h}_1(G)$. Consequently, the derived Lie subalgebra, $\mathfrak{h}'(G)$, coincides with $\mathfrak{h}_{\geq 2}(G)$. If we drop the torsion-freeness assumption on $H_1(G)$, we may still define the rational holonomy Lie algebra, $\mathfrak{h}_{\mathbb{Q}}(G)$, using the rational homology groups of G .

To a group G , Quillen associates in a functorial way a Malcev filtered Lie algebra, M_G ; see [24, Appendix A], and also [21] for further details. A finitely presented group G is said to be 1-formal if M_G is isomorphic to the rational holonomy Lie algebra, $\mathfrak{h}_{\mathbb{Q}}(G)$, completed with respect to the bracket length filtration. Equivalently, M_G is a quadratic Malcev Lie algebra. If the group G is 1-formal, then $\text{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{h}_{\mathbb{Q}}(G)$, as follows from [24]. Moreover, $\text{gr}(G/G'') \otimes \mathbb{Q} \cong \mathfrak{h}_{\mathbb{Q}}(G)/\mathfrak{h}_{\mathbb{Q}}(G)''$, as shown in [21].

Assume now G is finitely presented, and $H_1(G)$ is torsion-free. Then, the canonical projection $\text{Lie}(H_1(G)) \twoheadrightarrow \text{gr}(G)$ factors through an epimorphism of graded Lie algebras, $\Psi_G: \mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$, which in turn descends to an epimorphism

$$\Psi_G^{(2)}: \mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \text{gr}(G/G'').$$

If the group G is 1-formal, then the maps $\Psi_G \otimes \mathbb{Q}$ and $\Psi_G^{(2)} \otimes \mathbb{Q}$ are isomorphisms, see [21].

By construction, the resonance variety $\mathcal{R}_1(G)$ depends only on the holonomy Lie algebra $\mathfrak{h}_{\mathbb{Q}}(G)$. More precisely, if $\mathfrak{h}_{\mathbb{Q}}(G_1) \cong \mathfrak{h}_{\mathbb{Q}}(G_2)$, as graded Lie algebras, then there is a linear isomorphism $H^1(G_1, \mathbb{C}) \cong H^1(G_2, \mathbb{C})$, restricting to an isomorphism $\mathcal{R}_1(G_1) \cong \mathcal{R}_1(G_2)$.

For a right-angled Artin group G_{Γ} , it is easily seen that $\mathfrak{h}(G_{\Gamma}) = \mathfrak{h}_{\Gamma}$, cf. [22]. Moreover, as shown by Kapovich and Millson [13], the group G_{Γ} is 1-formal. We now prove an analogous result for the Bestvina-Brady groups.

PROPOSITION 6.1. *If the flag complex Δ_{Γ} is simply-connected, then N_{Γ} is 1-formal.*

Proof. Consider the Dicks-Leary presentation (2.3) for N_{Γ} . It follows from [19] that the Malcev Lie algebra of N_{Γ} is the quotient of $\widehat{\text{Lie}}(\mathbf{E})$, the free Malcev Lie algebra on \mathbf{E} , by the closed Lie ideal generated by the elements (e, f) , (f, g) , (e, g) , efg^{-1} , for all directed triangles (e, f, g) as in Figure 1. Here multiplication denotes the Campbell-Hausdorff product in the underlying Malcev group.

Now use [20, Lemma 2.5] to replace the CH commutators (e, f) , (f, g) , (e, g) by the corresponding Lie commutators, $[e, f]$, $[f, g]$, $[e, g]$. It follows from the definition of the CH product that we may also replace efg^{-1} by $e + f - g$. This shows that $M_{N_{\Gamma}}$ is a quadratic Malcev Lie algebra, and so, N_{Γ} is 1-formal. \square

Let $\iota: N \rightarrow G$ be a homomorphism between finitely presented groups with torsion-free abelianizations. Denote by $\iota_*: H_*(N) \rightarrow H_*(G)$ the induced homomorphism in homology. We then have a commuting diagram,

$$\begin{array}{ccc} H_2(N) & \xrightarrow{\nabla_N} & H_1(N) \wedge H_1(N) \\ \downarrow \iota_* & & \downarrow \iota_* \wedge \iota_* \\ H_2(G) & \xrightarrow{\nabla_G} & H_1(G) \wedge H_1(G) \end{array} \quad (6.1)$$

Consequently, there is an induced morphism of graded Lie algebras, $\mathfrak{H}(\iota): \mathfrak{H}(N) \rightarrow \mathfrak{H}(G)$.

