SYSTOLIC FREEDOM OF LOOP SPACE

MIKHAIL G. KATZ[1] AND ALEXANDER I. SUCIU

ABSTRACT. Given a pair of integers m and n such that $1 < m < n$, we show that every n-dimensional manifold admits metrics of arbitrarily small total volume, and possessing the following property: every m-dimensional submanifold of less than unit m-volume is necessarily torsion in homology.

This result is different from the case of a pair of complementary dimensions, for which a direct geometric construction works and gives the analogous theorem in complete generality. In the present paper, we use Sullivan's telescope model for the rationalisation of a space to observe systolic freedom.

1. INTRODUCTION

Does the total volume of a Riemannian manifold impose a constraint upon the volume of its submanifolds? This question has interested differential geometers since the work of M. Berger (see [8] and [9, p. 192]) in the seventies, and of M. Gromov [13], [14] in the eighties. In this note we answer this question in the negative (for all but 1-dimensional submanifolds), as follows.

Theorem 1.1. *Let m and n be integers such that $2 \leq m < n$. Let X be an n-dimensional compact smooth manifold. Then X admits metrics of arbitrarily small total volume, such that every m-dimensional orientable submanifold of less than unit m-volume is null-homologous as a cycle with rational coefficients.*

In other words, the manifold X admits a sequence of metrics (X, g_j) with $vol(g_j) \to 0$ as $j \to \infty$, and such that for every orientable $M \subset X$ we have

$$vol_m(M) < 1 \implies [M] = 0 \text{ in } H_m(X, \mathbb{Q}),$$

where $vol_m(M)$ is the volume induced by g_j. Such a phenomenon is called *systolic freedom (modulo torsion)*. For $m = 1$ the theorem is false: the length of the shortest noncontractible loop is typically constrained by total volume (see section 2).

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Our theorem generalizes a previous result by I. Babenko and the authors [6], concerning systolic freedom in middle dimension (the case $n = 2m$). The next step toward the proof of Theorem 1.1 was accomplished in [21], where we reduced the problem to the examination of a finite list of CW complexes (the notion of systolic freedom can be extended in this setting). These complexes are the successive skeleta (up to the n^{th} skeleton) of the based loop space, $\Omega(S^{m+1})$, of the sphere S^{m+1}.

The main idea of the present paper is to use rational homotopy theory to establish the systolic freedom of the skeleta of the loop space.

The starting point of our proof is a map from the p-fold Cartesian product $(S^m)^{\times p}$, where $p = [\frac{n}{m}]$, to the n-skeleton of the Eilenberg-Maclane space $K(\mathbb{Z}, m)$ which induces an epimorphism in rational homology in all dimensions through n. We construct a CW complex W by attaching cells to $(S^m)^{\times p}$ so as to replace the epimorphisms by isomorphisms. By Sullivan [26], the localisation at 0 (rationalisation) W_0 of W may be thought of as the n-skeleton of $K(\mathbb{Q}, m)$. Now a map from a compact complex X has image in a finite subcomplex of $K(\mathbb{Q}, m)$ and hence in a finite piece of W_0. If W_0 admits a telescope model, we can conclude that the image of X may be deformed into a copy of W inside W_0. In general, the argument is more delicate; here we need to overcome two difficulties:

(a) lack of control of the higher homotopy groups of the skeleta of James's model of the spherical loop space (*cf.* Remark 4.3);
(b) the extra dimension of cells present in a 'rational cell' being attached in the process of rationalizing a complex (*cf.* Remark 5.2).

Finally, the pullback arguments of [2], [6], and [21] reduce the freedom of skeleta of the loop space to the freedom of $(S^m)^{\times p}$, known since [18].

Remark 1.2 (Pair of complementary dimensions)**.** The m-systole $sys_m(X)$ of X can be briefly defined as the least mass of a rectifiable m-current representing a nonzero class in integer homology (*cf.* section 3). Our main theorem states that X admits a sequence of metrics $\{g_j\}$ with the following asymptotic behavior as $j \to \infty$:

$$\frac{vol_n(g_j)}{sys_m(g_j)^{\frac{n}{m}}} \to 0,$$

at least if the m-dimensional homology of X is torsion free.

A related problem is that of systolic freedom in a pair of complementary dimensions m and $n - m$. Recall that X is called $(m, n - m)$-systolically free if there exist metrics g_j which behave as follows:

$$\frac{vol_n(g_j)}{sys_m(g_j) sys_{n-m}(g_j)} \to 0.$$

While our technique relies on classifying spaces and rational homotopy theory, the problem of a pair of complementary dimensions can be solved by a direct geometric construction, with the input from algebraic topology reduced to Poincaré duality. See [3, 4, 20].

The paper is organized as follows. Section 2 discusses systolic constraint and mentions a recent result of M. Freedman on freedom with $\mathbb{Z}_2$-coefficients. Section 3 defines systolic freedom and the concept of an 'm-meromorphic map'. Sections 4 and 5 contain the technical statement of Theorem 1.1 and its proof, and also describe the pertinent telescoping ideas from rational homotopy theory. Section 6 contains some speculations regarding an alternative proof using higher order Whitehead products. Section 7 outlines a program of study of systolic freedom with torsion coefficients, based on an analysis of 'torsion meromorphic maps'.

