

# NON-FINITENESS PROPERTIES OF FUNDAMENTAL GROUPS OF SMOOTH PROJECTIVE VARIETIES

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ABSTRACT. For each integer  $n \geq 2$ , we construct an irreducible, smooth, complex projective variety  $M$  of dimension  $n$ , whose fundamental group has infinitely generated homology in degree  $n + 1$  and whose universal cover is a Stein manifold, homotopy equivalent to an infinite bouquet of  $n$ -dimensional spheres. This non-finiteness phenomenon is also reflected in the fact that the homotopy group  $\pi_n(M)$ , viewed as a module over  $\mathbb{Z}\pi_1(M)$ , is free of infinite rank. As a result, we give a negative answer to a question of Kollár on the existence of quasi-projective classifying spaces (up to commensurability) for the fundamental groups of smooth projective varieties. To obtain our examples, we develop a complex analog of a method in geometric group theory due to Bestvina and Brady.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Let  $M$  be an irreducible, smooth complex projective variety, with fundamental group  $G = \pi_1(M)$ . Groups  $G$  arising in this fashion are called *projective groups*. Except for the obvious fact that projective groups are finitely presentable, very little is known about their finiteness properties. The aim of this note is to answer several questions in this direction.

The classical finiteness conditions in group theory that we have in mind are:

(i) Wall's property  $\mathcal{F}_n$  ( $n \leq \infty$ ), requiring the existence of a classifying space  $K(G, 1)$  with finite  $n$ -skeleton. Note that  $\mathcal{F}_1$  is equivalent to finite generation, whereas  $\mathcal{F}_2$  is equivalent to finite presentability of  $G$ .

(ii) Property  $FP_n$  ( $n \leq \infty$ ), requiring the existence of a projective  $\mathbb{Z}G$ -resolution of the trivial  $G$ -module  $\mathbb{Z}$ , which is finitely generated in all dimensions  $\leq n$ . Note that the  $FP_n$  condition implies the finite generation of the homology groups  $H_i(G, \mathbb{Z})$ , for all  $i \leq n$ . Clearly, if  $G$  is of type  $\mathcal{F}_n$ , then  $G$  is of type  $FP_n$ . It follows from [36] that the converse also holds, provided  $G$  is finitely presentable, and  $n < \infty$ .

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(iii) Finiteness conditions related to the cohomological dimension of  $G$ , defined as  $\text{cd}(G) = \sup \{i \mid H^i(G, A) \neq 0\}$ , where  $A$  ranges over all  $\mathbb{Z}G$ -modules.

We refer to the works of Bieri [5], Brown [7], and Serre [29] for detailed information on finiteness properties of groups.

1.2. The first question we consider here was formulated by J. Kollár in [19, §0.3.1]: *Is a projective group  $G$  commensurable (up to finite kernels) with another group  $G'$ , admitting a  $K(G', 1)$  which is a quasi-projective variety?*

Note that necessarily  $G'$  must have a finite  $K(G', 1)$ , since every quasi-projective variety has the homotopy type of a finite CW-complex, see e.g. [10, p. 27]. For a discussion of various commensurability notions, we refer to §2.6.

The second problem we examine here is related to the Shafarevich conjecture [30], as reformulated in geometric finiteness terms by Kollár, in [19, 0.3.1.1–0.3.1.2]: *What other kind of finiteness properties are imposed on the group  $G = \pi_1(M)$  by the Stein property of the universal cover,  $\widetilde{M}$ ?*

Recall that a *Stein manifold* is a complex manifold which can be biholomorphically embedded as a closed subspace of some affine space  $\mathbb{C}^r$ . A classical result of Andreotti and Frankel [2] asserts that a Stein manifold of (complex) dimension  $n$  has the homotopy type of a CW-complex of dimension at most  $n$ .

1.3. The first example of a finitely presented group with infinitely generated third homology group is due to J. Stallings [33]. A systematic way of constructing groups  $N$  of type  $\mathcal{F}_n$ , but not of type  $FP_{n+1}$ , was found by M. Bestvina and N. Brady [4]. These authors start with a finite graph  $\Gamma = (\mathbf{V}, \mathbf{E})$ , and consider the associated right-angled Artin group  $G_\Gamma$ , with a generator  $v$  for each vertex in  $\mathbf{V}$ , and with a relation  $uv = vu$  for each edge in  $\mathbf{E}$ . The *Bestvina–Brady group*  $N_\Gamma$  is then defined as the kernel of the homomorphism  $\nu: G_\Gamma \rightarrow \mathbb{Z}$ , which sends each generator  $v$  to 1.

The group  $G_\Gamma$  admits as classifying space a subcomplex  $K_\Gamma$  of the torus of dimension  $|\mathbf{V}|$ , with cells in one-to-one correspondence with the simplices of the flag complex  $\Delta_\Gamma$ . Bestvina and Brady had the remarkable idea of exploiting the natural affine cell structure of the universal cover,  $\widetilde{K}_\Gamma$ , and to do a geometric and combinatorial version of real Morse theory on  $\widetilde{K}_\Gamma$ . In this way, they were able to establish a spectacular connection between the finiteness properties of the group  $N_\Gamma$ , and the homotopical properties of the simplicial complex  $\Delta_\Gamma$ .

It was noticed in [28] that the Stallings group may be realized as the fundamental group of the complement of a complex line arrangement in  $\mathbb{P}^2$ . In [14], we identified a large class of Bestvina–Brady groups which are *quasi-projective*, yet are not commensurable to any group admitting a classifying space which is a quasi-projective variety. Starting from a group  $G_\Gamma$  which is a product of  $r \geq 3$  free groups on at least two generators, we showed that  $N_\Gamma = \pi_1(H)$ , where  $H$  is the generic fiber of an explicit

polynomial map,  $h: X \rightarrow \mathbb{C}^*$ , and  $X = \mathbb{C}^r \setminus \{h = 0\}$ . Thus,  $N_\Gamma$  is the fundamental group of an irreducible, smooth complex affine variety of dimension  $r - 1$ . On the other hand,  $H_r(N_\Gamma; \mathbb{Z})$  is not finitely generated, and so  $N_\Gamma$  is not of type  $FP_r$ .

1.4. In this paper, we develop a complex analog of the Bestvina–Brady method, well adapted to construct projective groups with controlled finiteness properties. This leads to answers to the two questions mentioned in §1.2, as follows.