LEMMA 6.2. *If $\pi_1(\Delta_\Gamma) = 0$, then the map $\mathfrak{H}'_{\mathbb{Q}}(\iota): \mathfrak{H}'_{\mathbb{Q}}(N_\Gamma) \rightarrow \mathfrak{H}'_{\mathbb{Q}}(G_\Gamma)$ is an isomorphism of graded Lie algebras.*

Proof. The inclusion $\iota: N_\Gamma \rightarrow G_\Gamma$ induces Lie algebra maps $\mathfrak{H}(\iota): \mathfrak{H}(N_\Gamma) \rightarrow \mathfrak{H}(G_\Gamma)$ and $\text{gr}(\iota): \text{gr}(N_\Gamma) \rightarrow \text{gr}(G_\Gamma)$, which commute with the natural surjections from the holonomy to the associated graded Lie algebras. Passing to derived Lie subalgebras, and tensoring with \mathbb{Q} , we obtain the following commuting diagram:

$$\begin{array}{ccc} \mathfrak{H}'(N_\Gamma) \otimes \mathbb{Q} & \xrightarrow{\mathfrak{H}'(\iota) \otimes \mathbb{Q}} & \mathfrak{H}'(G_\Gamma) \otimes \mathbb{Q} \\ \downarrow \Psi_{N_\Gamma} \otimes \mathbb{Q} & & \downarrow \Psi_{G_\Gamma} \otimes \mathbb{Q} \\ \text{gr}'(N_\Gamma) \otimes \mathbb{Q} & \xrightarrow{\text{gr}'(\iota) \otimes \mathbb{Q}} & \text{gr}'(G_\Gamma) \otimes \mathbb{Q} \end{array} \quad (6.2)$$

The vertical arrows are isomorphisms, by the 1-formality of G_Γ and N_Γ , insured by [13] and Proposition 6.1, respectively. The bottom arrow is an isomorphism, by Theorem 5.6(i). Hence, the top arrow, $\mathfrak{H}'(\iota) \otimes \mathbb{Q} = \mathfrak{H}'_{\mathbb{Q}}(\iota)$, is also an isomorphism. \square

7. Cohomology ring

In this section, we exploit the 1-formality property of a finitely presented Bestvina-Brady group N_Γ , in order to give a purely combinatorial description of $H^{\leq 2}(N_\Gamma, \mathbb{Q})$.

7.1. Homology of N_Γ

Fix a coefficient field \mathbb{k} . The extension $1 \rightarrow N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\nu} \mathbb{Z} \rightarrow 0$ defines a natural $\mathbb{k}\mathbb{Z}$ -module structure on $H_*(N_\Gamma, \mathbb{k})$. The next result gives a combinatorial description of this structure. (See also [12], [16] for related computations.)

PROPOSITION 7.1. *Let Γ be a finite graph, with flag complex Δ_Γ . For each $r > 0$, we have an isomorphism of $\mathbb{k}\mathbb{Z}$ -modules,*

$$H_r(N_\Gamma, \mathbb{k}) \cong (\mathbb{k}\mathbb{Z})^{\dim \tilde{H}_{r-1}(\Delta_\Gamma, \mathbb{k})} \oplus (\epsilon \mathbb{k})^{\dim B_{r-1}(\Delta_\Gamma, \mathbb{k})},$$

where $\epsilon \mathbb{k}$ denotes the trivial $\mathbb{k}\mathbb{Z}$ -module \mathbb{k} , and $B_\bullet(\Delta_\Gamma, \mathbb{k})$ are the simplicial boundaries.

Proof. By Shapiro's Lemma, $H_*(N_\Gamma, \mathbb{k}) = H_*(\mathbb{k}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} \tilde{C}_\bullet)$, where \tilde{C}_\bullet is the equivariant chain complex from Proposition 3.1, and the change of rings $\mathbb{Z}G_\Gamma \rightarrow \mathbb{k}\mathbb{Z}$

is induced by $\nu: G_\Gamma \rightarrow \mathbb{Z}$. Write $P := \mathbb{k}\mathbb{Z} = \mathbb{k}[\tau^{\pm 1}]$. Using (3.1), we find that

$$H_r(N_\Gamma, \mathbb{k}) = \frac{P \otimes Z_{r-1}(\Delta_\Gamma, \mathbb{k})}{(\tau - 1)P \otimes B_{r-1}(\Delta_\Gamma, \mathbb{k})}, \quad (7.1)$$

as modules over P , where $Z_\bullet(\Delta_\Gamma, \mathbb{k})$ denotes the reduced simplicial cycles.