2. Systolic freedom versus constraint

Our theorem shows that the total volume imposes no upper bounds on the size of the least-area homologically nontrivial submanifold. In contrast, let us recall two results on the existence of such bounds for related invariants.

The first result is M. Gromov's inequality [13] for essential manifolds, which we will state in the particular case of real projective space $\mathbb{R}P^n$. Namely, all metrics g on $\mathbb{R}P^n$ obey the inequality

$$\min_{\gamma\sim\mathbb{R}P^1} g\text{-}length(\gamma) \le C_n \sqrt[n]{vol(g)}, \tag{1}$$

where the minimum in the lefthand side is over all closed curves in $\mathbb{R}P^n$ which are homotopic to $\mathbb{R}P^1 \subset \mathbb{R}P^n$. Here the constant C_n is independent of g, and the n-th root ensures scale invariance. Gromov's theorem generalizes classical results of Loewner and Pu in dimension 2 (see also [17]).

The second result is an inequality similar to the above, but with length replaced by area. Namely, all metrics g on complex projective space $\mathbb{C}P^n$ obey Gromov's inequality (see [14, p. 262], or [21, Theorem 2.6]):

$$\inf_{\gamma_{\mathbb{Q}}\sim\mathbb{C}P^1} g\text{-area}(\gamma_{\mathbb{Q}}) \le C_n \sqrt[n]{vol(g)}, \tag{2}$$

where the infimum is over all *rational* 2-cycles $\gamma_{\mathbb{Q}} = \sum_k r_k\sigma_k$ ($r_k \in \mathbb{Q}$) representing the generator $[\gamma_{\mathbb{Q}}] = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^n, \mathbb{Q}) = \mathbb{Q}$ (the sharp value of the constant is $\sqrt[n]{n!}$, as discussed in [22, section 6.3] and [11, 1.8.2]). Here by definition

$$g\text{-area}(\textstyle\sum_k r_k\sigma_k) = \sum_k |r_k| g\text{-area}(\sigma_k), \tag{3}$$

and the area of a piecewise smooth singular 2-simplex σ_k is the integral of the pullback of g to the standard 2-simplex. This inequality was generalized by J. Hebda [16].

Remark 2.1 (Torsion coefficients)**.** All of our results ultimately rely on a calibration technique using differential forms (*cf.* [21, Appendix A]). A new technique using instead a lower bound on the first eigenvalue λ_1 of the Laplacian, was recently invented by M.

Freedman [12], who proved that $S^1 \times S^2$ is $(1,2)$-systolically free even if one uses $\mathbb{Z}_2$-coefficients in homology when defining the systoles. This result implies in particular that the 4-manifold $S^2 \times S^2$ is free in middle dimension even with $\mathbb{Z}_2$-coefficients (*cf.* section 7).

On the other hand, the question whether inequality (2) remains valid for the complex projective plane if we work with coefficients in $\mathbb{Z}_2$ is still open:

$$\inf_{\gamma_{\mathbb{Z}_2} \sim \mathbb{C}P^1} g\text{-area}(\gamma_{\mathbb{Z}_2}) \overset{??}{\leq} C_n \sqrt[n]{vol(g)}, \tag{4}$$

where $\gamma_{\mathbb{Z}_2}$ is a 2-cycle with $\mathbb{Z}_2$-coefficients (*e.g.* a possibly nonorientable surface) representing the nonzero element in $H_2(\mathbb{C}P^n, \mathbb{Z}_2) = \mathbb{Z}_2$. See also section 7.

It is still unknown whether $\mathbb{R}P^3$ is (1,2)-systolically free modulo 2 (this question was originally posed [7, p. 622] in 1994), or in formulas:

$$\inf_{\gamma_{\mathbb{Z}_2},\, \sigma_{\mathbb{Z}_2}} g\text{-length}(\gamma_{\mathbb{Z}_2}) g\text{-area}(\sigma_{\mathbb{Z}_2}) \overset{??}{\leq} \text{Const}\, vol(g), \tag{5}$$

for all metrics $(\mathbb{R}P^3, g)$, where the infimum is over all curves $\gamma_{\mathbb{Z}_2} \sim \mathbb{R}P^1$ and surfaces $\sigma_{\mathbb{Z}_2} \sim \mathbb{R}P^2$.

3. PULLING BACK METRICS BY m-MEROMORPHIC MAPS

In [21], we reduced the proof of the m-systolic freedom modulo torsion (see definition below) of all smooth n-manifolds, to the examination of a finite list of objects: the successive skeleta (up to the n^{th} skeleton) of the based loop space $\Omega(S^{m+1})$. The price one has to pay is the enlargement of the category to that of CW complexes (see Definition 3.2).

Definition 3.1. Let (X, g) be a finite n-dimensional simplicial complex, endowed with a piecewise smooth metric g. Let $m \leq n$. Let $\alpha \in H_m(X, \mathbb{Z})$. Define

$$v(\alpha) = \inf_{M \in \alpha} vol_m(M),$$

where the infimum is taken over all piecewise smooth integer cycles M representing the class α. Here the volume of a (smooth) singular simplex is that of the pullback of the quadratic form g to the simplex, and we take absolute values of the coefficients to obtain the volume of the cycle, as in formula (3).