**Theorem A.** *For each  $r \geq 3$ , there is an  $(r - 1)$ -dimensional, smooth, irreducible, complex projective variety  $H$ , with fundamental group  $G$ , such that:*

- (1) *The homotopy groups  $\pi_i(H)$  vanish for  $2 \leq i \leq r - 2$ , while  $\pi_{r-1}(H) \neq 0$ .*
- (2) *The universal cover  $\tilde{H}$  is a Stein manifold.*
- (3) *The group  $G$  is of type  $\mathcal{F}_{r-1}$ , but not of type  $FP_r$ .*
- (4) *The group  $G$  is not commensurable (up to finite kernels) to any group having a classifying space of finite type.*

Our method actually yields stronger results (see Corollary 5.4): the first non-vanishing higher homotopy group  $\pi_{r-1}(H)$  from (1) is a free  $\mathbb{Z}\pi_1(H)$ -module, with geometrically computable system of free generators, and the universal cover  $\tilde{H}$  in (2) has the homotopy type of a wedge of  $(r - 1)$ -spheres. Similar results were obtained in [12], for *open*, smooth algebraic varieties  $H$ .

Theorem A gives a negative answer to Kollár’s question: by Part (4), the group  $G$  is not commensurable (up to finite kernels) to any group  $G'$  admitting a  $K(G', 1)$  which is a quasi-projective variety.

As for the second question, it is easy to show that the Stein property of the universal cover of a smooth projective variety  $M$  forces the cohomological dimension of the projective group  $G = \pi_1(M)$  to be larger than the complex dimension of  $M$ ; see Proposition 5.7. On the other hand, there is no implication of the Stein condition at the level of  $FP_*$  finiteness properties of  $G$ . To see this, compare smooth projective curves  $C$  of positive genus, for which  $\tilde{C}$  is Stein and  $C$  is aspherical, to the varieties  $H$  in Theorem A, for which  $\tilde{H}$  is Stein, yet  $\pi_1(H)$  is not of type  $FP_r$ , for  $r = \dim H + 1$ .

Finally, let us note that Theorem A also sheds light on the following question of Johnson and Rees [18]: are fundamental groups of compact Kähler manifolds Poincaré duality groups of even cohomological dimension? In [35], Toledo answered this question, by producing examples of smooth projective varieties  $M$  with  $\pi_1(M)$  of *odd* cohomological dimension. Our results (see Theorem 5.2(3)) show that fundamental groups of smooth projective varieties need not be Poincaré duality groups of *any* cohomological dimension: their Betti numbers need not be finite.

1.5. We start by establishing a general relationship between the finiteness properties of proper normal subgroups  $N$  of a finitely generated group  $G$ , with  $G/N$  torsion-free abelian, and the structure of certain subsets of the complex algebraic torus  $\mathbb{T}_G = \text{Hom}(G, \mathbb{C}^*)$ .

By definition, the *characteristic varieties* of  $G$  are the jumping loci for the homology of  $G$  with coefficients in rank one complex local systems:

$$(1) \quad \mathcal{V}_t^s(G) = \{\rho \in \mathbb{T}_G \mid \dim_{\mathbb{C}} H_s(G, \mathbb{C}_\rho) \geq t\}.$$

If  $G$  is of type  $FP_n$ , it is readily seen that  $\mathcal{V}_t^s(G)$  is an algebraic subvariety of  $\mathbb{T}_G$ , for  $s < n$ . If  $G$  is finitely presented, the varieties  $\mathcal{V}_t^1(G)$  may be computed directly from a presentation of  $G$ , using the Fox free differential calculus, see e.g. [17].

The importance of these varieties emerged from work of S.P. Novikov [27] on Morse theory for closed 1-forms on manifolds. As shown by Arapura [3], the characteristic varieties  $\mathcal{V}_1^1(G)$  provide powerful obstructions for deciding the realizability of  $G$  as the fundamental group of a smooth quasi-projective variety. We refer to [13] for various refinements in the case of 1-formal groups, in particular, projective groups.

**Theorem B.** *Let  $G$  be a finitely generated group. Suppose  $\nu: G \rightarrow A$  is a non-trivial homomorphism to a torsion-free abelian group  $A$ , and set  $N = \ker(\nu)$ . If  $\mathcal{V}_1^r(G) = \mathbb{T}_G$  for some integer  $r \geq 1$ , then:*

- (1)  $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) = \infty$ .
- (2)  $N$  is not commensurable (up to finite kernels) to any group of type  $FP_r$ .

The proof is given in Section 2. The theorem applies to groups of the form  $G = \times_{j=1}^r \pi_1(C_j)$ , with each  $C_j$  a smooth complex curve of negative Euler characteristic.

1.6. As noted by Deligne [9], every finitely presentable group can be realized as the fundamental group of an algebraic variety  $X$  (which can be chosen as the union of an arrangement of affine subspaces in some  $\mathbb{C}^n$ ). Insisting that  $X$  be a *smooth* variety, though, puts some stringent conditions on what groups can occur; see the monograph [1] as a reference, and [13] for some recent developments.

We set out here to construct new, interesting examples of projective groups. With Theorem B in mind, we adopt the viewpoint of realizing these new groups as subgroups of known groups, rather than extensions of known groups.

A convenient setup is provided by *irrational pencils*, that is, holomorphic maps  $h: X \rightarrow E$  between compact, connected, complex analytic manifolds, with target a curve  $E$  with  $\chi(E) \leq 0$ , and with connected generic fiber. Let  $p: \tilde{E} \rightarrow E$  be the universal cover, and denote by  $\hat{h}: \hat{X} \rightarrow \tilde{E}$  the pull-back of  $h$  via  $p$ . Clearly, the maps  $h$  and  $\hat{h}$  have the same fibers; let  $H$  be the common *smooth* fiber. Using complex Morse theory on  $\hat{X}$ , we obtain the following.

**Theorem C.** *Let  $h: X \rightarrow E$  be an irrational pencil. Suppose  $h$  has only isolated singularities. Then:*

- (1)  $\pi_i(\widehat{X}, H) = 0$ , for all  $i < \dim X$ .
- (2) *If, moreover,  $\dim X \geq 3$ , the induced homomorphism  $h_\#: \pi_1(X) \rightarrow \pi_1(E)$  is surjective, with kernel isomorphic to  $\pi_1(H)$ .*

The proof is given in Section 3. For more on complex Morse theory, see Looijenga's book [23], especially Section 5B.