Set $B := B_{r-1}(\Delta_\Gamma, \mathbb{k})$, $Z := Z_{r-1}(\Delta_\Gamma, \mathbb{k})$, and $H := \tilde{H}_{r-1}(\Delta_\Gamma, \mathbb{k})$. It is straightforward to check that the natural maps $\mathbb{k} \otimes B \hookrightarrow P \otimes Z$ and $P \otimes Z \rightarrow P \otimes H$ give rise to the following split exact sequence of P -modules:

$$0 \longrightarrow \epsilon \mathbb{k} \otimes B \longrightarrow \frac{P \otimes Z}{(\tau - 1)P \otimes B} \longrightarrow P \otimes H \longrightarrow 0. \quad (7.2)$$

The conclusion follows by putting together (7.1) and (7.2). \square

7.2. Cohomology ring in low degrees

Recall from Section 3 that $K_\Gamma = K(G_\Gamma, 1)$ is a subcomplex of the standard torus $(S^1)^V$. This readily implies that the cohomology ring $H^*(G_\Gamma, \mathbb{k})$ is the quotient of the exterior \mathbb{k} -algebra on \mathbb{V} by the ideal generated by the monomials vw corresponding to non-edges $\{v, w\} \in \bar{\mathbb{E}}$.

LEMMA 7.2. *If $\pi_1(\Delta_\Gamma) = 0$, then the following hold.*

- (i) *The cup-product map $\cup_{N_\Gamma}: \bigwedge^2 H^1(N_\Gamma, \mathbb{Q}) \rightarrow H^2(N_\Gamma, \mathbb{Q})$ is surjective.*
- (ii) *The map $\iota^*: H^2(G_\Gamma, \mathbb{Q}) \rightarrow H^2(N_\Gamma, \mathbb{Q})$ is surjective.*

Proof. (i) It is enough to show that $\dim_{\mathbb{Q}} H_2(N_\Gamma, \mathbb{Q}) = \dim_{\mathbb{Q}} \text{im}(\nabla_{N_\Gamma})$. We know from Proposition 7.1 that $\dim H_2(N_\Gamma, \mathbb{Q}) = \dim Z_1(\Delta_\Gamma, \mathbb{Q}) = |\mathbb{E}| - |\mathbb{V}| + 1$, since Δ_Γ is simply-connected. On the other hand,

$$\begin{aligned} \dim \text{im}(\nabla_{N_\Gamma}) &= \dim \bigwedge^2 H_1(N_\Gamma, \mathbb{Q}) - \dim \mathfrak{H}_2(N_\Gamma) \otimes \mathbb{Q} \\ &= \binom{|\mathbb{V}| - 1}{2} - \left(\binom{|\mathbb{V}|}{2} - |\mathbb{E}| \right) \\ &= |\mathbb{E}| - |\mathbb{V}| + 1, \end{aligned}$$

by the definition of the holonomy Lie algebra, Lemma 6.2, and the injectivity of ∇_{G_Γ} .

(ii) Follows from Part (i), since we know from Lemma 3.2 that $\iota^*: H^1(G_\Gamma, \mathbb{Q}) \rightarrow H^1(N_\Gamma, \mathbb{Q})$ is surjective. \square

7.3. Proof of Theorem 1.3

Clearly, ι^* factors through the quotient by the ideal generated by ν , since $\nu \iota = 0$. The isomorphism claim in degree 1 follows immediately from Proposition 4.2(i). The surjectivity property in degree 2 is a direct consequence of Lemma 7.2(ii). We are left with proving that

$$\ker(\iota^*: H^2(G_\Gamma, \mathbb{Q}) \rightarrow H^2(N_\Gamma, \mathbb{Q})) \subset \text{im}(\cdot \nu: H^1(G_\Gamma, \mathbb{Q}) \rightarrow H^2(G_\Gamma, \mathbb{Q})).$$