We define the *systole modulo torsion*, sys^∞, of (X, g) by setting

$$sys_m^\infty(g) = \inf_{\substack{\alpha \in H_m(X,\mathbb{Z}) \\ \alpha \neq \text{torsion}}} v(\alpha).$$

Definition 3.2. An n-dimensional CW complex X is *m-systolically free (modulo torsion)* if

$$\inf_g \frac{vol_n(g)^{\frac{m}{n}}}{sys_m^\infty(g)} = 0,$$

where the infimum is taken over all piecewise smooth metrics g on a finite simplicial complex X' homotopy equivalent to X.

Here the choice of X' is immaterial by virtue of the pullback Lemma 3.6 below, which applies in particular to homotopy equivalences. The idea of the proof of the systolic freedom of the skeleta of the loop space is a reduction to the case of the product of spheres, for which systolic freedom was established in [18] for $m \geq 3$, and in [21, Lemma 4.5 and Corolary 7.9] in the remaining case:

Proposition 3.3 ([18], [21])**.** *The Cartesian product $(S^m)^{\times p}$ of p copies of the m-sphere is m-systolically free, for all $m \geq 2$ and $p \geq 2$.*

We will carry out such a reduction by means of constructing a morphism from a skeleton of $\Omega(S^{m+1})$ to a product of spheres, and applying pullback arguments developed in [2], [6], and [21].

The morphisms from X to Y best suited to our problem are more flexible than continuous maps from X to Y. We still work with continuous maps, but we allow certain enlargements of the target Y of the map, which we will refer to as 'meromorphic extensions'.

Definition 3.4. Let Y be an n-dimensional CW complex. A CW complex W is called a *meromorphic extension* of Y if W has the homotopy type of a finite CW complex obtained from Y by attaching cells e^d of strictly smaller than the top dimension, or in formulas:

$$W \simeq Y \cup \bigcup_i e^{d_i}, \text{ where } d_i \leq n-1.$$

Definition 3.5. Let X^n and Y^n be finite CW complexes of dimension n. An *m-meromorphic map*, $X \dashrightarrow Y$, is a pair (W, f) where W is a meromorphic extension of Y and $f : X \to W$ is a continuous map which induces a monomorphism in m-dimensional rational homology:

$$f_* : H_m(X, \mathbb{Q}) \to H_m(W, \mathbb{Q}) \text{ is injective.}$$

Our terminology is inspired by the observation that the blow-up map $Y = \hat{X} \to X$ of, say, a complex analytic manifold X admits a kind of an inverse: $X \dashrightarrow Y$. Here the space W is obtained by coning off the exceptional divisor in $Y = \hat{X}$, while the inverse is perturbed in a neighborhood of the blown up point (see [21, Example 7.4] for details). Such maps are useful because of the following lemma on pulling back systolic freedom (*cf.* [21, Proposition 7.10]).

Lemma 3.6 (Pullback Lemma)**.** *Let $X \dashrightarrow Y$ be an m-meromorphic map. If Y is m-systolically free modulo torsion, then so is X.*

Example 3.7. Let us show how one can apply the pullback lemma to reduce the 2-systolic freedom of the complex projective plane $\mathbb{C}P^2$ to that of the product $S^2 \times S^2$.

Consider the decomposition $\mathbb{C}P^2 = S^2 \cup_h D^4$, where $h = \frac{1}{2}[e,e]$ is the Hopf map. Here e is the identity map of S^2, and $[e,e]$ is the Whitehead product. Also consider the CW complex $W = S^2 \times S^2 \cup_{e_1 - e_2} D^3$. Let $f : \mathbb{C}P^1 \to W$, $f(e) = de_1$ where $d \geq 2$ is an even integer. Then

$$\begin{aligned} f(h) &= f(\tfrac{1}{2}[e,e]) = \tfrac{1}{2}[f(e), f(e)] = \tfrac{1}{2}[de_1, de_1] \\ &= \tfrac{d^2}{2}[e_1, e_1] = \tfrac{d^2}{2}[e_1, e_2] = 0 \text{ in } \pi_3(W), \end{aligned}$$

since $e_1 = e_2$ in W, and so $[e_1, e_1] = [e_1, e_2]$. Thus f extends to a map $f : \mathbb{C}P^2 \to W$ which is injective in 2-dimensional homology, and hence may be viewed as a 2-meromorphic map $\mathbb{C}P^2 \dashrightarrow S^2 \times S^2$. Now Lemma 3.6 reduces the 2-systolic freedom of $\mathbb{C}P^2$ to that of $S^2 \times S^2$.

See also section 6 for a possible generalisation to $\mathbb{C}P^n$ using higher order Whitehead products.

The above argument is valid in view of the absence of torsion in $\pi_{2m-1}(S^m)$ for $m = 2$. For $m > 2$, we use the following argument to find an m-meromorphic map to $S^m \times S^m$ (see proof of Lemma 4.4 in [21]).