A concrete family of examples is constructed in Section 4. Starting with an elliptic curve  $E$ , we take 2-fold branched covers  $f_j: C_j \rightarrow E$  ( $1 \leq j \leq r$ ), so that each curve  $C_j$  has genus at least 2. Setting  $X = \prod_{j=1}^r C_j$ , we see that  $X$  is a smooth, projective variety, whose universal cover is a contractible, Stein manifold. Moreover,  $\mathcal{V}_1^r(\pi_1(X)) = \mathbb{T}_{\pi_1(X)}$ . Using the group law on  $E$ , we can define a map  $h: X \rightarrow E$  by  $h = \sum_{j=1}^r f_j$ , for  $r \geq 2$ . Under certain assumptions on the branched covers  $f_j$ , we show that the smooth fiber of  $h$  is connected, and  $h$  has only isolated singularities.

Invoking now Theorems B and C completes the proof of Theorem A. Details and further discussion are given in Section 5.

## 2. CHARACTERISTIC VARIETIES AND FINITENESS PROPERTIES OF SUBGROUPS

This section is devoted to the proof of Theorem B.

2.1. Fix a positive integer  $m$ . Let  $\mathbb{T}_m = \text{Hom}(\mathbb{Z}^m, \mathbb{C}^*)$  be the character torus of  $\mathbb{Z}^m$ , and let  $\Lambda = \mathbb{C}\mathbb{Z}^m$  be its coordinate ring. Note that  $\Lambda$  is isomorphic to the ring of Laurent polynomials in  $m$  variables. In particular,  $\Lambda$  is a noetherian ring, of dimension  $m$ .

**Lemma 2.2.** *Let  $A$  be a  $\Lambda$ -module which is finite-dimensional as a  $\mathbb{C}$ -vector space. Then, for each  $j \geq 0$ , the set*

$$A_j := \{\rho \in \mathbb{T}_m \mid \text{Tor}_j^\Lambda(\mathbb{C}_\rho, A) = 0\}$$

*is a Zariski open, non-empty subset of the algebraic torus  $\mathbb{T}_m$ .*

*Proof.* Pick a free  $\Lambda$ -resolution  $F_\bullet \xrightarrow{\varepsilon} A$ , with  $F_j = \Lambda^{c_j}$ , and view the differentials  $d_j: \Lambda^{c_j} \rightarrow \Lambda^{c_{j-1}}$  as matrices with entries in  $\Lambda$ . For a character  $\rho \in \mathbb{T}_m$ , let  $d_j(\rho): \mathbb{C}^{c_j} \rightarrow \mathbb{C}^{c_{j-1}}$  be the evaluation of  $d_j$  at  $\rho$ . Clearly,  $\rho \in A_j$  if and only if

$$\text{rank } d_{j+1}(\rho) + \text{rank } d_j(\rho) \geq c_j,$$

a Zariski open condition on  $\rho$ . Assuming  $A_j$  to be empty, we derive a contradiction, as follows.

Let  $f \in \text{Ann}_\Lambda(A)$ . Denote by  $\mu_f$  the homothety induced by  $f$  on  $\Lambda$ -modules. Since  $\mu_f = 0$  on  $A$ ,  $\mu_f$  must induce the zero map on  $\text{Tor}_j^\Lambda(\mathbb{C}_\rho, A)$ , for any  $\rho \in \mathbb{T}_m$ . In turn, this map may be computed by using  $\mu_f$  on  $F_\bullet$ . It follows that the homothety  $\mu_{f(\rho)}$

on  $\mathbb{C}^{c_j}$  induces the zero map on  $\mathrm{Tor}_j^\Lambda(\mathbb{C}_\rho, A)$ , which is a non-zero  $\mathbb{C}$ -vector space, by our assumption on  $A_j$ . Hence,  $f(\rho) = 0$  for any  $\rho \in \mathbb{T}_m$ , i.e.,  $f = 0$  in  $\Lambda$ . This shows that  $\mathrm{Ann}_\Lambda(A) = 0$ .

Now recall that  $\dim_{\mathbb{C}} A < \infty$ , which implies that the  $\Lambda$ -module  $A$  is both noetherian and artinian, therefore of finite length. So,  $\dim(\Lambda/\mathrm{Ann}_\Lambda(A)) = 0$ , by standard commutative algebra. But  $\dim(\Lambda) = m > 0$ , a contradiction.  $\square$

2.3. Let  $G$  be a finitely generated group, and suppose  $\nu: G \rightarrow \mathbb{Z}^m$  is an epimorphism. Writing  $N = \ker(\nu)$ , we have an exact sequence

$$(2) \quad 1 \longrightarrow N \longrightarrow G \xrightarrow{\nu} \mathbb{Z}^m \longrightarrow 0 .$$

Denote by  $\nu^*: \mathbb{T}_m = \mathrm{Hom}(\mathbb{Z}^m, \mathbb{C}^*) \rightarrow \mathbb{T}_G = \mathrm{Hom}(G, \mathbb{C}^*)$  the induced map between character tori.

**Theorem 2.4.** *Assume that  $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) < \infty$ . Then there is a Zariski open, non-empty subset  $U \subset \mathbb{T}_m$  such that  $H_{\leq r}(G, \mathbb{C}_{\nu^*\rho}) = 0$ , for any  $\rho \in U$ .*

*Proof.* By Shapiro's Lemma,  $H_*(N, \mathbb{C}) = H_*(G, \Lambda)$ . Let us examine the spectral sequence associated to the base change  $\rho: \Lambda \rightarrow \mathbb{C}$ , for a fixed character  $\rho \in \mathbb{T}_m$ :

$$E_{st}^2 = \mathrm{Tor}_s^\Lambda(\mathbb{C}_\rho, H_t(G, \Lambda)) \Rightarrow H_{s+t}(G, \mathbb{C}_{\nu^*\rho}) ,$$

see [24, Theorem XII.12.1]. A finite number of applications of Lemma 2.2 guarantees the existence of a Zariski open, non-empty subset  $U \subset \mathbb{T}_m$ , such that  $E_{st}^2$  vanishes, provided  $s, t \leq r$ , and  $\rho \in U$ . The conclusion follows.  $\square$

**Remark 2.5.** If  $G$  admits a finite  $K(G, 1)$ , Theorem 2.4 also follows from Novikov-Morse theory; see [16], Proposition 1.30 and Theorem 1.50. See also [15], Theorem 1 for a related result, under the same finiteness assumption on  $G$ .

