

CHEN LIE ALGEBRAS

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ABSTRACT. The Chen groups of a finitely-presented group G are the lower central series quotients of its maximal metabelian quotient, G/G'' . The direct sum of the Chen groups is a graded Lie algebra, with bracket induced by the group commutator. If G is the fundamental group of a formal space, we give an analog of a basic result of D. Sullivan, by showing that the rational Chen Lie algebra of G is isomorphic to the rational holonomy Lie algebra of G modulo the second derived subalgebra. Following an idea of W.S. Massey, we point out a connection between the Alexander invariant of a group G defined by commutator-relators, and its integral holonomy Lie algebra.

As an application, we determine the Chen Lie algebras of several classes of geometrically defined groups, including surface-like groups, fundamental groups of certain link complements in S^3 , and fundamental groups of complements of hyperplane arrangements in \mathbb{C}^ℓ . For link groups, we sharpen Massey and Traldi's solution of the Murasugi conjecture. For arrangement groups, we prove that the rational Chen Lie algebra is combinatorially determined.

1. INTRODUCTION

1.1. A classical construction of W. Magnus associates to a group G a graded Lie algebra over \mathbb{Z} ,

$$\mathrm{gr}(G) = \bigoplus_{k \geq 1} \Gamma_k G / \Gamma_{k+1} G,$$

where $\{\Gamma_k G\}_{k \geq 1}$ is the lower central series of the group, defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = (\Gamma_k G, G)$, and the Lie bracket $[x, y]$ is induced from the group commutator $(x, y) = xyx^{-1}y^{-1}$.

Many properties of a group are reflected in properties of its associated graded Lie algebra. For instance, if G is finitely generated, then the abelian groups $\mathrm{gr}_k(G)$ are also finitely generated; their ranks, $\phi_k(G)$, are important numerical invariants of G . In the case when $G = \mathbf{F}_n$, the free group of rank n , Magnus showed that $\mathrm{gr}(\mathbf{F}_n) = \mathbf{L}_n$, the free Lie algebra on n generators, whose graded ranks were computed by E. Witt. In general, though, the computation of the LCS ranks $\phi_k(G)$ can be exceedingly difficult.

1.2. In his thesis [6], K.T. Chen introduced a more manageable approximation to the LCS ranks. Let G/G'' be the maximal metabelian quotient of G . The Chen groups are

2000 *Mathematics Subject Classification*. Primary 20F14, 20F40; Secondary 17B70, 52C35, 55P62, 57M25.

Key words and phrases. Lower central series, derived series, associated graded Lie algebra, Chen groups, Alexander invariant, Malcev completion, holonomy Lie algebra, link in S^3 , Murasugi conjecture, hyperplane arrangement.

the graded pieces of the associated graded Lie algebra $\text{gr}(G/G'')$. Assume G is finitely-generated, and let $\theta_k(G) = \text{rank}(\text{gr}_k(G/G''))$ be the rank of the k -th Chen group. Then $\theta_k(G) = \phi_k(G)$, for $k \leq 3$, and $\theta_k(G) \leq \phi_k(G)$, for $k > 3$.

K.T. Chen showed that the Chen groups of \mathbf{F}_n are torsion-free, and computed their ranks. He also gave an algorithm for computing the ranks $\theta_k(G)$ for an arbitrary finitely presented group G , but that algorithm is highly impractical.

1.3. In a subsequent paper [8], Chen introduced the *rational holonomy Lie algebra* of a space X . Assuming X is a connected CW-complex, with finite 2-skeleton, this Lie algebra is

$$\mathfrak{H}(X; \mathbb{Q}) := \mathbf{L}^*(H_1(X; \mathbb{Q})) / \text{ideal}(\text{im}(\partial_X^{\mathbb{Q}})),$$

where $\mathbf{L}^*(H_1(X; \mathbb{Q}))$ is the free (graded) Lie algebra over \mathbb{Q} , generated in degree 1 by $H_1(X; \mathbb{Q})$, and $\partial_X^{\mathbb{Q}}$ is the dual of the cup-product map, $\cup_X^{\mathbb{Q}}: H^1(X; \mathbb{Q}) \wedge H^1(X; \mathbb{Q}) \rightarrow H^2(X; \mathbb{Q})$. It is readily seen that $\mathfrak{H}(X; \mathbb{Q}) = \mathfrak{H}(G; \mathbb{Q})$, where G is the fundamental group of X , and $\mathfrak{H}(G; \mathbb{Q})$ is the rational holonomy Lie algebra of the Eilenberg-MacLane space $K(G, 1)$.

1.4. Now suppose X is a *formal* space, in the sense of Sullivan [41]. Loosely speaking, this means that the rational homotopy type of X is a formal consequence of the rational cohomology algebra of X . Let $G = \pi_1(X)$. Sullivan showed that there is an isomorphism of graded Lie algebras over \mathbb{Q} ,

$$(1.1) \quad \text{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{H}(X; \mathbb{Q}).$$

There are many examples of formal spaces, for instance, spheres, compact Kähler manifolds (cf. [12]), and complements of complex hyperplane arrangements (as follows from [5]). Formality is preserved by taking products or wedges.

As proved in [41] and [31], the fundamental group of a formal space is 1-formal, in the sense of Definition 4.1. The isomorphism $\text{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{H}(G; \mathbb{Q})$ holds for arbitrary 1-formal groups. Fundamental groups of complements of complex projective hypersurfaces are always 1-formal (this was proved by Kohno [19], using resolution of singularities to reduce to the case of complements of normal-crossing divisors). In general though, complements of normal-crossing divisors may not be formal, see Morgan [31].

1.5. Assuming $H_1(G; \mathbb{Z})$ is torsion-free, one may define an integral form of Chen's holonomy Lie algebra. This graded \mathbb{Z} -Lie algebra, denoted by $\mathfrak{H}(G)$, comes equipped with a natural epimorphism of graded Lie algebras, $\Psi_G: \mathfrak{H}(G) \rightarrow \text{gr}(G)$. If G is 1-formal and $\mathfrak{H}(G)$ is torsion-free, the map Ψ_G gives an isomorphism

$$(1.2) \quad \mathfrak{H}(G) \cong \text{gr}(G),$$

see [27], and also Section 5.

1.6. The purpose of this note is to produce analogs of isomorphisms (1.1) and (1.2) for the Chen Lie algebra $\text{gr}(G/G'')$. Without much more effort, we will do this for all the higher-order Chen Lie algebras,

$$\text{gr}(G/G^{(i)}),$$

$i \geq 2$, where $\{G^{(i)}\}_{i \geq 0}$ is the derived series of G , defined inductively by $G^{(0)} = G$ and $G^{(i+1)} = (G^{(i)}, G^{(i)})$.

In Sections 2–4, we treat the rational case. Our first main result is Theorem 3.5, where we describe the Malcev completion of $G/G^{(i)}$ by means of a functorial formula, in terms of the Malcev completion of G . Under a formality assumption, we deduce in Theorem 4.2 the following analog of Sullivan’s isomorphism (1.1).

Theorem A. *Let G be a finitely presented group. If G is 1-formal, then there is an isomorphism of graded Lie algebras,*

$$(1.3) \quad \mathrm{gr}(G/G^{(i)}) \otimes \mathbb{Q} \cong \mathfrak{H}(G; \mathbb{Q})/\mathfrak{H}^{(i)}(G; \mathbb{Q}), \quad \forall i \geq 2.$$

Formula (1.3) says that the rational associated graded Lie algebra of the i -th derived quotient of the group is isomorphic to the i -th derived quotient of the \mathbb{Q} -holonomy Lie algebra of the group. This formula does not hold for arbitrary G , see Example 4.4.

1.7. In Section 5, we treat the integral case. We first show that the map $\Psi_G: \mathfrak{H}(G) \rightarrow \mathrm{gr}(G)$ induces a natural epimorphism of graded Lie algebras, $\Psi_G^{(i)}: \mathfrak{H}(G)/\mathfrak{H}^{(i)}(G) \rightarrow \mathrm{gr}(G/G^{(i)})$, for each $i \geq 1$. Under some torsion-freeness and formality hypotheses, we deduce in Proposition 5.5 the following analog of isomorphism (1.2).

Theorem B. *Let G be a finitely presented group, with torsion-free abelianization. Suppose G is 1-formal. If $\mathfrak{H}(G)/\mathfrak{H}^{(i)}(G)$ is torsion-free, then the map $\Psi_G^{(i)}$ gives an isomorphism of graded Lie algebras over \mathbb{Z} ,*

$$(1.4) \quad \mathfrak{H}(G)/\mathfrak{H}^{(i)}(G) \cong \mathrm{gr}(G/G^{(i)}).$$

1.8. Our approach works particularly well for commutator-relators groups, that is, for finitely presented groups G defined by relators belonging to Γ_2 . In this case, the integral holonomy Lie algebra $\mathfrak{H}(G)$ may be described directly in terms of the defining relations of G , see Proposition 7.2.

The adjoint representation of $\mathfrak{H}(G)$ endows the *infinitesimal Alexander invariant*, $B(\mathfrak{H}(G)) = \mathfrak{H}(G)'/\mathfrak{H}(G)''$, with a natural graded module structure over the polynomial ring $S = \mathrm{Sym}(H_1 G)$. In Theorem 6.2, we derive from the defining presentation of G a finite presentation of $B(\mathfrak{H}(G))$ over the ring S , while in Proposition 9.6, we relate the module $B(\mathfrak{H}(G))$ to the classical Alexander invariant of the group, $B_G = G'/G''$.

1.9. In Theorem 7.3, we determine the Chen Lie algebras of “surface-like” groups. If $G = \langle x_1, \dots, x_n \mid r \rangle$ is 1-formal, with relator $r \in \Gamma_2 \mathbf{F}_n$ such that $\bar{r} = r \bmod \Gamma_3 \mathbf{F}_n$ satisfies certain non-degeneracy conditions, then:

$$\mathrm{gr}(G/G'') = \mathbf{L}_n / (\mathrm{ideal}(\bar{r}) + \mathbf{L}_n'').$$

Moreover, $\mathrm{gr}(G/G'')$ is torsion-free, with Hilbert series $1 + nt - (1 - nt + t^2)/(1 - t)^n$. This result is a Chen Lie algebra analog of work by Labute [22], where the Lie algebra $\mathrm{gr}(G)$ was determined for certain one-relator groups G .