Example 3.8. Let $X = S^m \cup_a D^{2m}$ be a CW complex of dimension $2m$ with 3 cells of dimensions 0, m, and $2m$, respectively. Let $W = S^m \times S^m \cup_{e_1 - e_2} D^{m+1}$. Recall that the Whitehead product $[e,e] \in \pi_{2m-1}(S^m)$ generates precisely the kernel of the suspension homomorphism. Suspension commutes with the homomorphism induced by the degree q self-map $\phi_q : S^m \to S^m$. Hence, if q is a multiple of the order of the (finite!) stable group $\pi_{2m}(S^{m+1})$, then the map ϕ_q sends $\pi_{2m-1}(S^m)$ to the subgroup generated by Whitehead products. In particular, $\phi_q(a) = \lambda[e,e]$. We now compose ϕ_q with the inclusion $\iota : S^m \to W$ of S^m as the first factor of $S^m \times S^m \subset W$. It follows that

$$\iota \circ \phi_q(a) = \iota(\lambda[e,e]) = \lambda[e_1, e_1] = \lambda[e_1, e_2] = 0 \in \pi_{2m-1}(W).$$

Thus, the map $\iota \circ \phi_q$ extends to a map $X \to W$. This proves the m-systolic freedom of X.

4. PROOF OF SYSTOLIC FREEDOM OF LOOP SPACE

The results of this section generalize the middle-dimensional systolic freedom, established in [5], [6], and [21, Theorem 1.5]. Let $n = mp$, where $m \geq 2$ and $p \geq 2$. Let $(S^m)^{\times p}$ denote the p-fold Cartesian product of m-spheres.

Theorem 4.1. *Let X be a finite n-dimensional CW complex with unit m-th Betti number: $b_m(X) = 1$. Then X admits an m-meromorphic map to $(S^m)^{\times p}$, and hence X is m-systolically free modulo torsion.*

Proof. Let $n = mp$, where m is even (the case of m odd is easier; see [21, section 10]). Let $S = (S^m)^{\times p}$. Let ϕ_j be the selfmap of S defined by a degree j map on each factor. The map ϕ_j induces a *scalar* homomorphism (namely, multiplication by a power of j) in each cohomology group of S. Namely, $\phi_j^* = \wedge^\ell(j\cdot \mathrm{Id}) = j^\ell : H^{\ell m}(S) \to H^{\ell m}(S)$. Let $w = x_1 + \cdots + x_p$ be the sum of the standard generators x_i of the cohomology group $H^m(S) = \mathbb{Z}^p$. Then

$$w^p = p!\, x_1 \cup \cdots \cup x_p \neq 0 \text{ in } H^n(S).$$

Now the class w defines a map $f : S \to K(\mathbb{Z}, m)$ such that $f^*(u) = w$, where $u \in H^m(K(\mathbb{Z}, m)) = \mathbb{Z}$ is a generator. Recall that u is a multiplicative generator of the polynomial ring $H^*(K(\mathbb{Z}, m))\otimes\mathbb{Q} \cong \mathbb{Q}[u]$ (m even). Our proof will rely on the following lemma.

Lemma 4.2. *The map $f : (S^m)^{\times p} \to K(\mathbb{Z}, m)$ extends to a map $f : W \to K(\mathbb{Z}, m)$ which induces an isomorphism in rational homology through dimension n, where W is a finite CW complex obtained from S by attaching cells of dimension at most $n-m+1 < n$. Furthermore, the maps ϕ_j^2 extend to W for j divisible by a sufficiently large j_0, in the following two situations:*

(a) *if $p = 2$ or 3;*
(b) *if $m = 2$.*

Proof. Since $w^p \neq 0$, the map $f : (S^m)^{\times p} \to K(\mathbb{Z}, m)$ induces an epimorphism

$$H_k(S, \mathbb{Q}) \to H_k(K(\mathbb{Z}, m), \mathbb{Q}) \tag{6}$$

in each dimension less than n, and an isomorphism in dimension n. We will use the relative Hurewicz theorem to eliminate the successive kernels of the homomorphisms (6), and construct W by skeleta $W^{(k)}$ inductively on k (*cf.* [26, p. 27]).

Let $S = (S^m)^{\times p}$ and let $W^{(m)} = S^{(m)} = S^m \vee \cdots \vee S^m$.

Next, let $W^{(m+1)} = S^{(m)} \cup_{e_1-e_2} D^{m+1} \cup_{e_2-e_3} \cdots \cup_{e_{p-1}-e_p} D^{m+1}$. Then $W^{(m+1)} \simeq S^m$.

Since the map ϕ_j acts by $j\cdot\mathrm{Id}$ on $\pi_m(W^{(m)}) = H_m(S)$, it extends to $W^{(m+1)}$.