2.6. Two groups,  $G$  and  $G'$ , are said to be *commensurable* if there is a group  $\pi$  and a diagram

$$\begin{array}{ccc} G & & G' \\ & \swarrow & \nearrow \\ & \pi & \end{array}$$

with arrows injective and of cofinite image. The two groups are said to be *commensurable up to finite kernels* if there is a zig-zag of such diagrams, connecting  $G$  to  $G'$ , with arrows of finite kernel and cofinite image. Commensurability implies commensurability up to finite kernels, but the converse is not true in general. Nevertheless, the two notions coincide if one of the two groups is residually finite. For details on all this, see the book by de la Harpe [8, §IV.B.27–28].

**Proposition 2.7.** *Suppose  $G$  and  $G'$  are commensurable up to finite kernels. Then  $G$  is of type  $FP_n$  if and only if  $G'$  is of type  $FP_n$ .*

This is an immediate consequence of the following two results of Bieri.

**Lemma 2.8** ([5], Proposition 2.5). *Let  $\pi$  be a finite-index subgroup of  $G$ . Then  $G$  is of type  $FP_n$  if and only if  $\pi$  is.*

**Lemma 2.9** ([5], Proposition 2.7). *Let  $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$  be an exact sequence of groups, and assume  $N$  is of type  $FP_\infty$ . Then  $G$  is of type  $FP_n$  if and only if  $Q$  is.*

2.10. We are now in position to finish the proof of Theorem B. Recall we are given a finitely generated group  $G$ , and a non-trivial homomorphism  $\nu: G \rightarrow A$  to a torsion-free abelian group  $A$ . Write  $N = \ker(\nu)$ . Note that  $\text{im}(\nu) \cong \mathbb{Z}^m$ , for some  $m > 0$ . Without loss of generality, we may assume  $\nu$  is surjective, so that we have the exact sequence (2).

Part (1). By assumption, there is an integer  $r > 0$  such that  $\mathcal{V}_1^r(G) = \mathbb{T}_G$ ; that is to say,  $H_r(G, \mathbb{C}_\rho) \neq 0$ , for all  $\rho \in \mathbb{T}_G$ . By Theorem 2.4,  $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) = \infty$ .

Part (2). By Part (1), the group  $N$  is not of type  $FP_r$ . The conclusion follows from Proposition 2.7.  $\square$

2.11. We conclude this section with some simple examples of groups  $G$  to which Theorem B applies.

**Lemma 2.12.** *Let  $G = \times_{i=1}^r G_i$  be a product of finitely generated groups. If  $\mathcal{V}_1^1(G_i) = \mathbb{T}_{G_i}$ , for all  $i$ , then  $\mathcal{V}_1^r(G) = \mathbb{T}_G$ .*

*Proof.* For a character  $\rho \in \mathbb{T}_G$ , denote by  $\rho_i \in \mathbb{T}_{G_i}$  the restriction of  $\rho$  to  $G_i$ . By the Künneth formula,  $H_r(G, \mathbb{C}_\rho) \supset \bigotimes_{i=1}^r H_1(G_i, \mathbb{C}_{\rho_i})$ , and the tensor product is non-zero, by hypothesis.  $\square$

**Example 2.13.** Let  $G$  be the fundamental group of a smooth (not necessarily compact) complex curve  $C$ , with  $\chi(C) < 0$ . Then  $\mathcal{V}_1^1(G) = \mathbb{T}_G$ . Indeed, for any  $\rho \in \mathbb{T}_G$ , the Euler characteristic  $\chi(C, \mathbb{C}_\rho) := \dim H_0(C, \mathbb{C}_\rho) - \dim H_1(C, \mathbb{C}_\rho) + \dim H_2(C, \mathbb{C}_\rho)$  equals  $\chi(C)$ , and the claim follows.

Using Lemma 2.12, Example 2.13, and Theorem B, we obtain the following.

**Corollary 2.14.** *Let  $C_1, \dots, C_r$  be smooth, complex curves with  $\chi(C_j) < 0$ , and let  $G = \pi_1(\prod_{j=1}^r C_j)$ . Then:*

- (1)  $\mathcal{V}_1^r(G) = \mathbb{T}_G$ .
- (2) *If  $N$  is a normal subgroup of  $G$ , with  $G/N \cong \mathbb{Z}^m$ , for some  $m > 0$ , then  $N$  is not commensurable (up to finite kernels) to any group of type  $FP_r$ .*

In this context, we should note that the  $FP_n$  finiteness and non-finiteness properties of subgroups of finite products of surface groups were analyzed by Bridson, Howie, Miller, and Short [6], using different methods. The fact that the subgroup  $N$  from Corollary 2.14(2) above cannot be of type  $FP_r$  may be deduced from the results in [6]; our Theorem B improves this to  $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) = \infty$ .

**Remark 2.15.** As is well-known, fundamental groups of smooth complex curves are residually finite. Hence, the product groups  $G$  (and thus, their subgroups  $N$ ) from above are also residually finite. Consequently, the two notions of commensurability are equivalent for such groups. Note however that there do exist projective groups which are not residually finite, see [1].

### 3. A COMPLEX ANALOG OF BESTVINA–BRADY THEORY

In this section, we prove Theorem C from the Introduction.

3.1. Let  $X$  and  $E$  be compact, connected, complex analytic manifolds. Assume  $r := \dim X > 1$  and  $\dim E = 1$ . Let  $h: X \rightarrow E$  be a holomorphic map, and denote by  $C(h)$  the set of critical points of  $h$ . Write  $E^* = E \setminus h(C(h))$  and  $X^* = h^{-1}(E^*)$ .

Since  $h$  is a proper map, the restriction  $h^*: X^* \rightarrow E^*$  is a topologically locally trivial fibration. The fibers  $H_t = h^{-1}(t)$ , with  $t \in E^*$ , are called the *smooth fibers* of  $h$ . Clearly, such fibers are homeomorphic to each other. The map  $h$  is called an *irrational pencil* if  $E$  is a curve of positive genus, and the smooth fiber of  $h$ , denoted  $H$ , is connected.