There is a rich variety of one commutator-relator groups G satisfying the hypotheses of our Theorem 7.3. Among them, we mention the class of fundamental groups of irreducible complex plane projective curves of positive genus (see Example 7.4), and the algebraically defined class of surface-like groups from Example 7.5.

1.10. Let $K = (K_1, \dots, K_n)$ be a link in S^3 , with complement X , and fundamental group $G = \pi_1(X)$. The Chen groups $\text{gr}_k(G/G'')$, first considered by Chen in [6], [7], were intensively studied in the 70's and 80's, see for example [23], [25], [28], [29], [32], [42]. The most complete computation of Chen groups of links was done by Murasugi [32], in the case of 2-component links. No general formula is known for the Chen groups of an arbitrary link. Nevertheless, Murasugi proposed a simple formula for the Chen ranks θ_k and LCS ranks ϕ_k of a link group, in the case when all the linking numbers are equal to ± 1 . Murasugi's conjecture was proved in particular cases by Kojima [21] and Maeda [25], and in full generality by Massey and Traldi [29]. Further work in this direction was done in [23], [27], and [2].

As an application of our results, we prove an even stronger form of Murasugi's conjecture, both rationally and integrally. The main contribution is at the multiplicative level of the Chen Lie algebras of the links under consideration. See Theorem 10.4 (f), and Theorem 10.8 (b)–(c) respectively.

1.11. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in \mathbb{C}^ℓ , with complement X , and group $G = \pi_1(X)$. The study of lower central series quotients of arrangement groups was initiated by Kohno [20], who computed the LCS ranks of the pure braid groups. Falk and Randell [13] extended Kohno's computation to the broader class of fiber-type arrangement groups, expressing the LCS ranks in terms of the exponents of the arrangement. See [18], [36] for further generalizations of the LCS formula, and [40], [39] for other formulas, and computations in low ranks, even when the LCS formula does not hold.

Another direction was started in [9], [10], with the study of Chen groups of arrangements. The Chen ranks $\theta_k(G)$ can provide stronger information than the LCS ranks $\phi_k(G)$. For example, the Chen ranks distinguish the pure braid group on $\ell \geq 4$ strands from the corresponding direct product of free groups, whereas the LCS ranks don't.

As a direct application of our results, we show, in Theorem 11.1, that the rational Chen Lie algebra of a complex hyperplane arrangement is combinatorially determined. Further applications can be found in [35].

Acknowledgments. The first author was partially supported by CERES grant 152/2003 of the Romanian Ministry of Education and Research. The second author was partially supported by NSF grants DMS-0105342 and DMS-0311142; he is grateful to the “Simion Stoilow” Institute of Mathematics of the Romanian Academy for hospitality during the completion of this work.

2. MALCEV LIE ALGEBRAS AND EXPONENTIAL GROUPS

Our approach to the various graded Lie algebras associated to a group G , such as $\text{gr}(G)$ and $\text{gr}(G/G^{(i)})$, is based on the Malcev completion functor of Quillen [38, Appendix A]. We review in this section Malcev Lie algebras and their associated exponential groups, leading to the general definition of the Malcev completion of a group.

Definition 2.1. A *Malcev Lie algebra* is a rational Lie algebra L , together with a complete, descending \mathbb{Q} -vector space filtration, $\{F_r L\}_{r \geq 1}$, such that:

- (1) $F_1 L = L$;
- (2) $[F_r L, F_s L] \subset F_{r+s} L$, for all r and s ;
- (3) the associated graded Lie algebra, $\text{gr}(L) = \bigoplus_{r \geq 1} F_r L / F_{r+1} L$, is generated in degree 1.

Completeness of the filtration means that the topology on L induced by $\{F_r L\}_{r \geq 1}$ is Hausdorff, and that every Cauchy sequence converges. In other words, the canonical map to the inverse limit, $\pi: L \rightarrow \varprojlim_r L/F_r L$, is a vector space isomorphism.

For example, any nilpotent Lie algebra L , with lower central series filtration $\{\Gamma_r L\}_{r \geq 1}$, is a Malcev Lie algebra.

Definition 2.2. Let L be a Malcev Lie algebra, with filtration $\{F_r L\}_{r \geq 1}$. The *exponential group* associated to L is the filtered group $\exp(L)$, with underlying set L , group multiplication given by the Campbell-Hausdorff formula

$$(2.1) \quad x \cdot y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \cdots, \quad \text{for } x, y \in L,$$

and filtration provided by the normal subgroups $\{\exp(F_r L)\}_{r \geq 1}$.

The convergence of the series (2.1) follows from condition (2) above, together with the completeness of the filtration topology.

If $\rho: G \rightarrow \exp(L)$ is a group homomorphism, then, as shown by Lazard [24], ρ induces a graded Lie algebra map,

$$(2.2) \quad \text{gr}(\rho): \text{gr}(G) \otimes \mathbb{Q} \longrightarrow \text{gr}(\exp(L)) = \text{gr}(L).$$

Definition 2.3. A *Malcev completion* of a group G is a group homomorphism, $\rho: G \rightarrow \exp(L)$, where L is a Malcev Lie algebra, such that, for each $r \geq 1$, the factorization $\rho_r: G/\Gamma_r G \rightarrow \exp(L/F_r L)$ has the property that: given a homomorphism $f_r: G/\Gamma_r G \rightarrow \exp(\tilde{L})$, where \tilde{L} is a nilpotent Malcev Lie algebra, there is a unique lift \tilde{f}_r to $\exp(L/F_r L)$.

The above universality property of Malcev completion is embodied in the commuting diagram

$$(2.3) \quad \begin{array}{ccc} G & \xrightarrow{\rho} & \exp(L) \\ \downarrow & & \downarrow \\ G/\Gamma_r G & \xrightarrow{\rho_r} & \exp(L/F_r L) \\ & \searrow f_r & \vdots \tilde{f}_r \\ & & \exp(\tilde{L}) \end{array}$$

The Malcev Lie algebra L is uniquely determined by the above universality property. The canonical Malcev completion, \hat{G} , can be constructed as follows (see Quillen [37], [38]).

The group algebra $\mathbb{Q}G$ has a natural Hopf algebra structure, with comultiplication given by $\Delta(g) = g \otimes g$, and counit the augmentation map. Let I be the augmentation ideal, and let $\widehat{\mathbb{Q}G} = \varprojlim_r \mathbb{Q}G/I^r$ be the completion of $\mathbb{Q}G$ with respect to the I -adic filtration. The Hopf algebra structure extends to the completion. Moreover, the Lie

algebra of primitive elements in $\widehat{\mathbb{Q}G}$, endowed with the inverse limit filtration, is a Malcev Lie algebra, denoted by M_G . Set:

$$(2.4) \quad \widehat{G} = \exp(M_G).$$

For example, if G is a nilpotent group, then M_G is a nilpotent Lie algebra, and so $\widehat{G} = G \otimes \mathbb{Q}$, the classical Malcev completion of G . In general, $\widehat{G} = \varprojlim_n ((G/\Gamma_n G) \otimes \mathbb{Q})$.

For an arbitrary group G , Quillen also constructs a natural group homomorphism,

$$(2.5) \quad \rho_G: G \longrightarrow \widehat{G} = \exp(M_G),$$

called the *functorial* Malcev completion homomorphism, which satisfies the universality property (2.3). Moreover,

$$(2.6) \quad \text{gr}(\rho_G): \text{gr}(G) \otimes \mathbb{Q} \xrightarrow{\sim} \text{gr}(M_G)$$

is an isomorphism of graded Lie algebras.

Now, let L be a Malcev Lie algebra as in Definition 2.3. Then, by the above discussion, $L \cong M_G$. Hence, by (2.6),

$$(2.7) \quad \text{gr}(L) \cong \text{gr}(G) \otimes \mathbb{Q}.$$

3. MALCEV COMPLETION AND DERIVED SERIES

We now investigate the relationship between Malcev completion and derived series. We will need a few lemmas.

Lemma 3.1. *Let $\rho: G \rightarrow \exp(L)$ be a group homomorphism. Then $\rho(G^{(i)}) \subset \exp(\overline{L^{(i)}})$, for all $i \geq 1$, where $\overline{L^{(i)}}$ is the closure of $L^{(i)}$ with respect to the filtration topology on L .*

Proof. Suppose $\rho(g) = x$ and $\rho(h) = y$. By the Campbell-Hausdorff formula, we have:

$$\rho(ghg^{-1}h^{-1}) = xyx^{-1}y^{-1} = [x, y] + \cdots \in \exp(\overline{L'}).$$

Thus, $\rho(G') \subset \exp(\overline{L'})$. The general case follows by induction on i . \square

In particular, the above Lemma implies that ρ factors through a homomorphism

$$\rho^{(i)}: G/G^{(i)} \longrightarrow \exp(L/\overline{L^{(i)}}),$$

for all $i \geq 1$. Here, $L/\overline{L^{(i)}}$ is endowed with the filtration induced from L ; this is a Malcev filtration (in the sense of Definition 2.1), since $\overline{L^{(i)}}$ is a closed ideal of L .

Lemma 3.2. *Let L be a nilpotent Malcev Lie algebra. Then $\exp(L)^{(i)} = \exp(L^{(i)})$, for all $i \geq 1$.*

Proof. It is enough to prove the equality for $i = 1$; the general case follows by induction on i . To prove the inclusion $\exp(L)' \subset \exp(L')$, use the Campbell-Hausdorff formula to express the group commutator (x, y) as a sum of Lie brackets, as above. To prove the reverse inclusion, use the Zassenhaus formula (cf. [24]) to express the Lie bracket $[x, y]$ as a product of group commutators. \square

We will also need the following criterion of Quillen [38], which characterizes the Malcev completion of a nilpotent group. In the statement below, one has to take into account the fact that the exponential construction described in Section 2 establishes a categorical equivalence between nilpotent \mathbb{Q} -Lie algebras, and nilpotent uniquely divisible groups; see [38].

Proposition 3.3 (Quillen). *Let N be a nilpotent group. A homomorphism $\varphi: N \rightarrow M$ defines a Malcev completion of N if and only if:*

- (1) M is a uniquely divisible, nilpotent group;
- (2) $\ker \varphi$ is a torsion group;
- (3) for all $y \in M$, there is an integer $n \neq 0$ such that $y^n \in \text{im } \varphi$.