By dualizing, it is enough to check that the inclusion

$$\ker(\mu_\nu^\top \circ \nabla_{G_\Gamma}: H_2(G_\Gamma, \mathbb{Q}) \rightarrow H_1(G_\Gamma, \mathbb{Q})) \subset \text{im}(\iota_*: H_2(N_\Gamma, \mathbb{Q}) \rightarrow H_2(G_\Gamma, \mathbb{Q})),$$

holds, where μ_ν^\top is the transpose of $\mu_\nu: H^1(G_\Gamma, \mathbb{Q}) \rightarrow \bigwedge^2 H^1(G_\Gamma, \mathbb{Q})$, the right-multiplication by ν . It follows from Proposition 4.2(i) that $\ker(\mu_\nu^\top) = \text{im}(\bigwedge^2 \iota_*)$.

Hence, $\ker(\mu_\nu^\top \circ \nabla_{G_\Gamma}) = \nabla_{G_\Gamma}^{-1}(\text{im}(\bigwedge^2 \iota_*))$. Now recall from Lemma 6.2 that $\bigwedge^2 \iota_*$ induces an isomorphism

$$\bigwedge^2 \iota_*: \bigwedge^2 H_1(N_\Gamma, \mathbb{Q})/\text{im}(\nabla_{N_\Gamma}) \xrightarrow{\cong} \bigwedge^2 H_1(G_\Gamma, \mathbb{Q})/\text{im}(\nabla_{G_\Gamma}). \quad (7.3)$$

The desired inclusion follows at once from (7.3) and diagram (6.1).

This finishes the proof of Theorem 1.3 from the Introduction. In [16], Leary and Saadetoğlu, using a different approach, obtain a similar description of the truncated cohomology ring $H^{\leq r}(N_\Gamma)$, in the situation when $\tilde{H}_{< r}(\Delta_\Gamma) = 0$.

8. Characteristic and resonance varieties

In previous work [22], [5], we determined the resonance and characteristic varieties of right-angled Artin groups. In this section, we do the same for finitely presented Bestvina-Brady groups, thus proving Theorem 1.4 from the Introduction.

8.1. Jumping loci for G_Γ

Let $\Gamma = (V, E)$ be a finite graph, and let G_Γ be the corresponding right-angled Artin group. Write $H_V = H^1(G_\Gamma, \mathbb{C})$ and $\mathbb{T}_V = \text{Hom}(G_\Gamma, \mathbb{C}^*)$. If W is a subset of V , write H_W and \mathbb{T}_W for the corresponding coordinate subspaces, respectively, subtori.

THEOREM 8.1 ([22], [5]). *Let Γ be a finite graph. Then:*

$$\mathcal{R}_1(G_\Gamma) = \bigcup_{\substack{W \subset V \\ \Gamma_W \text{ disconnected}}} H_W \quad \text{and} \quad \mathcal{V}_1(G_\Gamma) = \bigcup_{\substack{W \subset V \\ \Gamma_W \text{ disconnected}}} \mathbb{T}_W.$$

It follows that the irreducible components of the above varieties are indexed by the subsets $W \subset V$, maximal among those for which Γ_W is disconnected. In particular, if Γ is disconnected, then $\mathcal{R}_1(G_\Gamma) = H_V$ and $\mathcal{V}_1(G_\Gamma) = \mathbb{T}_V$.

EXAMPLE 8.2. Let Γ be a tree on $n > 2$ vertices. Then $\mathcal{R}_1(G_\Gamma)$ is a union of coordinate hyperplanes, one for each cut point (that is, non-extremal vertex) of Γ . In particular, the irreducible components of $\mathcal{R}_1(G_\Gamma)$ are in general position.

8.2. A map between jumping loci

Let $N_\Gamma = \ker(\nu: G_\Gamma \rightarrow \mathbb{Z})$, and let $\iota: N_\Gamma \hookrightarrow G_\Gamma$ be the natural inclusion.

Assuming Γ is connected, we infer from Proposition 4.2(i) that ι induces a vector space epimorphism $\iota^*: H^1(G_\Gamma, \mathbb{C}) \rightarrow H^1(N_\Gamma, \mathbb{C})$. Identifying $H^1(G_\Gamma, \mathbb{C})$ with \mathbb{C}^V , the kernel of ι^* gets identified with the diagonal line.