So let  $h: X \rightarrow E$  be an irrational pencil, and consider the exact homotopy sequence of the fibration  $H \hookrightarrow X^* \xrightarrow{h^*} E^*$ . Since  $H$  is connected,  $h_{\#}^*$  is an epimorphism. Clearly, the inclusion  $\iota: E^* \rightarrow E$  induces an epimorphism  $\iota_{\#}: \pi_1(E^*) \rightarrow \pi_1(E)$ . It follows that  $h_{\#}$  is an epimorphism as well. Hence,  $h_{\#}$  induces an isomorphism

$$(3) \quad \pi_1(X)/\ker(h_{\#}) \xrightarrow{\cong} \pi_1(E) .$$

3.2. Now let  $p: \tilde{E} \rightarrow E$  be the universal covering of  $E$ , and let  $\hat{h}: \hat{X} \rightarrow \tilde{E}$  be the pull-back of  $h$  along  $p$ . We get an induced mapping,  $\hat{p}: \hat{X} \rightarrow X$ , which is the Galois cover associated to the normal subgroup  $\ker(h_{\#}) \subset \pi_1(X)$ , as in diagram (4).

$$(4) \quad \begin{array}{ccccc} H & \xrightarrow{\hat{i}} & \hat{X} & \xrightarrow{\hat{h}} & \tilde{E} \\ \parallel & & \downarrow \hat{p} & & \downarrow p \\ H & \xrightarrow{i} & X & \xrightarrow{h} & E \\ \parallel & & \uparrow & & \uparrow \iota \\ H & \xrightarrow{i^*} & X^* & \xrightarrow{h^*} & E^* \end{array}$$

Note that  $\tilde{E}$  is either the complex affine line  $\mathbb{C}$  (when  $\text{genus}(E) = 1$ ), or the open unit disc  $\mathbb{D}$  (when  $\text{genus}(E) > 1$ ). Moreover,  $\hat{h}: \hat{X} \rightarrow \tilde{E}$  is a proper complex analytic mapping, with smooth fiber  $H$ . Assuming  $h$  has only isolated critical points, we infer that  $\hat{h}$  has countably many isolated singularities.

**Lemma 3.3.** *The space  $\widehat{X}$  is the union of an increasing sequence of open subsets  $X_n$ , such that, for each  $n \geq 1$ , the set  $X_n$  contains  $H$ , and the inclusion  $i_n: H \rightarrow X_n$  is an  $(r-1)$ -homotopy equivalence.*

*Proof.* Let  $S = \widetilde{E} \setminus \widehat{h}(C(\widehat{h}))$  be the set of regular values for  $\widehat{h}$ . By applying a suitable automorphism to  $\widetilde{E}$ , we may assume that  $b = 0$  belongs to  $S$ . For a fixed  $n$ , define

$$X_n = \widehat{h}^{-1}(D_n),$$

with  $D_n$  an open disc in  $\widetilde{E}$  centered at  $b$  and of radius  $r_n = n$  (when  $\widetilde{E} = \mathbb{C}$ ), or  $r_n = 1 - 1/n$  (when  $\widetilde{E} = \mathbb{D}$ ). Clearly,  $X_n$  contains  $H = \widehat{h}^{-1}(b)$ .

Consider a finite family of embeddings,  $\gamma_c: [0, 1] \rightarrow D_n$ , parametrized by the critical values  $c \in \widehat{h}(C(\widehat{h})) \cap D_n$ , such that

- (i)  $\gamma_c(0) = b$ ,  $\gamma_c(1) = c$ ,  $\gamma_c([0, 1]) \subset S$ ;
- (ii)  $\gamma_c([0, 1]) \cap \gamma_{c'}([0, 1]) = b$ , for  $c \neq c'$ ;
- (iii) for each  $c \in \widehat{h}(C(\widehat{h})) \cap D_n$ , one can find a small closed disc  $D_c \subset D_n$  centered at  $c$ , disjoint from the paths  $\gamma_{c'}([0, 1])$  for  $c \neq c'$ ;
- (iv)  $\gamma_c([0, 1]) \cap \partial D_c = \gamma_c(1 - \delta)$ , for the same  $\delta$ , with  $1 \gg \delta > 0$ .

Let  $K_n = \bigcup_c \gamma_c([0, 1]) \cup \bigcup_c D_c$ . Since  $\widehat{h}$  is a fibration over  $S$ , it follows that  $X_n$  has the same homotopy type as  $\widehat{h}^{-1}(K_n)$ . Similarly, let  $L_n = \bigcup_c \gamma_c([0, 1 - \delta])$ . Then  $L_n$  is a contractible space and hence  $\widehat{h}^{-1}(L_n)$  has the homotopy type of  $H = \widehat{h}^{-1}(b)$ . Note that  $\widehat{h}^{-1}(K_n)$  is obtained from  $\widehat{h}^{-1}(L_n)$  by replacing the fibers  $\widehat{h}^{-1}(\gamma_c(1 - \delta))$  by the corresponding tubes  $\widehat{h}^{-1}(D_c)$ . Since we are in a proper situation, with finitely many isolated singularities, each such tube has the homotopy type of the central singular fiber  $\widehat{h}^{-1}(c)$ , which in turn is obtained from the nearby smooth fiber  $\widehat{h}^{-1}(\gamma_c(1 - \delta))$  by attaching a finite number of  $r$ -dimensional cells. (These cells are the cones over the corresponding  $(r-1)$ -dimensional vanishing cycles; see for instance the very similar proof in [23, pp. 72–73]).

The above argument shows that each  $X_n$  has the homotopy type of a space obtained from the smooth fiber  $H$  by attaching finitely many  $r$ -cells. Therefore,  $\pi_i(X_n, H) = 0$ , for all  $i < r$ , and the conclusion follows.  $\square$

3.4. We are now ready to finish the proof of Theorem C.

Part (1). From Lemma 3.3, and the fact that homotopy groups commute with direct limits, it follows that  $\pi_i(\widehat{X}, H) = 0$ , for all  $i < r$ .

Part (2). Since, by assumption,  $r \geq 3$ , the exact homotopy sequence of the pair  $(\widehat{X}, H)$  shows that the inclusion  $\widehat{i}: H \rightarrow \widehat{X}$  induces an isomorphism on fundamental groups. Referring to diagram (4), it follows that  $\ker(h_{\#}) \cong \pi_1(\widehat{X}) \cong \pi_1(H)$ . Combining this isomorphism with (3) yields the desired conclusion.  $\square$

**Remark 3.5.** In the case when all the isolated singularities of  $h$  are non-degenerate, the above proof essentially goes back to Lefschetz [21], see Lamotke [20], Section 5 and Section 8, particularly claim (8.3.2) and its proof. The situation considered there corresponds to rational pencils, for which one may decompose the base of the pencil,  $E = \mathbb{P}^1$ , as the union of two discs glued along their common boundary. Hence, there is no need to pass to the universal cover  $\tilde{E}$ , as for irrational pencils, thus avoiding the difficulty of having to handle infinitely many critical values.