Lemma 3.4. *Let N be a nilpotent group. Suppose $\varphi: N \rightarrow M$ is a Malcev completion homomorphism. Then the restrictions to derived series terms, $\varphi^{(i)}: N^{(i)} \rightarrow M^{(i)}$, are also Malcev completion homomorphisms.*

Proof. By [17], the restriction of φ to lower central series terms, $\Gamma_k \varphi: \Gamma_k N \rightarrow \Gamma_k M$, is a Malcev completion homomorphism, for each $k \geq 1$. Since $\varphi^{(1)} = \Gamma_2 \varphi$, the claim is proved for $i = 1$. The general case follows by induction on i . \square

Theorem 3.5. *Let G be a group, with Malcev completion $\rho: G \rightarrow \exp(L)$, as in Definition 2.3. Then, the Malcev completion of the derived quotient $G/G^{(i)}$ is given by*

$$(3.1) \quad \widehat{G/G^{(i)}} = \exp(L/\overline{L^{(i)}}), \quad \text{for all } i \geq 1.$$

Proof. Fix $i \geq 1$. By Lemma 3.1, ρ factors through a homomorphism $\rho^{(i)}: G/G^{(i)} \rightarrow \exp(L/\overline{L^{(i)}})$. It remains to verify that this homomorphism satisfies the universality property (2.3).

So fix $r \geq 1$, and consider the following diagram:

$$\begin{array}{ccccc}
 G & \xrightarrow{\rho} & \exp(L) & & \\
 \swarrow & & \downarrow & \searrow & \\
 G/G^{(i)} & \xrightarrow{\rho^{(i)}} & \exp(L/\overline{L^{(i)}}) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 G/\Gamma_r G & \xrightarrow{\rho_r} & \exp(L/F_r L) & & \\
 \downarrow & \searrow f_r & \downarrow & \swarrow \tilde{f}_r & \\
 G/\Gamma_r G \cdot G^{(i)} & \xrightarrow{\rho_r^{(i)}} & \exp(L/(\overline{L^{(i)}} + F_r L)) & \xleftarrow{\tilde{f}} & \exp(\tilde{L}) \\
 \downarrow & \nearrow f & & \swarrow \tilde{f} & \\
 G/G^{(i)} & & & &
 \end{array}$$

The maps $\rho^{(i)}$ and ρ_r factor through a common quotient map, $\rho_r^{(i)}$. We are given a map f and must lift it to \tilde{f} , as indicated in the diagram. Let f_r be the composite of f with the projection map from $G/\Gamma_r G$. The map f_r has a unique lift \tilde{f}_r , by the universality property of ρ . To construct the lift \tilde{f} , it is enough to show that $(L^{(i)} + F_r L)/F_r L$ is

contained in the kernel of \tilde{f}_r . (Since the topology of L/F_rL is discrete, $(\overline{L^{(i)} + F_rL})/F_rL = (L^{(i)} + F_rL)/F_rL$.)

By Lemmas 3.2 and 3.4, we have the following: For each $x \in (L^{(i)} + F_rL)/F_rL$, there is a $g \in (G^{(i)} \cdot \Gamma_r G)/\Gamma_r G$ such that $x^n = \rho_r(g)$, for some $n \neq 0$. Hence:

$$\tilde{f}_r(x)^n = \tilde{f}_r(\rho_r(g)) = f_r(g) = f(g) = 1 \in \exp(\tilde{L})$$

and so $\tilde{f}_r(x) = 1$, since $\exp(\tilde{L})$ is uniquely divisible.

To show that the lift \tilde{f} is unique, let $x \in \exp(L/F_rL)$. As before, write $x^m = \rho_r(h)$, for some $m \neq 0$ and $h \in G/\Gamma_r G$. Then $\tilde{f}(x)^m = f(h)$, and the uniqueness of \tilde{f} follows from the unique divisibility of $\exp(\tilde{L})$. \square

4. HOLONOMY LIE ALGEBRAS AND FORMALITY

In this section, we establish our main result. Under a formality assumption, we identify the derived quotients of the rational holonomy Lie algebra of a group G with the associated graded rational Lie algebra of the derived quotients of the group.

We will only consider spaces X having the homotopy type of a connected CW -complex with finite 2-skeleton. Accordingly, if G is the fundamental group of X , then G is finitely-presented, and the classifying space $K(G, 1)$ can be chosen to have finite 2-skeleton.

Let $\mathfrak{H}(G; \mathbb{Q})$ be Chen's *rational holonomy Lie algebra* of the space $K(G, 1)$,

$$(4.1) \quad \mathfrak{H}(G; \mathbb{Q}) := \mathbf{L}^*(H_1(G; \mathbb{Q})) / \text{ideal}(\text{im}(\partial_G^{\mathbb{Q}})).$$

Here \mathbf{L}^* denotes the free, graded Lie algebra functor, where the grading is given by bracket length, and $\partial_G^{\mathbb{Q}}: H_2(G; \mathbb{Q}) \rightarrow \mathbf{L}^2(H_1(G; \mathbb{Q}))$ is the dual of the rational cup-product map, $\cup_G^{\mathbb{Q}}: H^1(G; \mathbb{Q}) \wedge H^1(G; \mathbb{Q}) \rightarrow H^2(G; \mathbb{Q})$, with the standard identification of $\bigwedge^2 H_1$ with $\mathbf{L}^2(H_1)$.

The next definition will be crucial for our purposes.

Definition 4.1. A group G is called *1-formal* if the Malcev Lie algebra of G is isomorphic to the rational holonomy Lie algebra of G , completed with respect to bracket length.

In other words, G is 1-formal if

$$(4.2) \quad \widehat{G} = \exp\left(\widehat{\mathfrak{H}(G; \mathbb{Q})}\right).$$

We are now in position to state and prove our main result (Theorem A from the Introduction).

Theorem 4.2. *Let G be a 1-formal group. Then, for each $i \geq 1$, there is an isomorphism of rational graded Lie algebras,*

$$(4.3) \quad \text{gr}(G/G^{(i)}) \otimes \mathbb{Q} \cong \mathfrak{H}(G; \mathbb{Q}) / \mathfrak{H}(G; \mathbb{Q})^{(i)}.$$

Proof. Set $\mathfrak{H} = \mathfrak{H}(G; \mathbb{Q})$ and $L = \widehat{\mathfrak{H}}$. By 1-formality, $\widehat{G} = \exp(L)$. By Theorem 3.5, we have $\widehat{G/G^{(i)}} = \exp(L/\overline{L^{(i)}})$. Using the isomorphism (2.7), we find

$$(4.4) \quad \text{gr}(G/G^{(i)}) \otimes \mathbb{Q} \cong \text{gr}(L/\overline{L^{(i)}}).$$

Now consider the canonical Lie algebra map, $\iota: \mathfrak{H} \rightarrow L$, which sends the bracket length filtration of \mathfrak{H} to the Malcev filtration of L , and embeds \mathfrak{H} into its completion. For each $i \geq 1$, there is an induced map of filtered Lie algebras, $\iota^{(i)}: \mathfrak{H}/\mathfrak{H}^{(i)} \rightarrow L/\overline{L^{(i)}}$. It is straightforward to check that

$$(4.5) \quad \text{gr}(\iota^{(i)}): \text{gr}(\mathfrak{H}/\mathfrak{H}^{(i)}) \longrightarrow \text{gr}(L/\overline{L^{(i)}})$$

is an isomorphism. Moreover, $\text{gr}(\mathfrak{H}/\mathfrak{H}^{(i)}) \cong \mathfrak{H}/\mathfrak{H}^{(i)}$, since the filtration of $\mathfrak{H}/\mathfrak{H}^{(i)}$ is induced by the grading. Combining this isomorphism with those from (4.4) and (4.5) completes the proof. \square

Here is a quick application of our result.

Corollary 4.3. *Let \mathbf{L}_n be the free \mathbb{Z} -Lie algebra on n generators (in degree 1). Then $\mathbf{L}_n/\mathbf{L}_n''$ is a graded, free abelian group, with Hilbert series*

$$(4.6) \quad \text{Hilb}(\mathbf{L}_n/\mathbf{L}_n'', t) = nt + \sum_{k \geq 2} (k-1) \binom{n+k-2}{k} t^k.$$

Proof. The torsion-freeness of $\mathbf{L}_n/\mathbf{L}_n''$ follows from the fact that the derived subalgebra of a free \mathbb{Z} -Lie algebra is again free; see [4, Exercise 11, p.77].

Let $\mathbf{F}_n = \pi_1(\bigvee_n S^1)$ be the free group on n generators. Since wedges of circles are formal spaces, Theorem 4.2 applies to \mathbf{F}_n . Note that $\mathfrak{H}(\mathbf{F}_n; \mathbb{Q}) = \mathbf{L}_n \otimes \mathbb{Q}$, as follows directly from definition (4.1). Therefore, the coefficients of the Hilbert series of $\mathbf{L}_n/\mathbf{L}_n''$ are equal to the Chen ranks, $\theta_k(\mathbf{F}_n)$. Those ranks were computed by Chen [6] (see also [32, 29, 10]). Putting things together yields (4.6). \square

Theorem 4.2 can be used as a 1-formality test. We illustrate with a simple example. Another example (for link groups) will be given in §10.2.

Example 4.4. Let $G = \langle x_1, x_2 \mid ((x_1, x_2), x_2) \rangle$. Plainly, $\theta_3 = \phi_3 \leq 1$. On the other hand, $\mathfrak{H}(G; \mathbb{Q}) = \mathbf{L}_2 \otimes \mathbb{Q}$ (this follows from Proposition 7.2). By Corollary 4.3, the rank of the degree 3 piece of $\mathfrak{H}(G; \mathbb{Q})/\mathfrak{H}(G; \mathbb{Q})''$ is 2. We infer that G cannot be 1-formal, since this would contradict Theorem 4.2.

5. INTEGRAL HOLONOMY LIE ALGEBRAS

Our next objective is to find an integral version of Theorem 4.2. We start by recalling the definition and basic properties of \mathbb{Z} -holonomy Lie algebras. As before, let X be a space having the homotopy type of a connected CW-complex with finite 2-skeleton.