Similarly, ι induces an algebraic group epimorphism $\iota^*: \mathbb{T}_{G_\Gamma} \rightarrow \mathbb{T}_{N_\Gamma}$. Identifying \mathbb{T}_{G_Γ} with $(\mathbb{C}^*)^V$, the kernel of ι^* gets identified with the diagonal 1-dimensional subtorus.

LEMMA 8.3. *If the flag complex Δ_Γ is simply-connected, and $|V| > 1$, then*

- (i) *The map $\iota^*: H^1(G_\Gamma, \mathbb{C}) \rightarrow H^1(N_\Gamma, \mathbb{C})$ restricts to a surjection $\iota^*: \mathcal{R}_1(G_\Gamma) \rightarrow \mathcal{R}_1(N_\Gamma)$.*
- (ii) *The map $\iota^*: \mathbb{T}_{G_\Gamma} \rightarrow \mathbb{T}_{N_\Gamma}$ restricts to a surjection $\iota^*: \mathcal{V}_1(G_\Gamma) \rightarrow \mathcal{V}_1(N_\Gamma)$.*

Proof. Part (i). There is an intimate connection between the resonance variety and the infinitesimal Alexander invariant of a finitely presented group G . More precisely, $\mathcal{R}_1(G)$ coincides, away from the origin, with the zero set of the annihilator of the $\text{Sym}(H_1(G)) \otimes \mathbb{C}$ -module $\mathfrak{B}(G) \otimes \mathbb{C}$. This follows from [5, Lemma 4.2], together with the description of radicals of Fitting ideals in terms of annihilators, see [10, pp. 511–513].

It follows from Lemma 6.2 that $\mathfrak{B}(\iota) \otimes \mathbb{C}: \mathfrak{B}(N_\Gamma) \otimes \mathbb{C} \rightarrow \mathfrak{B}(G_\Gamma) \otimes \mathbb{C}$ is an isomorphism of modules over $\text{Sym}(H_1(N_\Gamma)) \otimes \mathbb{C}$. Hence,

$$\text{ann}(\mathfrak{B}(N_\Gamma) \otimes \mathbb{C}) = (\iota_* \otimes \mathbb{C})^{-1}(\text{ann}(\mathfrak{B}(G_\Gamma) \otimes \mathbb{C})). \quad (8.1)$$

Taking the complex varieties defined by the ideals on both sides of (8.1) finishes the proof.

Part (ii). Similarly, we know from [5, Lemma 4.5] that $\mathcal{V}_1(G)$ coincides, away from the origin, with the zero set of the annihilator of the $\mathbb{C}H_1(G)$ -module $B(G) \otimes \mathbb{C}$. By Proposition 4.2(iii), the map $B(\iota) \otimes \mathbb{C}: B(N_\Gamma) \otimes \mathbb{C} \rightarrow B(G_\Gamma) \otimes \mathbb{C}$ is an isomorphism of modules over $\mathbb{C}H_1(N_\Gamma)$. As above, we conclude by taking the complex varieties defined by the annihilators of these two modules. \square

LEMMA 8.4. *Suppose W is a proper subset of the vertex set V of Γ . Then the restriction of $\iota^*: H^1(G_\Gamma, \mathbb{C}) \rightarrow H^1(N_\Gamma, \mathbb{C})$ to the coordinate subspace H_W is injective. Similarly, the restriction of $\iota^*: \mathbb{T}_{G_\Gamma} \rightarrow \mathbb{T}_{N_\Gamma}$ to the coordinate subtorus \mathbb{T}_W is injective.*

Proof. Let $\{e_v\}_{v \in V}$ be the standard basis for the vector space $H^1(G_\Gamma, \mathbb{C}) = \mathbb{C}^V$. Suppose $\iota^*(\sum_{w \in W} c_w e_w) = 0$. Then $\sum_{w \in W} c_w e_w = c \sum_{v \in V} e_v$, for some scalar $c \in \mathbb{C}$. Picking $v \in V \setminus W$, and comparing the coefficient of e_v on both sides of this equation, we see that $c = 0$. This finishes the proof of the first claim; the proof of the second claim follows along the same lines. \square

For a subset $W \subset V$, let H'_W denote the subspace $\iota^*(H_W) \subset H^1(N_\Gamma, \mathbb{C})$, and let \mathbb{T}'_W denote the subtorus $\iota^*(\mathbb{T}_W) \subset \mathbb{T}_{N_\Gamma}$.