We set $W^{(2m)} = W^{(m+1)} \cup S^{(2m)}$ and note that, by patching the two pieces, ϕ_j extends to $W^{(2m)}$. Note that $W^{(2m)} \simeq S^m \cup_i D_i^{2m}$, where $i = 1, \ldots, \binom{p}{2}$. We have $W^{(2m)}/W^{(m+1)} \simeq \vee_i S^{2m}$, $i = 1, \ldots, \binom{p}{2}$. Consider the exact sequence

$$\pi_{2m}(W^{(m+1)}) \to \pi_{2m}(W^{(2m)}) \to \pi_{2m}(W^{(2m)}, W^{(m+1)}). \tag{7}$$

By Hurewicz's theorem, the map ϕ_j induces multiplication by $j^2\cdot\mathrm{Id}$ on the group

$$\pi_{2m}(W^{(2m)}, W^{(m+1)}) = H_{2m}(W^{(2m)}, W^{(m+1)}) = H_{2m}(\vee_i S^{2m}) = \mathbb{Z}^{\frac{p}{2}}. \tag{8}$$

Now consider the diagram

$$\begin{tikzcd}
H_{2m+1}(K(\mathbb{Z},m),W^{(2m)}) \arrow[r, "\partial"] & H_{2m}(W^{(2m)}) \arrow[r, "f_*"] & H_{2m}(K(\mathbb{Z},m)) \\
\pi_{2m+1}(K(\mathbb{Z},m),W^{(2m)}) \arrow[u, "h_1"'] \arrow[r, "\partial"] & \pi_{2m}(W^{(2m)}), \arrow[u, "h_2"'] &
\end{tikzcd}$$

where h_i are the Hurewicz maps. By the relative Hurewicz theorem, the homomorphism h_1 is surjective. Thus $\ker(f_*) = \operatorname{im}(\partial \circ h_1) = \operatorname{im}(h_2 \circ \partial)$. Hence, every element in $\ker(f_*)$ is spherical.

Let $\alpha_i \in \pi_{2m}(W^{(2m)})$, $i = 1, \dots \frac{p}{2} - 1$, be a set of lifts via the Hurewicz homomorphism h_2, of a set of generators of the kernel of f. From the exact sequence (7), we have $\phi_j(\alpha_i) = j^2\alpha_i + t$, where $t \in \pi_{2m}(S^m)$ is of finite order. By Sullivan's result [26, p. 19] on the nilpotence of the finite homotopy groups of spheres, there exists a j_0 such that if $j_0 | j$ then $\phi_j = 0$ on $\pi_{2m}(S^m)$. Assume also that j_0 is a multiple of the order of $\pi_{2m}(S^m)$. Then

$$\phi_j^2(\alpha_i) = \phi_j(j^2\alpha_i + t) = j^2(\phi_j(\alpha_i)) + \phi_j(t) = j^4\alpha_i + j^2 t = j^4\alpha_i,$$

i.e. $\phi_j^2(\alpha_i)$ is proportional to α_i.

It follows that ϕ_j^2 extends to the next skeleton

$$W^{(2m+1)} = W^{(2m)} \cup_{\alpha_i} D_i^{2m+1}$$

and therefore also to $W^{(3m)} = W^{(2m+1)} \cup S^{(3m)}$, proving part (a) of the lemma.

We can similarly define $W^{(\ell m)} = W^{((\ell-1)m+1)} \cup S^{(\ell m)}$, and use the Serre form of the relative Hurewicz theorem (see [23, pp. 95–98]) to define

$$W^{(\ell m+1)} = W^{(\ell m)} \cup_{\alpha_i} D_i^{\ell m+1} \text{ for } i = 1, \dots, \tbinom{p}{\ell} - 1.$$

Proceeding by induction, we extend f to the space $W = W^{((p-1)m+1)} \cup S$, which is the desired meromorphic extension of S (*cf.* Definition 3.4).

Remark 4.3. The above argument does not yield an extension of ϕ_j^2 to $W^{(3m+1)}$ (when $p \geq 4$) because already the group $\pi_{3m}(W^{(2m+1)}, W^{(m+1)})$ cannot be shown to be finite merely from Hurewicz's theorem, as in the calculation (8) above. An alternative approach given below gives an easy proof only for $m = 2$ (*cf.* Remark 4.4).

For $m = 2$, we can understand the effect of ϕ_j on $\pi_{2\ell}(W^{(2\ell)})$ easily using the Hopf fibration over $\mathbb{C}P^n$. We argue inductively. To the extent that ϕ_j already acts on $W^{(2\ell)}$, the family (ϕ_j) localizes homology of the space $W^{(2l)}$. Therefore by [26, p. 19], the family also localizes homotopy, so that we have the same corollary as for the spheres: the map on d torsion of $\pi_i(W^{(2l)})$ induced by ϕ_d is nilpotent. To show that the action of ϕ_j is scalar on $\pi_{2\ell}(W^{(2\ell)})$, it therefore suffices to show that this group has rank precisely

$$\operatorname{rank}(H_{2\ell}(W^{(2\ell)})) - 1 = \tbinom{p}{\ell} - 1.$$

Consider the exact sequence of pair $(W^{(2\ell)}, W^{(2l-1)})$:

$$\pi_{2\ell}(W^{(2\ell-1)}) \to \pi_{2\ell}(W^{(2\ell)}) \to \pi_{2\ell}(W^{(2\ell)}, W^{(2\ell-1)}) = H_{2\ell}(W^{(2\ell)}, W^{(2\ell-1)}) = \mathbb{Z}^{\binom{p}{\ell}}$$