#### 4. BRANCHED COVERS AND ELLIPTIC PENCILS

In this section, we construct a family of irrational pencils that satisfy the hypothesis of Theorem C. To obtain these examples, we will replace the products of free groups on at least 2 generators appearing within the framework of Bestvina–Brady theory (as in §1.3) by products of fundamental groups of smooth projective curves of genus at least 2 (as in §2.11).

4.1. The starting point is a classical branched covering construction. Let  $E$  be an arbitrary complex elliptic curve. Let  $B \subset E$  be a finite subset, of cardinality  $|B| = 2g - 2$ , with  $g > 1$ . Then

$$(5) \quad H_1(E \setminus B, \mathbb{Z}) \cong H_1(E, \mathbb{Z}) \oplus H_1^B,$$

where the group  $H_1^B$  is generated by the homology classes  $\{\alpha_b\}_{b \in B}$  of elementary small positive loops around the points of  $B$ , subject to the single relation  $\sum_{b \in B} \alpha_b = 0$ .

Let  $\varphi \in \text{Hom}(\pi_1(E \setminus B), \mathbb{Z}/2\mathbb{Z})$  be any homomorphism with the property that, with respect to decomposition (5),

$$(6) \quad \varphi(\alpha_b) = 1, \quad \forall b \in B.$$

The next result is of a well-known type. We include a sketch of proof, for the reader's convenience.

**Proposition 4.2.** *For any choice of  $B \subset E$  and  $\varphi$  as above, there is a projective, smooth curve  $C$  of genus  $g$ , together with a ramified Galois  $\mathbb{Z}/2\mathbb{Z}$ -cover,  $f: C \rightarrow E$ . Furthermore, the map  $f$  induces a bijection between the ramification locus  $R \subset C$  and the branch locus  $B \subset E$ ; the restriction  $f: C \setminus R \rightarrow E \setminus B$  is the Galois cover corresponding to  $\varphi$ ; and  $f$  has ramification index 2 at each point of  $R$ .*

*Proof.* Set  $E^* = E \setminus B$ , and let  $f^*: C^* \rightarrow E^*$  be the Galois  $\mathbb{Z}/2\mathbb{Z}$ -cover associated to  $\ker(\varphi)$ . In view of a classical result of Stein [34], this cover extends uniquely to a ramified covering between the respective compactifications,  $f: C \rightarrow E$ .

By construction, the ramification locus  $R$  coincides with the critical set  $C(f)$ . The assertions on the restriction of  $f$  to  $R$ , and on ramification indices, are straightforward consequences of covering space theory. It follows that the ramification divisor of  $f$

is  $\sum_{c \in R} c$ , whence  $\text{genus}(C) = g$ , by the Riemann-Hurwitz formula; see for instance [26, p. 142].  $\square$

4.3. Let  $E$  be an elliptic curve, and fix an integer  $r > 1$ . For each index  $j$  from 1 to  $r$ , let  $B_j \subset E$  be a finite set with  $|B_j| = 2g_j - 2 > 0$ . Choose a homomorphism  $\varphi_j: H_1(E \setminus B_j) \rightarrow \mathbb{Z}/2\mathbb{Z}$  satisfying (6), and denote by  $f_j: C_j \rightarrow E$  the corresponding branched cover (with ramification locus  $R_j$  and branch locus  $B_j$ ), as constructed in Proposition 4.2. Note that  $\text{genus}(C_j) = g_j \geq 2$ .

Write  $X = C_1 \times \cdots \times C_r$  and  $E^{\times r} = E \times \cdots \times E$ , and consider the product mapping

$$f = f_1 \times \cdots \times f_r: X \rightarrow E^{\times r}.$$

Set  $X_1 = \prod_{j=1}^r (C_j \setminus R_j)$  and  $Y_1 = \prod_{j=1}^r (E \setminus B_j)$ . It follows that  $f$  restricts to a  $(\mathbb{Z}/2\mathbb{Z})^r$ -covering,  $f: X_1 \rightarrow Y_1$ , which is determined by the homomorphism

$$(7) \quad \varphi := \bigotimes_{j=1}^r \varphi_j: H_1(Y_1) \rightarrow (\mathbb{Z}/2\mathbb{Z})^r.$$

4.4. Let  $s_2: E^{\times 2} \rightarrow E$  be the group law of the elliptic curve, and extend it by associativity to a map  $s_r: E^{\times r} \rightarrow E$ . Using these maps, we may define a holomorphic map  $h$  as the composite

$$(8) \quad h = s_r \circ f: X \rightarrow E.$$

The next result shows that  $h$  is an *elliptic pencil*, that is, an irrational pencil over an elliptic curve.

**Lemma 4.5.** *The smooth fiber of  $h$  is connected.*

*Proof.* Let  $H_t = h^{-1}(t)$  be a smooth fiber of  $h$ . In order to show that  $H_t$  is connected, it is enough to check that  $H_1 := H_t \cap X_1$  is connected. This is due to the fact that no component of  $H_t$  is contained in a hypersurface of the form  $\{(x_1, \dots, x_r) \in X \mid x_j = c\}$ . Indeed, such an inclusion would force equality, whence  $\sum_{i \neq j} f_i(x_i) = t - f_j(c)$ , for all  $x_i \in C_i$  ( $i \neq j$ ). Clearly, this is impossible, since  $r \geq 2$ .

Set  $Z_t = s_r^{-1}(t)$ . Note that  $f: H_1 \rightarrow Z_t \cap Y_1$  is the pull-back of the covering  $f: X_1 \rightarrow Y_1$ , along the inclusion  $\iota: Z_t \cap Y_1 \hookrightarrow Y_1$ . Therefore,  $H_1$  is connected, provided the composition  $H_1(Z_t \cap Y_1) \xrightarrow{\iota_*} H_1(Y_1) \xrightarrow{\varphi} (\mathbb{Z}/2\mathbb{Z})^r$  is onto. To check this condition, it is enough to verify that each generator  $\varepsilon_j \in (\mathbb{Z}/2\mathbb{Z})^r$ , with 1 in the  $j$ -th position and 0 elsewhere, lies in the image of  $\varphi \circ \iota_*$ .