LEMMA 8.5. *Suppose W_1 and W_2 are two subsets of V , of size at most $|V| - 2$. If $H'_{W_1} \subset H'_{W_2}$, or $\mathbb{T}'_{W_1} \subset \mathbb{T}'_{W_2}$, then $W_1 \subset W_2$.*

Proof. Assume there is a vertex $v_1 \in W_1 \setminus W_2$. Since $|W_2| \leq |V| - 2$, there must be another vertex $v_2 \in V \setminus W_2$, distinct from v_1 . Suppose $H'_{W_1} \subset H'_{W_2}$. Then $e_{v_1} = \sum_{v \in W_2} c_v e_v + c \sum_{v \in V} e_v$. Comparing coefficients of e_{v_2} on both sides, we find $c = 0$; hence $e_{v_1} \in H_{W_2}$, a contradiction. The case $\mathbb{T}'_{W_1} \subset \mathbb{T}'_{W_2}$ is treated similarly. \square

8.3. Proof of Theorem 1.4

It follows from Theorem 8.1 and Lemma 8.3 that

$$\mathcal{R}_1(N_\Gamma) = \bigcup_{W} H'_W \quad \text{and} \quad \mathcal{V}_1(N_\Gamma) = \bigcup_{W} \mathbb{T}'_W, \quad (8.2)$$

where, in both cases, the union is taken over all subsets $W \subset V$, maximal among those for which Γ_W is disconnected. Lemma 8.4 guarantees that $H'_W \subset H^1(N_\Gamma, \mathbb{C})$

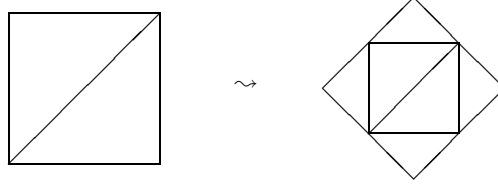


FIGURE 3. *Building an extra-special triangulation of the disk*

is a vector subspace of dimension $|W|$, and $\mathbb{T}'_W \subset \mathbb{T}_{N_\Gamma}$ is a subtorus of dimension $|W|$.

First assume $\kappa(\Gamma) = 1$, that is, Γ has a cut point. This means there is a $v \in V$ such that $\Gamma_{V \setminus v}$ is disconnected. A dimension count shows that $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$ and $\mathcal{V}_1(N_\Gamma) = \mathbb{T}_{N_\Gamma}$.

Now assume $\kappa(\Gamma) > 1$. We infer from Lemma 8.5 that (8.2) gives indeed the irreducible decompositions of the respective varieties. This ends the proof of Theorem 1.4.

EXAMPLE 8.6. Let Γ be a tree on $n > 2$ vertices. Then $\kappa(\Gamma) = 1$, and so $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C}) = \mathbb{C}^{n-1}$, and $\mathcal{V}_1(N_\Gamma) = \mathbb{T}_{N_\Gamma} = (\mathbb{C}^*)^{n-1}$ (this computation also follows from the fact that $N_\Gamma = F_{n-1}$).

9. Comparison with Artin groups and arrangement groups

In this section, we use the methods developed above to compare the Bestvina-Brady groups to two other classes of groups: Artin groups and arrangement groups.

9.1. Extra-special triangulations

Recall we defined a triangulation Δ of the disk D^2 to be *special* if Δ can be obtained from a triangle by adding one triangle at a time, along a unique boundary edge.

LEMMA 9.1. *Let Δ be a special triangulation of the 2-disk, and $\Gamma = (V, E)$ its 1-skeleton. Then $\mathcal{R}_1(N_\Gamma)$ is a proper subset of $H^1(N_\Gamma, \mathbb{C})$.*

Proof. Recall from Lemma 2.7(ii) that $\Delta_\Gamma = \Delta$; in particular, Theorem 1.4 applies. It is also readily seen that the graph Γ has no cut points, i.e, $\kappa(\Gamma) > 1$. Thus, $\mathcal{R}_1(N_\Gamma) \subsetneq H^1(N_\Gamma, \mathbb{C})$. \square

DEFINITION 9.2. A triangulation of D^2 is called *extra-special* if it is obtained from a special triangulation, by adding one triangle along each boundary edge. (See Figure 3.)