by Hurewicz. We need to show that the group $\pi_{2\ell}(W^{(2\ell-1)})$ is finite. Now the map $W^{(2\ell-1)} \to \mathbb{C}P^{\ell-1}$, where $\mathbb{C}P^{\ell-1} \subset \mathbb{C}P^{\infty} = K(\mathbb{Z}, 2)$, has already been constructed. This map induces an isomorphism in rational homology in all dimensions. By the Serre-Hurewicz theorem, the group $\pi_i(\mathbb{C}P^{\ell-1}, W^{(2\ell-1)})$ is finite for all i. The exact sequence of the pair shows that the groups $\pi_i(W^{(2\ell-1)})$ are all finite except π_2 and $\pi_{2\ell-1}$, since $\pi_{2\ell-1}(\mathbb{C}P^{\ell-1}) = \pi_{2\ell-1}(S^{2\ell-1}) = \mathbb{Z}$ from the exact sequence of the Hopf fibration, proving part (b) of the Lemma. □

Remark 4.4. To generalize this proof to $m \geq 3$, one could replace the spaces $\mathbb{C}P^n$ by the skeleta of $\Omega(S^{m+1})$ (*cf.* formula (10)). However, the calculation of the homotopy groups of such skeleta is not as immediate as that of complex projective spaces.

5. Conclusion of proof

Lemma 4.2 provides a quick proof of Theorem 4.1 in the cases (a) and (b), as follows. We appeal to rational homotopy theory, to conclude that the rationalisation W_0 of W can be thought of as the n-skeleton of $K(\mathbb{Q}, m) = K(\mathbb{Z}, m)_0$.

Due to the existence of the selfmaps ϕ_j^2 of W, according to D. Sullivan [26], the space W_0 admits a model as an infinite telescope on W, whose j-th stage is the cylinder of the self-map ϕ_j^2 of W, where ϕ_j is induced by a degree j map on each of the factors in $S^m \times \cdots \times S^m = S$.

Now let X be a finite n-dimensional CW complex with $b_m(X) = 1$. Consider a map from X to $K(\mathbb{Q}, m)$ defined by any non-torsion m-dimensional cohomology class. The map may be assumed to have image in W_0, viewed as the skeleton of the classifying space. Being compact, the image of X lies in a finite piece of the telescope structure of W_0. Hence it can be retracted to the final stage, W, of the finite piece. Now W is just the space $(S^m)^{\times p}$, with some cells of dimension at most $n - m + 1 < n$ attached. Hence W is a meromorphic extension of S. We thus obtain an m-meromorphic map $X \dashrightarrow (S^m)^{\times p}$. The pullback lemma 3.6 completes the proof in the 2 cases mentioned in Lemma 4.2.

In the general case $p \geq 4, m \geq 3$, we need a more delicate argument. We rely on the following property of maps of compact spaces into rationalisations.

Lemma 5.1. *Let S be a CW complex admitting a telescope model (i.e. there are self-maps which localize homology). Assume that a CW complex W is obtained from S by attaching cells of dimension at most k. Then the image of a compact space mapping into the rationalisation W_0 of W may always be deformed into a subcomplex of W_0 which is homotopy equivalent to a copy of S with cells of dimension at most $k+1$ attached.*

Proof. Recall that S_0 is obtained as the direct limit in the following construction:

$$S_0 = \coprod_j S \times I \ / \ (x,1) \sim (\phi_j \circ id_j(x), 0). \tag{9}$$

Here $id_j : S \to S$ is the identification of the j-th and $(j+1)$-th levels.

A local CW complex is built inductively by attaching cones over the local sphere using maps of the local spheres into the lower 'local skeletons'. For each cell of dimension $\leq k$ attached to S, we attach a corresponding local cell to S_0, which contains cells σ^{d_i} of dimension $d_i \leq k+1$. Here the extra dimension is due to the presence of cylinders in formula (9). A map from a compact space X into W_0 has image in a finite subcomplex $W_0' \subset W_0$, which may be assumed to be of the following form: take a finite piece

$$S_0' = \coprod_j^N S \times I \ / \ (x,1) \sim (\phi_j \circ id_j(x), 0)$$

of the infinite telescope S_0, and attach, to S_0', at most finitely many cells from among the σ^{d_i}. Namely,

$$W_0' = S_0' \cup_i \sigma^{d_i}.$$

From the homotopy equivalence $S_0' \simeq S$, we obtain the equivalence $W_0' \simeq S \cup_i \sigma^{d_i}$, since each attaching map is an element of a homotopy group and hence a homotopy invariant, while homotopy groups commute with direct limits by a simple compactness argument. □

To complete the proof of Theorem 4.1, we argue as follows. Let W_0 be the rationalisation of the space W of the lemma. By the universal property of a localisation [26, p. 18], the map from W to the local space $K(\mathbb{Q}, m)$ induced by f extends to a map $W_0 \to K(\mathbb{Q}, m)$. Thus W_0 may be thought of as the n-skeleton of $K(\mathbb{Q}, m)$. Recall that W is obtained from S by adding cells of dimension at most $n - m + 1$.