Pick a point  $b_j^0 \in B_j$ . We may then write the generic point  $t \in E$  in the form  $t = \sum_{i=1}^r t_i^0$ , with  $t_j^0 = b_j^0$  and  $t_i^0 \notin B_i$ , if  $i \neq j$ . Choose  $i \neq j$  and define  $\gamma = (\gamma_1, \dots, \gamma_r): S^1 \rightarrow E^r$  as follows:  $\gamma_j$  is an elementary small positive loop around  $b_j^0$ ;  $\gamma_i = t_i^0 + t_j^0 - \gamma_j$ ; and  $\gamma_k$  is the constant loop at  $t_k^0$ , for  $k \neq i, j$ . By our choices,  $[\gamma]$  will be an element of  $H_1(Z_t \cap Y_1)$ , if  $\gamma_j$  is small enough. Using (6) and (7), it is readily seen that  $\varphi \iota_*([\gamma]) = \varepsilon_j$ .  $\square$

**Lemma 4.6.** *The map  $h: X \rightarrow E$  has only isolated singularities; more precisely,  $C(h) = R_1 \times \cdots \times R_r$ . Moreover, for each  $p \in C(h)$ , the induced function germ,  $h: (X, p) \cong (\mathbb{C}^r, 0) \rightarrow (E, h(p)) \cong (\mathbb{C}, 0)$ , is a non-degenerate quadratic singularity, i.e., an  $A_1$ -singularity.*

*Proof.* Let  $p = (p_1, \dots, p_r) \in X = C_1 \times \cdots \times C_r$ . Then, clearly,  $\text{im } d_p f = V_1 \times \cdots \times V_r$ , where  $V_j = 0$  if  $p_j \in C(f_j)$  and  $V_j = \mathbb{C}$  otherwise. This implies the first claim.

The second claim follows from the fact that each function germ  $f_j: (C_j, p_j) \cong (\mathbb{C}, 0) \rightarrow (E, f_j(p_j)) \cong (\mathbb{C}, 0)$ , where  $p_j \in R_j$ , is given in suitable coordinates by  $x_j \mapsto x_j^2$ , while the germ  $s_r: (E^r, (f_1(p_1), \dots, f_r(p_r))) \cong (\mathbb{C}^r, 0) \rightarrow (E, h(p)) \cong (\mathbb{C}, 0)$  is given, again in suitable coordinates, by  $(y_1, \dots, y_r) \mapsto y_1 + \cdots + y_r$ .  $\square$

**Remark 4.7.** Denote by  $H_t$  the generic smooth fiber of  $h$ , and assume  $r \geq 3$ . Then  $\pi_1(H_t) \cong \ker(h_{\#})$ , as a consequence of Theorem 1.1 from Shimada [31], combined with our Lemmas 4.5 and 4.6. Of course, this also follows from Theorem C, Part (2).

It is worth pointing out that our approach provides finer information, at the level of cell structures. Indeed, let  $h: X \rightarrow E$  be an elliptic pencil with only non-degenerate singularities, for instance, one of the pencils constructed above. In this case, the function  $g: \widehat{X} \rightarrow \mathbb{R}$ , given by  $g(z) = |\hat{h}(z)|^2$ , has only Morse singularities of index  $r = \dim X$  on  $\widehat{X} \setminus H_0$ , as can be seen from the expansion

$$\left(1 + \sum_{j=1}^r (x_j + \sqrt{-1} y_j)^2\right) \cdot \left(1 + \sum_{j=1}^r (x_j - \sqrt{-1} y_j)^2\right) = 1 + 2 \sum_{j=1}^r (x_j^2 - y_j^2) + \cdots$$

Since  $g$  is proper and the closed tube  $T_0 = g^{-1}([0, \epsilon])$  is homotopy equivalent to the central fiber  $H_0 = g^{-1}(0)$  for  $\epsilon > 0$  small enough, it follows from standard Morse theory (see [25, I.3]) that  $\widehat{X}$  has the homotopy type of the smooth fiber  $H_0$ , with countably many  $r$ -cells attached. Clearly, these cells are indexed by  $C(\hat{h})$ , the set of critical points of  $\hat{h}$ .

## 5. PROJECTIVE GROUPS WITH EXOTIC FINITENESS PROPERTIES

In this section, we put things together, and finish the proof of Theorem A.

5.1. We start by proving the following theorem.

**Theorem 5.2.** *Let  $X$  be an irreducible, smooth projective variety of dimension  $r \geq 3$ . Assume that the universal cover of  $X$  is an  $(r-2)$ -connected Stein manifold, and that  $\mathcal{V}_1^r(\pi_1(X)) = \mathbb{T}_{\pi_1(X)}$ . Let  $h: X \rightarrow E$  be a holomorphic map to an elliptic curve  $E$ , with connected smooth fiber  $H$ , and with isolated singularities. Then:*

- (1) *The homotopy groups  $\pi_i(H)$  vanish for  $2 \leq i \leq r-2$ , while  $\pi_{r-1}(H) \neq 0$ .*
- (2) *The universal cover  $\widetilde{H}$  is a Stein manifold.*

- (3) *The group  $N = \pi_1(H)$  has a  $K(N, 1)$  with finite  $(r-1)$ -skeleton, but  $H_r(N, \mathbb{Z})$  is not finitely generated.*
- (4) *The group  $N$  is not commensurable (up to finite kernels) to any group  $N'$  having a  $K(N', 1)$  of finite type.*

*Proof.* As before, let  $p: \tilde{E} \rightarrow E$  be the universal cover, and let  $\hat{h}: \hat{X} \rightarrow \tilde{E}$  be the pull-back of  $h$  along  $p$ . The universal cover  $\tilde{X} \rightarrow X$  factors as  $\hat{p} \circ q$ , where  $\hat{p}: \hat{X} \rightarrow X$  is the pull-back of  $p$  along  $h$ , and  $q: \tilde{X} \rightarrow \hat{X}$  is the universal cover of  $\hat{X}$ .