If Δ is extra-special, more can be said about the resonance variety of N_Γ . By definition, Δ is obtained by attaching triangles to the boundary edges of a special triangulation of the disk. Denote by (e_1, \dots, e_r) the circuit formed by these edges,

and write $W_i = V \setminus e_i$. Note that each edge e_i is a minimal cut set of Γ ; hence, H'_{W_i} is an irreducible component of $\mathcal{R}_1(N_\Gamma)$, for each $i = 1, \dots, r$.

LEMMA 9.3. *Let Γ be the 1-skeleton of an extra-special triangulation Δ of D^2 . Then the subspace $\bigcap_{i=1}^r H'_{W_i}$ has codimension $r - 1$ in $H^1(N_\Gamma, \mathbb{C})$. In particular, $\bigcap_{i=1}^r H'_{W_i} \neq 0$.*

Proof. We claim that

$$\iota^* \left(\bigcap_{i=1}^r H_{W_i} \right) = \bigcap_{i=1}^r H'_{W_i}. \quad (9.1)$$

The inclusion \subseteq is clear. The reverse inclusion is proved by induction on s ($0 < s < r$), with the case $s = 1$ being obvious. Set $P_k = \bigcap_{i=1}^k H_{W_i}$, $Q_k = H_{W_k}$, $P'_k = \bigcap_{i=1}^k H'_{W_i}$, and $Q'_k = H'_{W_k}$. We then have the commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_s \cap Q_{s+1} & \longrightarrow & P_s \oplus Q_{s+1} & \longrightarrow & P_s + Q_{s+1} \longrightarrow 0 \\ & & \downarrow \iota^* & & \downarrow \iota^* & & \downarrow \iota^* \\ 0 & \longrightarrow & P'_s \cap Q'_{s+1} & \longrightarrow & P'_s \oplus Q'_{s+1} & \longrightarrow & P'_s + Q'_{s+1} \longrightarrow 0 \end{array} \quad (9.2)$$

The middle arrow is an isomorphism, by Lemma 8.4 and the induction hypothesis. Clearly, the right arrow is an epimorphism. Note that $P_s + Q_{s+1}$ is a subspace of

$$H_{W_1 \cap \dots \cap W_s} + H_{W_{s+1}} = H_{(W_1 \cap \dots \cap W_s) \cup W_{s+1}}.$$

Since $(e_1 \cup \dots \cup e_s) \cap e_{s+1} \neq \emptyset$, this is a proper subspace of H_V . Thus, the right arrow in diagram (9.2) is injective, again by Lemma 8.4. Applying the 5-Lemma finishes the proof of the claim.

From (9.1), we see that

$$\text{codim} \bigcap_{i=1}^r H'_{W_i} = (|V| - 1) - \dim \bigcap_{i=1}^r H_{W_i} = r - 1.$$

Finally, if $\bigcap_{i=1}^r H'_{W_i} = 0$, then $r = |V|$. But clearly $|V| \geq 2r$. \square

9.2. Artin groups

A *weighted graph* is a graph $\Gamma = (V, E)$ endowed with a function $m: E \rightarrow \mathbb{Z}$ that assigns to each edge e an integer $m(e) \geq 2$. Such a weighted graph (Γ, m) determines an Artin group (of finite type),

$$G_{\Gamma, m} = \langle v \in V \mid \pi_m(v, w) = \pi_m(w, v) \text{ if } \{v, w\} \in E \rangle,$$

where $\pi_m(v, w) = v w v \dots$ has length $m(\{v, w\})$. When all edge weights are equal to 2, this is simply the right-angled Artin group G_Γ .

Associated to a weighted graph as above there is an ordinary (unlabeled) graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$, called the “odd contraction” of (Γ, m) , see [5, §10.9]. First define Γ_{odd} to be the unlabeled graph with vertex set V and edge set $\{e \in E \mid m(e) \text{ is odd}\}$. Then define \tilde{V} to be the set of connected components of Γ_{odd} , with two distinct components determining an edge $\{c, c'\} \in \tilde{E}$ if and only if there exist vertices $v \in c$ and $v' \in c'$ which are connected by an edge in E .