Definition 5.1 ([27]). Assume $H_1 X = H_1(X; \mathbb{Z})$ is torsion-free. The *integral holonomy Lie algebra* of X is

$$\mathfrak{H}(X) := \mathbf{L}^*(H_1 X) / \text{ideal}(\text{im}(\overline{\partial}_X)).$$

Here $\overline{\partial}_X := \partial_X \circ \kappa$, where $\partial_X: \text{Hom}(H^2 X, \mathbb{Z}) \rightarrow \bigwedge^2 H_1 X = \mathbf{L}^2(H_1 X)$ is dual to the cup-product map $\cup_X: \bigwedge^2 H^1 X \rightarrow H^2 X$, and $\kappa: H_2 X \rightarrow \text{Hom}(H^2 X, \mathbb{Z})$ is the canonical surjection induced by the Kronecker pairing. If G is a finitely-presented group with torsion-free abelianization, then the \mathbb{Z} -holonomy Lie algebra of G is

$$\mathfrak{H}(G) := \mathfrak{H}(K(G, 1)).$$

From the above definition, we see that $\mathfrak{h}(X)$ is a \mathbb{Z} -form of $\mathfrak{h}(X; \mathbb{Q})$, that is,

$$(5.1) \quad \mathfrak{h}(X; \mathbb{Q}) = \mathfrak{h}(X) \otimes \mathbb{Q},$$

as graded Lie algebras. Obviously, both $\bar{\partial}_X$ and $\partial_X^{\mathbb{Q}}$ are natural with respect to continuous maps. Consider now a classifying map, $f: X \rightarrow K(G, 1)$, where $G = \pi_1(X)$. Since $H_1 f$ is an isomorphism and $H_2 f$ is an epimorphism, we infer that f induces an isomorphism between \mathbb{Z} -holonomy Lie algebras:

$$(5.2) \quad \mathfrak{h}(X) \cong \mathfrak{h}(\pi_1(X)).$$

Similarly, $\mathfrak{h}(X; \mathbb{Q}) \cong \mathfrak{h}(\pi_1(X); \mathbb{Q})$.

By the universality property of free Lie algebras, the standard identification $H_1 G = \text{gr}_1(G)$ extends to a surjective map of graded \mathbb{Z} -Lie algebras,

$$\Psi_G: \mathbf{L}^*(H_1 G) \rightarrow \text{gr}(G),$$

natural with respect to group homomorphisms.

Proposition 5.2 ([27]). *The above map Ψ_G sends $\text{im}(\bar{\partial}_G) = \text{im}(\partial_G)$ (see Definition 5.1) to zero, thus inducing a natural epimorphism of graded \mathbb{Z} -Lie algebras,*

$$(5.3) \quad \Psi_G: \mathfrak{h}(G) \rightarrow \text{gr}(G),$$

for any finitely-presented group G with torsion-free abelianization.

We now show that Ψ_G further descends to the derived quotients.

Proposition 5.3. *For each $i \geq 1$, there is a natural epimorphism of graded \mathbb{Z} -Lie algebras, $\Psi_G^{(i)}: \mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} \rightarrow \text{gr}(G/G^{(i)})$, which fits into the commuting diagram*

$$\begin{array}{ccc} \mathfrak{h}(G) & \xrightarrow{\Psi_G} & \text{gr}(G) \\ \downarrow & & \downarrow \\ \mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} & \xrightarrow{\Psi_G^{(i)}} & \text{gr}(G/G^{(i)}) \end{array}$$

Proof. Clearly, $\Psi_G(\mathfrak{h}(G)^{(i)}) \subset \text{gr}(G)^{(i)}$. Furthermore, the canonical projection $\text{gr}(G) \rightarrow \text{gr}(G/G^{(i)})$ takes $\text{gr}(G)^{(i)}$ to 0, since the Lie bracket on $\text{gr}(G/G^{(i)})$ is induced by the group commutator. Thus, Ψ_G passes to the quotients. The induced map, $\Psi_G^{(i)}$, is automatically surjective. \square

Corollary 5.4. *Let G be a 1-formal group, with torsion-free abelianization. Then the natural Lie algebra maps*

$$\Psi_G^{(i)} \otimes \mathbb{Q}: \left(\mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} \right) \otimes \mathbb{Q} \longrightarrow \text{gr}(G/G^{(i)}) \otimes \mathbb{Q}$$

are isomorphisms, for all $i \geq 1$.

Proof. By Theorem 4.2, the graded Lie algebras $\text{gr}(G/G^{(i)}) \otimes \mathbb{Q}$ and $(\mathfrak{h}(G)/\mathfrak{h}(G)^{(i)}) \otimes \mathbb{Q}$ are isomorphic. Therefore, the linear map $\Psi_G^{(i)} \otimes \mathbb{Q}$ induces in each degree a surjection between \mathbb{Q} -vector spaces of the same (finite) dimension, hence an isomorphism. \square

The following Proposition, together with Corollary 5.4, implies Theorem B from the Introduction.

Proposition 5.5. *Suppose $\Psi_G^{(i)} \otimes \mathbb{Q}$ is an isomorphism. Then $\Psi_G^{(i)}$ induces a surjection on p -torsion, for all primes p . If, moreover, the Hilbert series of the graded vector space $(\mathfrak{H}(G)/\mathfrak{H}(G)^{(i)}) \otimes \mathbb{F}_p$ is independent of p , then $\Psi_G^{(i)}$ is an isomorphism of graded \mathbb{Z} -Lie algebras, and the graded abelian group $\text{gr}(G/G^{(i)})$ is torsion-free.*

Of course, the same statement also holds for $\Psi_G: \mathfrak{H}(G) \rightarrow \text{gr}(G)$; applications may be found in [2, 27]. Here is our first application for $\Psi_G^{(i)}$.

Corollary 5.6. *For each $i \geq 1$, there is an isomorphism $\text{gr}(\mathbf{F}_n/\mathbf{F}_n^{(i)}) \cong \mathbf{L}_n/\mathbf{L}_n^{(i)}$ of graded \mathbb{Z} -Lie algebras. Moreover, $\text{gr}(\mathbf{F}_n/\mathbf{F}_n^{(i)})$ is torsion-free, as a graded abelian group.*

Proof. As in the proof of Corollary 4.3, start by noting that \mathbf{F}_n is 1-formal, $\mathfrak{H}(\mathbf{F}_n) = \mathbf{L}_n$, and $\mathbf{L}_n/\mathbf{L}_n^{(i)}$ is torsion-free. Then apply Proposition 5.5. \square

6. THE INFINITESIMAL ALEXANDER INVARIANT OF A QUADRATIC LIE ALGEBRA

Let \mathbf{L}_Y be the free Lie algebra (over \mathbb{Z}) on a set Y . Denote by $S = \mathbb{Z}[Y]$ the symmetric algebra on Y , with variables in degree 1. By the Poincaré-Birkhoff-Witt Theorem, S is isomorphic to the universal enveloping algebra $U(\mathbf{L}_Y/\mathbf{L}'_Y)$.

Definition 6.1. Let E be a graded Lie algebra. Suppose $E = \mathbf{L}_Y/I$, where I is a homogeneous ideal, generated in degree ≥ 2 . The *infinitesimal Alexander invariant* of E is the graded S -module $B(E) := E'/E''$, with module structure given by the adjoint representation, via the exact sequence of Lie algebras

$$(6.1) \quad 0 \rightarrow E'/E'' \rightarrow E/E'' \rightarrow E/E' \rightarrow 0.$$

In the case when Y is finite and I is generated in degree 2, we can write down explicitly a finite presentation for $B(E)$. We shall use the Koszul resolution $(S \otimes \bigwedge_Y^k, d_k)$ of the trivial S -module \mathbb{Z} , where \bigwedge_Y^* denotes the exterior algebra on Y . We will also identify $\mathbf{L}_Y^1 = \bigwedge_Y^1$ and $\mathbf{L}_Y^2 = \bigwedge_Y^2$, with $[x, y]$ corresponding to $x \wedge y$.

Theorem 6.2. *Let E be a graded Lie algebra. Suppose E admits a quadratic, finite presentation of the form $E = \mathbf{L}/I$, where \mathbf{L} is the free Lie algebra on a finite set Y and I is a graded ideal generated by $J := I \cap \mathbf{L}^2$. Then*

$$(6.2) \quad S \otimes \left(\bigwedge_Y^3 \oplus J \right) \xrightarrow{d_3 + (\text{id} \otimes \partial)} S \otimes \bigwedge_Y^2 \longrightarrow B(E) \rightarrow 0$$

is a homogeneous, finite S -presentation of $B(E)$, where:

- $\deg(\bigwedge_Y^k) = k$, $\deg(J) = 2$,
- ∂ is the inclusion of J into \bigwedge_Y^2 ,
- $d_3(x \wedge y \wedge z) = x \otimes y \wedge z - y \otimes x \wedge z + z \otimes x \wedge y$, for $x, y, z \in Y$.

Proof. We start by finding a presentation for the infinitesimal Alexander invariant of the free Lie algebra $\mathbf{L} = \mathbf{L}_Y$ on the set $Y = \{y_1, \dots, y_n\}$. Define an S -linear map

$$(6.3) \quad \eta: S \otimes \bigwedge_Y^2 \longrightarrow \mathbf{L}'/\mathbf{L}'', \quad x \wedge y \mapsto [x, y] \text{ mod } \mathbf{L}''.$$

We claim that η is surjective. Indeed, let $[u, v] \in \mathbf{L}'$. If both u and v have bracket length greater than 1, then $[u, v] \in \mathbf{L}''$. So we may assume that u has length 1. Now, if v also has length 1, then $[u, v] = \eta(u \wedge v)$. The claim follows by induction on the bracket length of v , using commutator calculus.