If X is compact, by Lemma 5.1 the image of the map can be deformed into a space W_0' obtained from S by attaching cells of dimension at most $n - m + 2$.

Remark 5.2. The space W_0' is thus a meromorphic extension (*cf.* Definition 3.4) of S only if $m \geq 3$. The case $m = 2$ is handled as above (following Remark 4.3), using the Hopf fibration over $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

Since we rely on an existence theorem for rationalisations, what we lose control of in this version of the proof is the exact form of the meromorphic extension of S, which admits a continuous map from X. □

Corollary 5.3. *Every finite n-dimensional CW complex is m-systolically free modulo torsion, provided* $2 \leq m < n$.

Proof. Recall that Theorem 1.6 of [21] reduces the general case to that of the successive skeleta of the loop space $\Omega = \Omega(S^{m+1})$, arising from the cell decomposition of I. James,

$$\Omega \simeq S^m \cup e^{2m} \cup e^{3m} \cup \cdots, \tag{10}$$

with precisely one cell in each dimension divisible by m (see [27]). We now apply Theorem 4.1 to the n-skeleton of Ω, to prove its m-systolic freedom modulo torsion, completing the proof of the Corollary.

Our proof of Theorem 1.6 of [21] can be simplified by using the telescope model for the localisation $\Omega_0 = K(\mathbb{Q}, m)$ of the loop space $\Omega = \Omega(S^{m+1})$. Such a model exists by [26], since Ω admits self-maps defined by mapping a loop to its j-th iterate, which induce multiplication by j in all homology groups.

Namely, let X be a finite n-dimensional CW complex, and let $b = b_m(X)$ be its m-th Betti number. A choice of a basis for a maximal lattice in $H^m(X)$ defines a map $X \to K(\mathbb{Z}^b, m)$. The composition of this map with the canonical map $K(\mathbb{Z}^b, m) \to K(\mathbb{Z}^b, m)_0 = \Omega_0^{\times b}$, induces an isomorphism $H_m(X, \mathbb{Q}) \to H_m(\Omega_0^{\times b}, \mathbb{Q})$. Now we argue as above. We use the compactness of X to construct a projection to the final level of a finite piece of the telescope.

By the pullback lemma 3.6, it remains to prove the m-systolic freedom of the n-skeleton of $\Omega^{\times b}$. Next, we apply the carving up technique of [21], to reduce the problem to the m-systolic freedom of the closures of the top-dimensional cells in the n-skeleton $(\Omega^{\times b})^{(n)}$ of $\Omega^{\times b}$.

The cell decomposition of $(\Omega^{\times b})^{(n)}$ induced by the James decomposition (10) contains only cells of dimensions divisible by m. In particular, there are no cells of codimension 1.

The absence of codimension 1 cells in $(\Omega^{\times b})^{(n)}$ is the crucial ingredient of the carving up technique. It allows us to isolate the different cell closures in $(\Omega^{\times b})^{(n)}$ from each other. This is accomplished by inserting long cylinders based on the *boundary*, in $(\Omega^{\times b})^{(n-m)}$, of each top-dimensional cell. The absence of cells of codimension 1 guarantees that these cylinders have positive codimension, and thus make no contribution to total n-dimensional volume.

The effect of the long cylinders is to minimize the interaction between distinct top-dimensional cells, so that an m-cycle traveling from one to the other would have to pay a heavy price in terms of its m-volume. The formal argument, using coarea and isoperimetric inequalities, appears in [21, Appendix B].

Each of the top-dimensional cells in $(\Omega^{\times b})^{(n)}$ is a product of skeleta $\Omega^{(mp_i)}$ of Ω. Each $\Omega^{(mp_i)}$ admits a meromorphic map to $(S^m)^{\times p_i}$ by Theorem 4.1. Thus there exists a meromorphic map

$$\prod_i \Omega^{(mp_i)} \dashrightarrow \prod_i (S^m)^{\times p_i} = (S^m)^{\times(\Sigma_i p_i)} = (S^m)^{\times p}.$$

The m-systolic freedom of $\prod_i \Omega^{(mp_i)}$ now follows from the systolic freedom of products of spheres (*cf.* Proposition 3.3). □

6. Higher order Whitehead products

It is tempting to try to generalize the argument of Example 3.7 using higher order Whitehead products, so as to derive the systolic freedom of $\mathbb{C}P^n$, or of the appropriate skeleton of the loop space, from the systolic freedom of the Cartesian product $(S^2)^{\times n}$.