Part (1). The vanishing property for the higher homotopy groups is a consequence of Theorem C, given our connectivity assumptions on the universal cover  $\tilde{X}$ . If  $\pi_{r-1}(H)$  would also vanish, we could construct a classifying space  $K(N, 1)$  by attaching to  $H$  cells of dimension  $r+1$  and higher. In particular,  $N$  would be of type  $\mathcal{F}_r$ , with finitely generated  $r$ -th homology group, contradicting property (3), which is proved below.

Part (2). Since the pair  $(\hat{X}, H)$  is  $(r-1)$ -connected and  $r \geq 3$ , the universal cover  $\tilde{H}$  coincides with  $q^{-1}(H)$ . This is a closed analytic submanifold of the Stein manifold  $\tilde{X}$ , hence Stein as well.

Part (3). The group  $N$  is of type  $\mathcal{F}_{r-1}$ , by the same argument as in the proof of Part (1). Assuming  $H_r(N, \mathbb{Z})$  to be finitely generated, we infer that  $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) < \infty$ .

On the other hand, we know from Theorem C that  $h_{\#}: \pi_1(X) \rightarrow \pi_1(E) = \mathbb{Z}^2$  is surjective, and  $N \cong \ker(h_{\#})$ . But this contradicts Theorem B, due to our hypothesis on  $\mathcal{V}_1^r(\pi_1(X))$ .

Part (4). Follows from Part (3) and Proposition 2.7. □

5.3. As a by-product of our Morse-theoretical approach, we can give a precise description of both the  $\mathbb{Z}N$ -structure of  $\pi_{r-1}(H)$  and the homotopy type of  $\tilde{H}$ , in (1) and (2) above. We keep the notation and hypothesis from Theorem 5.2.

**Corollary 5.4.** *Assume that the universal cover of  $X$  is  $r$ -connected, and the map  $h: X \rightarrow E$  has only non-degenerate singularities. Let  $C(h)$  be the set of critical points of  $h$ , and let  $H$  be the smooth fiber of  $h$ . Then:*

- (1) *The homotopy group  $\pi_{r-1}(H)$  is a free  $\mathbb{Z}\pi_1(H)$ -module, with generators in one-to-one correspondence with  $C(h) \times \pi_1(E)$ .*
- (2) *The universal cover  $\tilde{H}$  has the homotopy type of a wedge of  $(r-1)$ -spheres, indexed by  $\pi_1(H) \times C(h) \times \pi_1(E)$ .*

*Proof.* Part (1). The exact homotopy sequence of the pair  $(\hat{X}, H)$  identifies  $\pi_{r-1}(H)$  with  $\pi_r(\hat{X}, H)$ , as modules over  $\mathbb{Z}\pi_1(H)$ . Now recall from Remark 4.7 that  $\hat{X}$  has the homotopy type of  $H$ , with some  $r$ -cells attached. Moreover, these cells are indexed by  $C(\hat{h}) = \hat{p}^{-1}(C(h)) \cong C(h) \times \pi_1(E)$ . Since  $r \geq 3$ , the claim follows from [32, Exercise 7.F.3].

Part (2). We know from Theorem 5.2(2) that  $\tilde{H}$  is an  $(r-1)$ -dimensional Stein manifold. Therefore,  $\tilde{H}$  has the homotopy type of a CW-complex of dimension at most  $r-1$ . Let  $W$  be the bouquet of  $(r-1)$ -spheres indexed by  $\pi_1(H) \times C(h) \times \pi_1(E)$ , and let  $\psi: W \rightarrow \tilde{H}$  be the map whose restriction to each sphere represents the corresponding generator of the free abelian group  $\pi_{r-1}(\tilde{H})$ , computed in Part (1). By Theorem 5.2(1),  $\pi_i(\tilde{H}) = \pi_i(W) = 0$ , for  $1 \leq i \leq r-2$ . Moreover,  $\psi$  induces an isomorphism on  $\pi_{r-1}$ , by construction. The claim follows from the Hurewicz and Whitehead theorems.  $\square$

Similar highly connected Stein spaces having the homotopy type of bouquets of spheres occur in the study of local complements of isolated non-normal crossing singularities, see [22], [11].

5.5. We can now finish the proof of Theorem A.

Fix an integer  $r \geq 3$ , and let  $E$  be an elliptic curve. For each index  $j$  from 1 to  $r$ , construct a 2-fold branched cover  $f_j: C_j \rightarrow E$ , with  $C_j$  a curve of genus at least 2, as in §4.3. The product  $X = \prod_{j=1}^r C_j$  is a smooth, projective variety of dimension  $r$ , whose universal cover is a contractible, Stein manifold. By Corollary 2.14(1), the characteristic variety  $\mathcal{V}_1^r(\pi_1(X))$  coincides with the character torus  $\mathbb{T}_{\pi_1(X)}$ .

Now define a holomorphic map  $h: X \rightarrow E$  as in (8). By Lemmas 4.5 and 4.6, the smooth fiber of  $h$  is connected, and  $h$  has only isolated, non-degenerate singularities. Thus, the hypotheses of Theorem 5.2 hold. The conclusions of Theorem A follow at once from Theorem 5.2.  $\square$

For this class of examples, the conclusions of Corollary 5.4 are valid as well.

5.6. The Stein condition influences another finiteness property of projective groups, namely, their cohomological dimension.

**Proposition 5.7.** *Let  $M$  be a compact connected,  $m$ -dimensional complex analytic manifold, and let  $G = \pi_1(M)$ . If the universal cover  $\tilde{M}$  is Stein, then  $\text{cd}(G) \geq m$ .*

*Proof.* Let  $\kappa: M \rightarrow K(G, 1)$  be a classifying map, with homotopy fiber  $\tilde{M}$ . Let us examine the associated Serre spectral sequence,

$$E_{st}^2 = H_s(G, H_t(\tilde{M}, \mathbb{Z})) \Rightarrow H_{s+t}(M, \mathbb{Z}).$$

Since  $\tilde{M}$  is Stein, it has the homotopy type of a CW-complex of dimension at most  $m$ . Therefore,  $E_{st}^2 = 0$ , for  $t > m$ . Assuming  $\text{cd}(G) < m$ , we infer that  $E_{st}^2 = 0$ , for  $s \geq m$ . These two facts together imply that  $H_{2m}(M, \mathbb{Z}) = 0$ , a contradiction.  $\square$

A related statement holds for the smooth fiber  $H$  from Theorem 5.2:  $\text{cd}(\pi_1(H)) \geq \dim H + 1$ , as follows from Part (3).

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