Now let $d_3: S \otimes \bigwedge_Y^3 \rightarrow S \otimes \bigwedge_Y^2$ be the Koszul differential. By the Jacobi identity, we have $\eta \circ d_3 = 0$. From Corollary 4.3, we know that the degree k component of \mathbf{L}'/\mathbf{L}'' is free abelian, of rank equal to $\theta_k(\mathbf{F}_n) = (k-1) \binom{n+k-2}{k}$, for each $k \geq 2$. A standard dimension-counting argument shows that this rank is the same as the rank of the degree k component of $\text{coker } d_3$, which is also free abelian. Thus, $\ker \eta = \text{im } d_3$ and the following sequence is exact:

$$(6.4) \quad S \otimes \bigwedge_Y^3 \xrightarrow{d_3} S \otimes \bigwedge_Y^2 \xrightarrow{\eta} B(\mathbf{L}_Y) \rightarrow 0.$$

We return now to the proof in the general case: $E = \mathbf{L}/I$, where I is generated in degree two. Let $\pi: \mathbf{L} \rightarrow E$ be the canonical projection. Note that $E'/E'' = \mathbf{L}'/(\mathbf{L}'' + I)$, and so we have an exact sequence

$$(6.5) \quad 0 \rightarrow (I + \mathbf{L}'')/\mathbf{L}'' \rightarrow \mathbf{L}'/\mathbf{L}'' \xrightarrow{\pi} E'/E'' \rightarrow 0.$$

It is readily seen that the S -linear surjection $\pi \circ \eta: S \otimes \bigwedge_Y^2 \rightarrow E'/E''$ has kernel equal to the image of $d_3 + (\text{id} \otimes \partial): S \otimes (\bigwedge_Y^3 \oplus J) \rightarrow S \otimes \bigwedge_Y^2$. This finishes the proof. \square

From the Hilbert-Serre Theorem, we also obtain the following corollary.

Corollary 6.3. *If E is a quadratic, finitely presented, graded Lie algebra over a field \mathbb{k} , then*

$$(6.6) \quad \text{Hilb}_{\mathbb{k}}(E/E'', t) = \text{Hilb}_{\mathbb{k}}(E'/E'', t) + \text{Hilb}_{\mathbb{k}}(E/E', t)$$

is a rational function in t .

The main example to keep in mind is that of the holonomy Lie algebra, $E = \mathfrak{H}(G)$, of a finitely presented group G with torsion-free abelianization. We simply call $B(\mathfrak{H}(G)) = \mathfrak{H}(G)'/\mathfrak{H}(G)''$ the *infinitesimal Alexander invariant* of G .

7. CHEN LIE ALGEBRAS OF ONE-RELATOR GROUPS

In this section, we isolate a class of groups G for which the holonomy Lie algebra $\mathfrak{H}(G)$ may be described directly in terms of the defining relations. If G has a single relator (of a certain type), we determine the Chen Lie algebra $\text{gr}(G/G'')$.

Definition 7.1. A group G is called a *commutator-relators* group if it admits a finite presentation of the form $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, where each relator r_j belongs to the commutator subgroup of the free group $\mathbf{F}_n = \langle x_1, \dots, x_n \rangle$.

Note that $G/G' = \mathbb{Z}^n$, with canonical \mathbb{Z} -basis, $Y = \{y_1, \dots, y_n\}$, given by the classes of the group generators. Recall that $\text{gr}(\mathbf{F}_n) = \mathbf{L}_Y$. Denote by $\{\bar{r}_j \in \mathbf{L}_Y^2\}_{1 \leq j \leq m}$ the images of the relators, modulo $\Gamma_3 \mathbf{F}_n$.

Proposition 7.2. *If $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ is a commutator-relators group, then*

$$(7.1) \quad \mathfrak{H}(G) = \mathbf{L}(y_1, \dots, y_n) / \text{ideal}(\bar{r}_1, \dots, \bar{r}_m).$$

Proof. Let $\Delta_G: H_2G \rightarrow H_2(G \times G)$ be the homomorphism induced by the diagonal map $G \rightarrow G \times G$, and let $\text{pr}: H_2(G \times G) \rightarrow H_1G \otimes H_1G$ be the projection given by the Künneth formula. Note that Δ_G commutes with the automorphism of $H_2(G \times G)$ induced by the flip map $G \times G \rightarrow G \times G$. Thus, the composite $\text{pr} \circ \Delta_G$ takes values in $\wedge^2 H_1G$, viewed as a subgroup of $\otimes^2 H_1G$ via the embedding $x \wedge y \mapsto x \otimes y - y \otimes x$. So we may define the reduced diagonal, $\overline{\Delta}_G: H_2G \rightarrow H_1G \wedge H_1G$, as the co-restriction of $\text{pr} \circ \Delta_G$. It is now straightforward to check that $\overline{\Delta}_G$ coincides with the map $\overline{\partial}_G$ from Definition 5.1. Equality (7.1) follows at once from [34, Lemma 2.8]. \square

Let X_g be a closed, orientable surface of genus $g \geq 1$, with fundamental group $\pi_1(X_g) = \langle x_1, x'_1, \dots, x_g, x'_g \mid \prod_{i=1}^g (x_i, x'_i) \rangle$. In [22], Labute showed that the associated graded Lie algebra of $\pi_1(X_g)$ has presentation

$$(7.2) \quad \text{gr}(\pi_1(X_g)) = \mathbf{L}(x_1, x'_1, \dots, x_g, x'_g) / ([x_1, x'_1] + \dots + [x_g, x'_g]),$$

with graded ranks $\phi_k = \phi_k(\pi_1(X_g))$ given by

$$(7.3) \quad \prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = 1 - 2gt + t^2.$$

(This formula can also be derived from [36], using the fact that X_g is formal, and $H^*(X_g; \mathbb{Q})$ is a Koszul algebra.)

In our next result, we provide analogs of (7.2) and (7.3) above, for the Chen Lie algebras of “surface-like” one-relator groups.

Theorem 7.3. *Let $G = \langle x_1, \dots, x_n \mid r \rangle$ be a commutator-relators group defined by a single relation. Denote by $\bar{r} \in \mathbf{L}^2(y_1, \dots, y_n)$ the class of r modulo Γ_3 . Assume that $\bar{r} \neq 0$ (modulo p), for every prime p . If G is 1-formal, then*

$$(7.4) \quad \text{gr}(G/G'') = \mathbf{L}(y_1, \dots, y_n) / (\text{ideal}(\bar{r}) + \mathbf{L}''(y_1, \dots, y_n)),$$

as graded \mathbb{Z} -Lie algebras. Moreover, $\text{gr}(G/G'')$ is torsion-free as a graded abelian group, and its Hilbert series, $\text{Hilb}(\text{gr}(G/G''), t) = \sum_{k \geq 1} \theta_k t^k$, is given by

$$(7.5) \quad \text{Hilb}(\text{gr}(G/G''), t) = 1 + nt - \frac{1 - nt + t^2}{(1 - t)^n}.$$

Proof. By Proposition 7.2, the graded \mathbb{Z} -Lie algebra on the right-hand side of equation (7.4) equals $\mathfrak{H}(G)/\mathfrak{H}(G)''$. By Corollary 5.4, Proposition 5.5 and Theorem 6.2, all we have to do is to check that

$$(7.6) \quad \text{Hilb}_{\mathbb{F}_p}(B(\mathfrak{H}(G)) \otimes \mathbb{F}_p, t) = 1 - \frac{1 - nt + t^2}{(1 - t)^n},$$

for all primes p . Fix then \mathbb{F}_p -coefficients (omitting them from notation).

From presentation (6.2) and the exactness of Koszul resolutions, we infer that

$$B(\mathfrak{H}(G)) = \text{coker} \{S \xrightarrow{\mu} \ker(d_1)\},$$

where μ denotes multiplication by $d_2(1 \otimes \bar{r}) \in \ker(d_1)$. Write $1 \otimes \bar{r} = \sum_{1 \leq i < j \leq n} c_{ij} \otimes y_i \wedge y_j$, with $c_{ij} \in \mathbb{F}_p$. Then

$$d_2(1 \otimes \bar{r}) = \sum_{i < j} c_{ij} (y_i \otimes y_j - y_j \otimes y_i),$$

and this is non-zero, since $\bar{r} \neq 0$ by assumption. Hence μ is injective, and formula (7.6) readily follows. \square

For many one-relator groups G , the above theorem determines the structure of the Chen Lie algebra $\text{gr}(G/G'')$, and computes the Chen ranks $\theta_k(G)$. We illustrate with several examples.

Example 7.4. Let G be the fundamental group of an irreducible curve in the complex projective plane. Denote by $\Sigma \subset X$ the set of singular points of X . For each $x \in \Sigma$, let m_x be the number of local branches passing through x . Set $m := \sum_{x \in \Sigma} (m_x - 1)$, and $n := 2g + m$, where g is the genus of X . Then, as is well-known, X has the homotopy type of the formal space $X_g \vee (\bigvee_m S^1)$. If $g = 0$, then $G = \mathbf{F}_m$, and Corollaries 4.3 and 5.6 apply. If $g \geq 1$, then $G = \pi_1(X_g) * \mathbf{F}_m$ has a presentation with n generators and one commutator relator, and Theorem 7.3 applies.

Example 7.5. Let $G = \langle x_1, \dots, x_n \mid r \rangle$ be a one-commutator-relator group. Assume that $\bar{r} \in \bigwedge^2(y_1, \dots, y_n)$ is non-degenerate, and $\bar{r} \not\equiv 0 \pmod{p}$, for every prime p . Denote by X the finite 2-complex associated to the given presentation of G . By non-degeneracy, $n = 2g$ and $H^*(X; \mathbb{Q}) = H^*(X_g; \mathbb{Q})$, as graded algebras. By the rigidity results from [3, §2.5], the group $G = \pi_1(X)$ is 1-formal. Hence, Theorem 7.3 applies.

8. ALEXANDER INVARIANT

We now turn to the relationship between the holonomy Lie algebra and the Alexander invariant. We start by reviewing some basic material on the Alexander module and the Alexander invariant of a group, as well as their associated graded modules, and their connection to the Chen groups.

8.1. Alexander modules. Let G be a finitely-presented group. Let $\mathbb{Z}G$ be the group-ring, $\epsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$ the augmentation map, given by $\epsilon(\sum n_g g) = \sum n_g$, and $I_G := \ker \epsilon$ the augmentation ideal. Finally, let G/G' be the abelianization of G , and $\alpha: G \rightarrow G/G'$ the canonical projection.

Associated to G there are two important modules over $\mathbb{Z}(G/G')$:

- (1) The *Alexander module*: $A_G = \mathbb{Z}(G/G') \otimes_{\mathbb{Z}G} I_G$, the module induced from I_G by the extension of α to group-rings.
- (2) The *Alexander invariant*: $B_G = G'/G''$, with G/G' acting on the cosets of G'' via conjugation: $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G, h \in G'$.