PROPOSITION 9.4. *Let Γ be the 1-skeleton of an extra-special triangulation of D^2 . Then the Bestvina-Brady group N_Γ is not isomorphic to any Artin group.*

Proof. Suppose N_Γ is isomorphic to an Artin group $G_{\Gamma',m}$. Let $\tilde{\Gamma}'$ be the odd contraction of Γ' . Lemma 10.11 from [5] guarantees that the respective Malcev Lie algebras, $M_{G_{\Gamma',m}}$ and $M_{G_{\tilde{\Gamma}'}}$, are filtered isomorphic. On the other hand, we know from Theorem 16.10 from [13] that both Artin groups, $G_{\Gamma',m}$ and $G_{\tilde{\Gamma}'}$, are 1-formal. Passing to associated graded Lie algebras, we obtain that $\mathfrak{H}_\mathbb{Q}(G_{\Gamma',m}) \cong \mathfrak{H}_\mathbb{Q}(G_{\tilde{\Gamma}'})$, as graded Lie algebras. Hence, $\mathfrak{H}_\mathbb{Q}(N_\Gamma) \cong \mathfrak{H}_\mathbb{Q}(G_{\tilde{\Gamma}'})$, as graded Lie algebras. This implies the existence of an ambient isomorphism

$$\mathcal{R}_1(N_\Gamma) \cong \mathcal{R}_1(G_{\tilde{\Gamma}'}).$$

From Lemma 9.1, we know that $\mathcal{R}_1(N_\Gamma) \subsetneq H^1(N_\Gamma, \mathbb{C})$. By Theorem 8.1, this forces $\tilde{\Gamma}'$ to be connected.

Let $\Gamma = (V, E)$, and write $v = |V|$, $e = |E|$. Similarly, let $\tilde{\Gamma}' = (V', E')$, and write $v' = |V'|$, $e' = |E'|$. We claim $v' = e' + 1$, and thus, $\tilde{\Gamma}'$ is a tree.

Note that $v' = b_1(G_{\tilde{\Gamma}'}) = b_1(N_\Gamma) = v - 1 \geq 5$. Moreover, $\binom{v'}{2} - e' = \text{rank } \mathfrak{H}_2(G_{\tilde{\Gamma}'}) = \text{rank } \mathfrak{H}_2(N_\Gamma)$. We also know that $\text{rank } \mathfrak{H}_2(N_\Gamma) = \text{rank } \mathfrak{H}_2(G_\Gamma)$, by Lemma 6.2. Since $\text{rank } \mathfrak{H}_2(G_\Gamma) = \binom{v}{2} - e$, we conclude that $e' = e - v + 1$. From Lemma 2.7(i), we know $2v - e = 3$; hence, $v' = e' + 1$, as claimed.

By the discussion from Example 8.2, the components of $\mathcal{R}_1(N_\Gamma) = \mathcal{R}_1(G_{\tilde{\Gamma}'})$ must intersect transversely. This contradicts Lemma 9.3. \square

9.3. Arrangement groups

Another widely studied class of groups are the fundamental groups of complements of complex hyperplane arrangements; see for instance [27] and references therein. Bestvina-Brady groups associated to simply-connected flag complexes share some common features with arrangement groups. Indeed, if G is a group in either class, then:

- G admits a finite presentation, with commutator relators only;
- G is 1-formal;
- $\mathcal{R}_1(G)$ is a union of linear subspaces.

There is a rather striking similarity between Bestvina-Brady groups associated to complete multipartite graphs and the fundamental groups of complements of “decomposable” arrangements. Indeed, if G is a group in either class, then the derived Lie algebra of $\text{gr}(G)$ is isomorphic to the derived Lie algebra of a product of free groups, and similarly for the derived Chen Lie algebra. For Bestvina-Brady groups, this was noted in Remark 5.8, while for decomposable arrangement groups, this is proved (by completely different methods) in Theorems 2.4 and 6.2 from [23].

Even so, there are many finitely presented Bestvina-Brady groups which are not arrangement groups.

PROPOSITION 9.5. *Let Γ be the 1-skeleton of an extra-special triangulation of D^2 . Then the Bestvina-Brady group N_Γ is not isomorphic to any arrangement group.*

Proof. If G is an arrangement group, then any two components of $\mathcal{R}_1(G)$ intersect only at 0, see [17]. By Lemma 9.3, this cannot happen for N_Γ . \square

Propositions 9.4 and 9.5 together yield Theorem 1.5 from the Introduction.

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