We form the analogue of the space W of Example 3.7 by defining W_n to be the CW complex obtained from $(S^2)^{\times n}$ by attaching $n-1$ copies of a 3-cell along differences of consecutive 2-spheres in the product $(S^2)^{\times n}$:

$$W_n = (S^2)^{\times n} \cup_{e_1-e_2} D^3 \cup_{e_2-e_3} \cdots \cup_{e_{n-1}-e_n} D^3.$$

The top-dimensional cell of $(S^2)^{\times n}$ is attached by means of the n^{th} order Whitehead product $a_n = [e_1, \dots, e_n]$ of the 2-dimensional generators. Thus

$$W_n = W_n^{(2n-2)} \cup_{a_n} D^{2n}$$

where $W^{(i)}$ denotes the i-skeleton. Furthermore, $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_h D^{2n}$ where

$$n!\, h = [e, e, \dots, e] \tag{11}$$

(*cf.* [24, p. 416], [1, p. 460], [15, p. 39]). Now let $f = \phi_d$ be a self-map of S^2 of degree d. By virtue of formula (11), we can write

$$f(n!\, h) = f([e, e, \dots, e]) \in [f(e), f(e), \dots, f(e)] = [de_1, de_1, \dots, de_1].$$

Since $e_i = e_j$ in $W_n^{(2n-2)}$ for all i and j, we have

$$f(n!\, h) \in [de_1, de_2, \dots, de_n] = d^n [e_1, e_2, \dots, e_n].$$

Since $f(n!\, h) = n!\, f(h)$, we obtain

$$f(h) \in \frac{d^n}{n!}[e_1, e_2, \dots, e_n] + t \subset \pi_{2n-1}(W_n^{(2n-2)})$$

assuming the fraction is an integer (*e.g.* if $n!$ divides d), where $t \in \pi_{2n-1}(W_n^{(2n-2)})$ is a torsion element of order dividing $(n!)$.

Now the problem of the torsion element t can be handled by replacing $\mathbb{C}P^n$ by the $2n$-skeleton of a model of $\Omega(S^3)$ (*cf.* (10)). Then the attaching map h' is a higher order Whitehead product and we may set $d = 1$, to obtain

$$h' \in [e_1, \dots, e_n] \subset \pi_{2n-1}(W_n^{(2n-2)}).$$

Now the set $[e_1, \dots, e_n] \subset \pi_{2n-1}(W_n)$ clearly contains zero, corresponding to the attaching map a_n of the top-dimensional cell in the product of spheres. However, this class may also contain other elements, due to ambiguity in the choice of the extension to the fat wedge. For instance, the 4-skeleton of the space $W_3 = S^2 \times S^2 \times S^2 \cup_{e_1-e_2} D^3 \cup_{e_2-e_3} D^3$ is homotopy equivalent to the 2-sphere with three distinct 4-cells attached along the same class $[e, e]$. There are thus 9 potentially nonhomotopic possibilities for extending the inclusion $S^2 \vee S^2 \vee S^2 \to W_3$ to the fat wedge $W_3^{(4)} \to W_3$.

Thus, the indeterminacy of the higher order Whitehead product is an obstruction to the attempted generalisation of the proof of Example 3.7. On the other hand, attaching higher-dimensional cells to reduce the rank of all homology groups to at most 1 may resolve the indeterminacy and yield an alternative proof of the main theorem.

7. Torsion coefficients for meromorphic maps

There now exists a first example of systolic freedom even with $\mathbb{Z}_2$ coefficients, due to M. Freedman [12] (*cf.* Remark 2.1). One could therefore propose to study systolic freedom over $\mathbb{Z}_2$ of manifolds. Essentially this would amount to proving the statement of Theorem 1.1, with the word 'orientable' deleted, as in the following question.

Question 7.1. Let m and n be integers such that $2 \leq m < n$. Let X be an n-dimensional compact smooth manifold. Does X admit metrics of arbitrarily small total volume, such that every m-dimensional (possibly nonorientable) submanifold of less than unit m-volume is null-homologous as a cycle with $\mathbb{Z}_2$ coefficients?

We could attack this question by studying 'torsion meromorphic maps', where we replace $\mathbb{Q}$ by $\mathbb{Z}_2$ in Definition 3.5. We could then study the partial order on the set of all manifolds of a given dimension, defined by the existence of a torsion meromorphic map between them.

This approach immediately yields the 2-systolic freedom over $\mathbb{Z}_2$ of $S^2 \times S^2$, since it admits a degree 1 map to $S^2 \times S^1 \times S^1$ with a pair of 2-cells attached (*cf.* [21, Corollary 7.9]), while the freedom of $S^2 \times S^1 \times S^1$ follows from Freedman's example $(S^2 \times S^1, g_F)$ by forming Cartesian product with a circle of length $\frac{sys_2^{\mathbb{Z}_2}(g_F)}{sys_1^{\mathbb{Z}_2}(g_F)}$.

Meanwhile, a simple analysis of the multiplicative structure of the pertinent cohomology rings suggests that the manifold $\mathbb{C}P^2$ ought not to admit a torsion meromorphic map to $S^2 \times S^2$. It is thus the simplest manifold for which Question 7.1 is open.

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Department of Mathematics and Computer Science, Bar Ilan University, Ramat Gan, 52900, Israel. Département de Mathématiques, Université Henri Poincaré, Vandoeuvre 54506, France
E-mail address: `katzmik@macs.biu.ac.il`
URL: `http://www.cs.biu.ac.il/~katzmik`

Department of Mathematics, Northeastern University, Boston, MA 02115, USA
E-mail address: `alexsuciu@neu.edu`
URL: `http://www.math.neu.edu/~suciu`