These two $\mathbb{Z}(G/G')$ -modules fit into the Crowell exact sequence,

$$(8.1) \quad 0 \rightarrow B_G \rightarrow A_G \rightarrow I_{G/G'} \rightarrow 0.$$

If X is a connected CW-complex with finite 2-skeleton and $G = \pi_1(X, *)$, and if $p: \bar{X} \rightarrow X$ is the maximal abelian cover, then the homology exact sequence of the pair $(\bar{X}, p^{-1}(*))$ splits off the sequence (8.1), with $B_G = H_1(\bar{X})$ and $A_G = H_1(\bar{X}, p^{-1}(*))$.

8.2. Associated graded modules. Set $I := I_{G/G'}$. The module B_G comes endowed with the I -adic filtration, $\{I^k B_G\}_{k \geq 0}$. Let $\text{gr}(B_G) = \bigoplus_{k \geq 0} I^k B_G / I^{k+1} B_G$ be the associated graded module over the ring $\text{gr}(\mathbb{Z}(G/G')) = \bigoplus_{k \geq 0} I^k / I^{k+1}$. Then, as shown by W.S. Massey [28],

$$(8.2) \quad \text{gr}_k(G/G'') = \text{gr}_{k-2}(B_G)$$

for all $k \geq 2$, where recall that the associated graded on the left side is taken with respect to the lower central series filtration. In particular,

$$(8.3) \quad \sum_{k \geq 0} \theta_{k+2}(G) \cdot t^k = \text{Hilb}(\text{gr}(B_G) \otimes \mathbb{Q}, t).$$

Example 8.3. Let $G = P_n$ be Artin's pure braid group on n strings. As shown by Kohno [20], the graded ranks $\phi_k = \phi_k(P_n)$ are given by:

$$(8.4) \quad \prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \prod_{j=1}^{n-1} (1 - jt).$$

The Hilbert series of the Alexander invariant of P_n was computed in [9]:

$$(8.5) \quad \text{Hilb}(B_{P_n}, t) = \frac{\binom{n+1}{4}}{(1-t)^2} - \binom{n}{4}.$$

It follows that $\theta_k(P_n) = (k-1)\binom{n+1}{4}$, for $k \geq 3$.

8.4. Presentations for Alexander modules. Suppose G/G' is torsion-free, of rank $n = b_1(G)$, and fix a basis $\{t_1, \dots, t_n\}$. This identifies the group ring $\mathbb{Z}(G/G')$ with the ring of Laurent polynomials $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Under this identification, the augmentation ideal I corresponds to the ideal $\mathcal{I} = (t_1 - 1, \dots, t_n - 1)$.

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a commutator-relators group. Note that $G/G' = \mathbb{Z}^n$. Pick as basis elements $t_i = \alpha\varphi(x_i)$, where $\varphi: \mathbf{F}_n \rightarrow G$ is the canonical projection. The Alexander module of G admits a finite presentation

$$(8.6) \quad \Lambda^m \xrightarrow{D_G = (\alpha\varphi\partial_i(r_j))} \Lambda^n \rightarrow A_G \rightarrow 0,$$

where $\partial_i = \frac{\partial}{\partial x_i}: \mathbf{F}_n \rightarrow \mathbb{Z}\mathbf{F}_n$ are the Fox free derivatives.

Let $(\Lambda^{\binom{n}{k}}, \delta_k)$ be the standard Koszul resolution of \mathbb{Z} over Λ . By the fundamental formula of Fox calculus, $\delta_1 \circ D_G = 0$; see [14]. A diagram chase (see [28]) gives a finite presentation for the Alexander invariant:

$$(8.7) \quad \Lambda^{\binom{n}{3}+m} \xrightarrow{\Delta_G = \delta_3 + \vartheta_G} \Lambda^{\binom{n}{2}} \rightarrow B_G \rightarrow 0,$$

where ϑ_G is a map satisfying $\delta_2 \circ \vartheta_G = D_G$. (Such a map always exists, by basic homological algebra.)

9. LINEARIZED ALEXANDER INVARIANT

In this section, we show that the linearized Alexander invariant of a commutator-relators group coincides with the infinitesimal Alexander invariant of the holonomy Lie algebra of the group.

9.1. Magnus embedding. The ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ can be viewed as a subring of the formal power series ring $P = \mathbb{Z}[[s_1, \dots, s_n]]$ via the Magnus embedding $\mu: \Lambda \hookrightarrow P$, defined by $\mu(t_i) = 1 + s_i$. Note that μ sends the ideal $\mathcal{I} = (t_1 - 1, \dots, t_n - 1)$ to the ideal $\mathfrak{m} = (s_1, \dots, s_n)$. Passing to associated graded rings (with respect to the filtrations by powers of \mathcal{I} and \mathfrak{m}), the homomorphism $\text{gr}(\mu)$ identifies $\text{gr}(\Lambda)$ with the polynomial ring $S = \text{gr}(P) = \mathbb{Z}[s_1, \dots, s_n]$.

As a result, we may view the associated graded Alexander invariant, $\text{gr}(B_G)$, as an S -module. To compute the Hilbert series of this module, one needs to know (at the very least) a finite presentation. But even if an explicit presentation for B_G is known, finding a presentation for $\text{gr}(B_G)$ involves an arduous Gröbner basis computation, see [9], [10]. We turn instead to a more manageable approximation for the Alexander invariant of a commutator-relators group G .

9.2. Linearized Alexander module. For each $q \geq 0$, let $\mu^{(q)}: \Lambda \rightarrow P/\mathfrak{m}^{q+1}$ be the q -th truncation of μ . Since all the relators of G are commutators, the entries of the Alexander matrix D_G from (8.6) are in the ideal \mathcal{I} , and so $\mu^{(0)}(D_G)$ is the zero matrix. Hence, all the entries of the linearized Alexander matrix, $D_G^{(1)} := \mu^{(1)}(D_G)$, belong to $\mathfrak{m}/\mathfrak{m}^2 = \text{gr}_1(P)$, and thus can be viewed as linear forms in the variables of S .

Definition 9.3. The *linearized Alexander module* of a commutator-relators group G is the S -module $\mathfrak{A}_G = \text{coker } D_G^{(1)}$.

The linearized Alexander matrix depends only on the relators of G , modulo length 3 commutators. Indeed, let $r \in \Gamma_2 \mathbf{F}_n$ be a commutator-relator, and set

$$r^{(3)} := \prod_{i < j} (x_i, x_j)^{\epsilon_{ij}(r)},$$

where $\epsilon_{ij} = \epsilon \partial_i \partial_j$. Then

$$(9.1) \quad r \equiv r^{(3)}, \quad \text{modulo } \Gamma_3 \mathbf{F}_n.$$

A Fox calculus computation shows that $\mu^{(1)} \alpha \varphi \partial_k(r) = \mu^{(1)} \alpha \varphi \partial_k(r^{(3)})$, see [14, 30]. Consequently, the entries of the $m \times n$ matrix $D_G^{(1)}$ are given by:

$$(9.2) \quad \left(D_G^{(1)} \right)_{k,j} = \sum_{i=1}^n \epsilon_{ij}(r_k) s_i.$$

9.4. Linearized Alexander invariant. As above, let G be an n -generator, commutator-relators group. Identify the symmetric algebra on $H_1(G) = \mathbb{Z}^n$ with the polynomial ring $S = \mathbb{Z}[s_1, \dots, s_n]$, and let $(S \otimes \bigwedge^k H_1(G), d_k)$ be the Koszul complex of S . The map $d_1: S^n \rightarrow S$ is simply the matrix of variables; its image is the ideal $\mathfrak{m} = (s_1, \dots, s_n)$. Notice that $d_1 \circ D_G^{(1)} = 0$. Hence, the map d_1 factors through the cokernel of $D_G^{(1)}$, giving an epimorphism $\mathfrak{A}_G \twoheadrightarrow \mathfrak{m}$.

Definition 9.5. The *linearized Alexander invariant* of a commutator-relators group G is the graded S -module $\mathfrak{B}_G = \ker(\mathfrak{A}_G \twoheadrightarrow \mathfrak{m})$.

Recall we also constructed (from the \mathbb{Z} -holonomy Lie algebra of G) a graded module over the polynomial ring $\mathbb{Z}[y_1, \dots, y_n]$: the infinitesimal Alexander invariant, $B(\mathfrak{H}(G)) = \mathfrak{H}(G)' / \mathfrak{H}(G)''$. Renaming variables, $B(\mathfrak{H}(G))$ becomes a module over $S = \mathbb{Z}[s_1, \dots, s_n]$.

Proposition 9.6. *Let G be a commutator-relators group. Then $\mathfrak{B}_G \cong B(\mathfrak{H}(G))$, as graded S -modules.*

Proof. By Theorem 6.2, Proposition 7.2, and exactness of the Koszul resolution,

$$(9.3) \quad B(\mathfrak{H}(G)) = \text{coker} \{d_2: S \otimes J \longrightarrow \ker(d_1)\},$$

where $J \subset \mathbf{L}_Y^2 \equiv \bigwedge_Y^2$ is generated by $\{\bar{r}_k\}_{1 \leq k \leq m}$. By definition,

$$(9.4) \quad \mathfrak{B}_G = \text{coker} \{D_G^{(1)}: S^m \longrightarrow \ker(d_1)\}.$$

By (9.2), (9.3), and (9.4), we are left with checking that

$$(9.5) \quad d_2(\bar{r}) = \sum_{1 \leq i, j \leq n} \epsilon_{ij}(r) s_i y_j,$$

for any $r \in \Gamma_2 \mathbf{F}_n$. We know from (9.1) that $\bar{r} = \overline{r^{(3)}}$. Hence:

$$(9.6) \quad \bar{r} = \sum_{1 \leq i < j \leq n} \epsilon_{ij}(r) y_i \wedge y_j.$$

Therefore, by the very definition of Koszul differentials,

$$(9.7) \quad d_2(\bar{r}) = \sum_{i < j} \epsilon_{ij}(r) (s_i y_j - s_j y_i).$$

Symmetry properties from the free differential calculus [14] show that (9.5) is equivalent to (9.7). \square

Corollary 9.7. *Let G be a commutator-relators group, with Alexander invariant B_G , linearized Alexander invariant \mathfrak{B}_G and infinitesimal Alexander invariant $B(\mathfrak{H}(G))$. Then*

$$(9.8) \quad \text{Hilb}(\mathfrak{B}_G \otimes \mathbb{Q}, t) = \text{Hilb}(B(\mathfrak{H}(G)) \otimes \mathbb{Q}, t).$$

If moreover G is 1-formal, then

$$(9.9) \quad t^2 \text{Hilb}(\text{gr}(B_G) \otimes \mathbb{Q}, t) = \sum_{k \geq 2} \theta_k(G) t^k = \text{Hilb}(B(\mathfrak{H}(G)) \otimes \mathbb{Q}, t).$$

Proof. The first equality from (9.9) follows from Massey's isomorphism (8.2), for an arbitrary finitely presented group G . If G is 1-formal, the second equality is provided by Theorem 4.2 (with $i = 2$), via (6.1). \square

The next example shows that the Hilbert series of the linearized Alexander invariant does depend in general on the characteristic of the field over which it is computed.

Example 9.8. Let G be the group with generators x_1, x_2 and a single relator $r = (x_1, x_2^p)$, where p is a prime. Since $\bar{r} = p[y_1, y_2]$ is non-degenerate, G is 1-formal, see

Example 7.5. By Proposition 7.2, we have $\mathfrak{H}(G) = \mathbf{L}(y_1, y_2)/\text{ideal}(p[y_1, y_2])$. By Theorem 6.2, the module $\mathfrak{B}_G = B(\mathfrak{H}(G))$ is isomorphic to $\mathbb{Z}[y_1, y_2]/(p)$, with degree shifted by 2. Hence:

$$\text{Hilb}(\mathfrak{B}_G \otimes \mathbb{k}, t) = \begin{cases} 0, & \text{if } \text{char}(\mathbb{k}) \neq p, \\ \binom{t}{1-t}^2, & \text{if } \text{char}(\mathbb{k}) = p. \end{cases}$$

10. LINKS IN S^3 AND THE MURASUGI CONJECTURE

10.1. **The holonomy Lie algebra of a link.** Let $K = (K_1, \dots, K_n)$ be a tame, oriented link in the 3-sphere, with complement $X = S^3 \setminus \bigcup_{i=1}^n K_i$, and fundamental group $G = \pi_1(X)$. By Alexander duality, $H^1(X) = \mathbb{Z}^n$, generated by classes e_1, \dots, e_n dual to the meridians of K . As is well-known, the cup-product map $\cup_X: H^1(X) \wedge H^1(X) \rightarrow H^2(X)$ is determined by the linking numbers, $l_{ij} := \text{lk}(K_i, K_j)$. Consequently, the holonomy Lie algebra is determined by the linking numbers of K . More precisely:

$$(10.1) \quad \mathfrak{H}(X) = \mathfrak{H}(G) = \mathbf{L}(y_1, \dots, y_n) / \left(\sum_{j=1}^n l_{ij}[y_i, y_j] = 0, 1 \leq i < n \right).$$

The information coming from linking numbers is conveniently encoded in the *linking graph* of K , denoted by \mathcal{G}_K . This is the subgraph of the complete graph on vertices $\{1, \dots, n\}$, having an edge $\{i, j\}$ whenever $l_{ij} \neq 0$, weighted by the corresponding linking number l_{ij} .

10.2. **Chen groups of 2-component links.** In [32], Murasugi computed the Chen groups of a 2-component link group G . Set $\ell := |\text{lk}(K_1, K_2)|$.

If $\ell = 0$, then G may well be non-1-formal. Indeed, in this case $\mathfrak{H}(G) = \mathbf{L}_2$ (cf. (10.1)), and thus, by Corollary 4.3, the degree 4 graded piece of $\mathfrak{H}(G)/\mathfrak{H}''(G)$ has rank 3. On the other hand, if G is the group of any 2-component link, then $\theta_4(G) = \phi_4(G)$. Take G to be the group of the Whitehead link. Then, $\text{gr}(G) = \mathbf{L}(y_1, y_2)/\text{ideal}([y_1, [y_2, [y_1, y_2]]])$, see [16]. It follows that $\theta_4(G) = 2$. Hence, by Theorem 4.2, the fundamental group of the Whitehead link is not 1-formal.

If $\ell \neq 0$, then $\text{gr}_k(G/G'') = \mathbb{Z}_\ell^{k-1}$, for all $k > 1$. In particular, if $\ell = 1$, then $\text{gr}_{>1}(G/G'') = 0$, and this easily implies $\text{gr}_{>1}(G) = 0$, too.

10.3. **The rational Murasugi conjecture.** In view of the above computations, Murasugi made the following conjecture: If G is the group of an n -component link with all linking numbers equal to ± 1 , then, for all $k > 1$,

$$(10.2) \quad \text{gr}_k(G) = \text{gr}_k(\mathbf{F}_{n-1}) \quad \text{and} \quad \text{gr}_k(G/G'') = \text{gr}_k(\mathbf{F}_{n-1}/\mathbf{F}_{n-1}'').$$

Murasugi's conjecture was proved, in a more general form, by Massey and Traldi [29]. These authors gave a sequence of necessary and sufficient conditions under which (10.2) holds, including the \mathbb{Z} -analogue of condition (b) below. Further conditions (for various choices of coefficients) were given in [23] and [2]. The following theorem is a rational version of the Murasugi conjecture, with one additional implication, that takes into account the Lie algebra structure on the direct sum of the Chen groups.

Theorem 10.4. *Let K be a link of n components, with complement X , group $G = \pi_1(X)$, holonomy Lie algebra $\mathfrak{H} = \mathfrak{H}(X)$, and linking graph $\mathcal{G} = \mathcal{G}_K$. The following are equivalent:*

- (a) *The linking graph \mathcal{G} is connected.*
- (b) *The rational cup-product map, $\cup_X^{\mathbb{Q}}: H^1(X; \mathbb{Q}) \wedge H^1(X; \mathbb{Q}) \rightarrow H^2(X; \mathbb{Q})$, is onto.*
- (c) *$\phi_k(G) = \phi_k(\mathbf{F}_{n-1})$, for all $k > 1$.*
- (d) *$\theta_k(G) = \theta_k(\mathbf{F}_{n-1})$, for all $k > 1$.*
- (e) *$\phi_2(G) = \phi_2(\mathbf{F}_{n-1})$.*

Moreover, any one of the above conditions implies that:

- (f) *$\text{gr}(G/G'') \otimes \mathbb{Q} = (\mathfrak{H}/\mathfrak{H}'') \otimes \mathbb{Q}$, as graded Lie algebras.*

Proof. The equivalence (a) \Leftrightarrow (b) is proved in [23], the implication (a) \Rightarrow (d) is proved in [29], (a) \Rightarrow (c) is proved in [2], while (c) \Rightarrow (e) and (d) \Rightarrow (e) are obvious.

To prove (e) \Rightarrow (b), set $\mathbf{L}_n^* = \mathbf{L}(y_1, \dots, y_n) \otimes \mathbb{Q}$. Let $\pi: \mathbf{L}_n^2 \rightarrow \text{gr}_2(G) \otimes \mathbb{Q}$ be the canonical surjection, induced by the Lie bracket of $\text{gr}(G) \otimes \mathbb{Q}$. From Sullivan [41], we know that $\ker(\pi) = \text{im}(\partial_X^{\mathbb{Q}})$, where $\partial_X^{\mathbb{Q}}$ is the dual of $\cup_X^{\mathbb{Q}}$. Moreover, $\dim \mathbf{L}_n^2 = \binom{n}{2}$, while $\text{rank}(\text{gr}_2(G)) = \phi_2(G) = \phi_2(\mathbf{F}_{n-1}) = \binom{n-1}{2}$, by assumption (e). Thus, $\dim(\ker \pi) = n - 1$, which equals $\dim H^2(X; \mathbb{Q})$.

We are left with proving (a) \Rightarrow (f). By Lemma 4.1 and Theorems 3.2 and 4.2 from [2], condition (a) implies $\text{gr}(G) \otimes \mathbb{Q} = \mathfrak{H} \otimes \mathbb{Q} = \mathbf{L}_{n-1} \rtimes \mathbf{L}_1$. Thus, $\mathfrak{H}' \otimes \mathbb{Q} = \mathfrak{H}^{>1} \otimes \mathbb{Q} = \mathbf{L}_{n-1}^{>1} = \mathbf{L}'_{n-1}$, and so $\mathfrak{H}'' \otimes \mathbb{Q} = \mathbf{L}''_{n-1}$. Consequently,

$$(10.3) \quad \dim(\text{gr}_k((\mathfrak{H}/\mathfrak{H}'') \otimes \mathbb{Q})) = \dim(\text{gr}_k((\mathbf{L}_{n-1}/\mathbf{L}''_{n-1}) \otimes \mathbb{Q})), \quad \text{for } k > 1.$$

Now recall that the map $\Psi_G^{(2)}: \mathfrak{H}/\mathfrak{H}'' \rightarrow \text{gr}(G/G'')$ is a surjection. Hence, by the equivalent condition (d) and by (10.3), the map $\Psi_G^{(2)} \otimes \mathbb{Q}$ is an isomorphism. \square

10.5. The integral Murasugi conjecture. To state an integral version of the previous theorem, we need to recall the following definition of Anick [1] (see also Labute [23]).

Definition 10.6. The linking graph \mathcal{G} is *connected modulo p* , where p is a prime number, if there is a spanning subtree of \mathcal{G} whose edges have linking numbers $l_{ij} \neq 0$ modulo p . The graph \mathcal{G} is *strongly connected* if it is connected modulo p , for all primes p .

We also need the following definition from [2, §3(2'')], slightly paraphrased for our purposes.

Definition 10.7. An n -component link is called \mathbb{Z} -*generic* if its \mathbb{Z} -holonomy Lie algebra splits as a semi-direct product $\mathbf{L}_{n-1} \rtimes \mathbf{L}_1$:

$$(10.4) \quad \mathfrak{H} = \mathbf{L}(y_1, \dots, y_n) / \left([y_i, y_n] + \sum_{j,k < n} c_{j,k}^i [y_j, y_k] = 0, 1 \leq i < n \right).$$

We now can improve on Theorem 10.4, as follows.

Theorem 10.8. *Let K be a link of n components, with complement X , group $G = \pi_1(X)$, holonomy Lie algebra $\mathfrak{H} = \mathfrak{H}(X)$, and linking graph $\mathcal{G} = \mathcal{G}_K$. Then:*

- (a) *The graph \mathcal{G} is strongly connected if and only if $\text{gr}(G/G'') = (\mathbf{L}_{n-1}/\mathbf{L}_{n-1}'') \oplus \mathbf{L}_1$ (as graded abelian groups).*
- (b) *If \mathcal{G} is strongly connected, then the Chen groups are torsion-free and $\text{gr}(G/G'') = \mathfrak{H}/\mathfrak{H}''$ (as graded Lie algebras).*
- (c) *The link K is \mathbb{Z} -generic if and only if the Chen Lie algebra splits as a semi-direct product, $\text{gr}(G/G'') = (\mathbf{L}_{n-1}/\mathbf{L}_{n-1}'') \rtimes \mathbf{L}_1$.*

Proof. (a) As shown by Labute [23], the graph \mathcal{G} is connected modulo p if and only if the cup-product map $\cup_X^{\mathbb{F}_p}: H^1(X; \mathbb{F}_p) \wedge H^1(X; \mathbb{F}_p) \rightarrow H^2(X; \mathbb{F}_p)$ is onto. Thus, \mathcal{G} is strongly connected if and only if $\cup_X: H^1(X; \mathbb{Z}) \wedge H^1(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ is onto. By Massey-Traldi [29, Theorem 1], this happens if and only if $\text{gr}_k(G/G'') = \text{gr}_k(\mathbf{F}_{n-1}/\mathbf{F}_{n-1}'')$, for all $k \geq 2$. Now use Corollary 5.6.

(b) If \mathcal{G} is strongly connected, then $\mathfrak{H}/\mathfrak{H}''$ is torsion-free, by [29, 3.4–3.6]. On the other hand, by the implication (a) \Rightarrow (f) from Theorem 10.4, we know that the surjection $\Psi_G^{(2)}: \mathfrak{H}/\mathfrak{H}'' \rightarrow \text{gr}(G/G'')$ induces an isomorphism $\Psi_G^{(2)} \otimes \mathbb{Q}: (\mathfrak{H}/\mathfrak{H}'') \otimes \mathbb{Q} \rightarrow \text{gr}(G/G'') \otimes \mathbb{Q}$. It follows that $\Psi_G^{(2)}$ is an isomorphism, and $\text{gr}(G/G'')$ is torsion-free.

(c) Suppose K is \mathbb{Z} -generic. The semi-direct product decomposition $\mathfrak{H} = \mathbf{L}_{n-1} \rtimes \mathbf{L}_1$ from (10.4) implies that $\partial_X: H_2(X; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z}) \wedge H_1(X; \mathbb{Z})$ is a split monomorphism. Therefore, \mathcal{G} is strongly connected, and so $\text{gr}(G/G'') = \mathfrak{H}/\mathfrak{H}''$, by Part (b). But now $\mathfrak{H}/\mathfrak{H}'' = (\mathbf{L}_{n-1}/\mathbf{L}_{n-1}'') \rtimes \mathbf{L}_1$, and this proves the forward implication.

Conversely, suppose $\text{gr}(G/G'') = (\mathbf{L}_{n-1}/\mathbf{L}_{n-1}'') \rtimes \mathbf{L}_1$. Then, in particular $\text{gr}_2(G) = \text{gr}_2(\mathbf{F}_{n-1})$. By Theorem 1 from [29], \mathcal{G} must be strongly connected; therefore $\mathfrak{H}/\mathfrak{H}'' = (\mathbf{L}_{n-1}/\mathbf{L}_{n-1}'') \rtimes \mathbf{L}_1$. Now,

$$(10.5) \quad (\mathbf{L}_{n-1}/\mathbf{L}_{n-1}'') \rtimes \mathbf{L}_1 = \mathbf{L}(y_1, \dots, y_n) / (\mathcal{I} + \mathbf{L}''(y_1, \dots, y_{n-1})),$$

where \mathcal{I} is the ideal generated by $[y_i, y_n] + \sum_{j, k < n} c_{j,k}^i [y_j, y_k]$, for $1 \leq i < n$, by the very definition of a semi-direct Lie product. On the other hand,

$$(10.6) \quad \mathfrak{H}/\mathfrak{H}'' = \mathbf{L}(y_1, \dots, y_n) / (\mathcal{J} + \mathbf{L}''(y_1, \dots, y_n)),$$

where $\mathcal{J} = \text{ideal}(\text{im}(\partial_X))$. By comparing (10.5) and (10.6) in degree 2, we obtain the semi-direct product decomposition (10.4) of \mathfrak{H} . This completes the proof. \square

Remark 10.9. The additive part (a) of the preceding theorem is basically contained in the main result of Massey and Traldi [29]. The other two multiplicative parts—related to the graded Lie algebra structure of $\text{gr}(G/G'')$ —are new.

Note also that the \mathbb{Z} -genericity condition in (c) is more restrictive than the strong connectivity condition in (a) and (b); see [2, Example 4.4].

11. CHEN GROUPS OF COMPLEX HYPERPLANE ARRANGEMENTS

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of n hyperplanes in \mathbb{C}^ℓ , with complement $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$. If \mathcal{A} is an affine arrangement in $\mathbb{C}^\ell \cong \{z \in \mathbb{C}^{\ell+1} \mid z_{\ell+1} = 1\}$, and $\text{c}\mathcal{A}$ is the cone on \mathcal{A} at the origin of $\mathbb{C}^{\ell+1}$, then $\text{c}\mathcal{A}$ is a central arrangement, and $\pi_1(X(\text{c}\mathcal{A})) \cong \pi_1(X(\mathcal{A})) \times \mathbb{Z}$. Therefore, there is no loss of generality in assuming, from now on, that \mathcal{A} is central. Set $X = X(\mathcal{A})$.

The fundamental group, $G = \pi_1(X)$, admits a finite presentation, with n generators (corresponding to the meridians), and commutator-relators.

The cohomology ring of the complement of a complex hyperplane arrangement was computed by Brieskorn [5], in answer to a conjecture of Arnol'd. A presentation for the ring $H^*(X; \mathbb{Z})$ in terms of the intersection lattice $\mathcal{L}(\mathcal{A}) = \{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A}\}$ was given by Orlik and Solomon [33].

An important consequence of Brieskorn's theorem is that every arrangement complement X is a formal space. In particular, the group G is 1-formal. Thus, our results from Sections 2–9 may be applied to G .

The following combinatorial description of $\mathfrak{H} = \mathfrak{H}(X) = \mathfrak{H}(G)$ follows from the Orlik-Solomon theorem. Let $[n] = \{1, \dots, n\}$ be the set of points of the underlying matroid \mathcal{M} of \mathcal{A} . Set $\mathcal{D} := \{d \subset [n] \mid d \text{ is a line of } \mathcal{M}\}$. Denote by J the abelian group freely generated by $\{(d, j) \mid d \in \mathcal{D} \text{ and } j \in d\}$, and define a homomorphism $\partial: J \rightarrow \mathbf{L}_Y^2$ (where $Y = \{y_1, \dots, y_n\}$) by:

$$(11.1) \quad \partial((d, j)) = [y_j, \sum_{i \in d} y_i].$$

Then the holonomy Lie algebra has presentation

$$(11.2) \quad \mathfrak{H} = \mathbf{L}_Y / \text{ideal}(\text{im}(\partial: J \rightarrow \mathbf{L}_Y^2)).$$

The linearized Alexander invariant, $\mathfrak{B}_G = B(\mathfrak{H})$, is a module over the polynomial ring $S = \mathbb{Z}[Y]$, with presentation matrix

$$\Delta_G^{(1)} = d_3 + \text{id} \otimes \bar{\partial}_G: S \otimes \left(\left(\bigwedge^3 H_1 G \right) \oplus H_2 G \right) \longrightarrow \bigwedge^2 H_1 G.$$

(This matrix first appeared in Theorem 4.6 and Remark 4.7 from [11].)

The next result (which is a direct consequence of Theorem 4.2) gives an affirmative answer to a conjecture formulated in [40, §8], regarding the combinatorial determination of the Chen ranks of complex hyperplane arrangements.

Theorem 11.1. *Let \mathcal{A} be a complex hyperplane arrangement, with complement X , fundamental group G , and holonomy Lie algebra \mathfrak{H} , with presentation given by (11.2) and (11.1). Then:*

$$(11.3) \quad \text{gr}(G/G'') \otimes \mathbb{Q} = (\mathfrak{H}/\mathfrak{H}'') \otimes \mathbb{Q}.$$

In particular, the rational Chen Lie algebra of the arrangement is combinatorially determined (as a graded Lie algebra), by the level 2 of the intersection lattice, $\mathcal{L}_2(\mathcal{A})$. Hence, the Chen ranks, $\theta_k(G) = \text{rank gr}_k(G/G'')$, are combinatorially determined.

Remark 11.2. If \mathcal{A} is hypersolvable, then both \mathfrak{H} and $\text{gr}(G)$ are torsion-free, see [18, §7]. In general, though, $\text{gr}(G)$ (and consequently, \mathfrak{H} , c.f. Proposition 5.5) may have torsion, see the example below.

The torsion-freeness of $\text{gr}(G/G'')$, also conjectured in [40, §8], is still open. In a stronger form, this conjecture would follow from the torsion-freeness of $\mathfrak{H}/\mathfrak{H}''$, but this is not the case, as the next example indicates.

Example 11.3. Let \mathcal{A} be a realization of the MacLane matroid, and G its group, see [10], [40] for details and references. As noted in [40, Example 10.7], the associated graded Lie algebra $\text{gr}(G)$ has 2- and 3-torsion.

Now let \mathfrak{B}_G be the linearized Alexander invariant. A computation with Macaulay 2 [15] reveals:

$$\text{Hilb}(\mathfrak{B}_G \otimes \mathbb{Q}) = t^2(8 + 5t - 10t^2 + 5t^3)/(1 - t)^2 = 8t^2 + 21t^3 + 24t^4 + 32t^5 + \dots,$$

$$\text{Hilb}(\mathfrak{B}_G \otimes \mathbb{F}_3) = t^2(8 + 5t - 7t^2 + 3t^3)/(1 - t)^2 = 8t^2 + 21t^3 + 27t^4 + 36t^5 + \dots.$$

Hence, \mathfrak{B}_G (and thus, $\mathfrak{H}/\mathfrak{H}''$) both have 3-torsion. Yet it is not clear whether this torsion survives in $\text{gr}(G/G'')$.

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