ON MILNOR FIBRATIONS OF ARRANGEMENTS

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Abstract. We use covering space theory and homology with local coefficients to study the Milnor fiber of a homogeneous polynomial. These techniques are applied in the context of hyperplane arrangements, yielding an explicit algorithm for computing the Betti numbers of the Milnor fiber of an arbitrary real central arrangement in $\mathbb{C}^3$, as well as the dimensions of the eigenspaces of the algebraic monodromy. We also obtain combinatorial formulas for these invariants of the Milnor fiber of a generic arrangement of arbitrary dimension using these methods.

0. Introduction

For any homogeneous polynomial $f = f(z_0, \dots, z_d)$, there is a global Milnor fibration [16] $F \longrightarrow M \xrightarrow{f} \mathbb{C}^*$, with total space $M = \mathbb{C}^{d+1} \setminus f^{-1}(0)$ and fiber $F = f^{-1}(1)$. The geometric monodromy, $h : F \to F$, of this fibration is given by $h(z_0, \dots, z_d) = (\xi z_0, \dots, \xi z_d)$, where n is the degree of f and $\xi = \exp(2\pi i/n)$. Since h has finite order n, the algebraic monodromy $h_* : H_q(F; \mathbb{C}) \to H_q(F; \mathbb{C})$ is diagonalizable, and the eigenvalues of h_* belong to the set of n^{th} roots of unity.

If f has an isolated singularity at the origin, then the homology of the Milnor fiber of f, and the eigenspaces of the algebraic monodromy are known ([16], [17]). However, if the singularity of f is not isolated, substantially less is known about these topological invariants of F. This problem has been the subject of a great deal of recent activity (see e.g. [1], [5], [7], [15]). In this note, we use covering space theory and homology with local coefficients to study the homology of F.

These techniques are particularly useful in the case where the polynomial f factors into n distinct linear forms. Such a polynomial defines a central arrangement of hyperplanes in $\mathbb{C}^{d+1}$. For any central arrangement in $\mathbb{C}^3$ that is defined by real forms, we obtain an explicit algorithm for computing the Betti numbers of the associated Milnor fiber, as well as the dimensions of the eigenspaces of the algebraic monodromy (Theorem 4.5). We also obtain combinatorial formulas for these invariants of the Milnor fiber of a generic arrangement of arbitrary dimension by these methods (Theorem 3.2), recovering a result due to Orlik and Randell [19].

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The structure of this note is as follows:

In section 1, we consider the Milnor fiber F of an arbitrary homogeneous polynomial f. The geometric monodromy $h : F \to F$ generates a cyclic group $\mathbb{Z}_n$ which acts freely on F. This free action gives rise to an n-fold covering $F \to F/\mathbb{Z}_n$. In the case of a square-free polynomial (so that the hypersurface $f^{-1}(0)$ is reduced), we determine the characteristic homomorphism Φ of this covering in Proposition 1.2. Using the Leray-Serre spectral sequence, we then identify $H_*(F;\mathbb{C})$, the homology of F with constant coefficients, with $H_*(F/\mathbb{Z}_n;\mathcal{V})$, the homology of $F/\mathbb{Z}_n$ with coefficients in the local system determined by Φ. Furthermore, we show that $\mathcal{V}$ decomposes into n rank one local systems $\mathcal{V}_k$, and, under the above identification, the ξ^k-eigenspace of the algebraic monodromy corresponds to $H_*(F/\mathbb{Z}_n;\mathcal{V}_k)$.

The remainder of the paper is devoted to arrangements of hyperplanes. We consider central arrangements in section 2 and generic arrangements in section 3. The algorithm for real arrangements in $\mathbb{C}^3$ is presented in section 4. In this case, M^* is homotopy equivalent to a 2-complex that, by results of Randell [22] and Falk [8], admits a combinatorial description. Applying the Fox calculus to this 2-complex, we obtain the aforementioned algorithm for computing the Betti numbers of the Milnor fiber, as well as the dimensions of the eigenspaces of the algebraic monodromy, of an arbitrary complexified real central arrangement in $\mathbb{C}^3$. The reader is referred to [1] and [7] for alternate means of computing the Betti numbers of the Milnor fiber in this situation, and is referred to [11] and [14] for a discussion of the eigenspaces of the algebraic monodromy in the context of Alexander polynomials.

Several detailed examples illustrating the algorithm are given in section 5. Here is a brief summary of the computations for the (complexified) Coxeter 3-arrangements:

	$b_0(F)$	$b_1(F)$	$b_2(F)$
A_3	1	7	18
B_3	1	8	79
H_3	1	14	493

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1. The Milnor Fiber of a Homogenous Polynomial

Let $f : \mathbb{C}^{d+1} \to \mathbb{C}$ be a homogeneous polynomial of degree $n \geq 2$. Since f is homogeneous, the associated *Milnor fibration* [16]

$$f : \mathbb{C}^{d+1} \setminus f^{-1}(0) \longrightarrow \mathbb{C}^*$$

is global, with total space $M = M(f) := \mathbb{C}^{d+1} \setminus f^{-1}(0)$, the complement of the zero set of f, and fiber $F := f^{-1}(1)$, the *Milnor fiber* of f. A characteristic homeomorphism (or geometric monodromy) for this fibration is the map $h : F \to F$ given by $h(z_0, \ldots, z_d) =$

$(\xi z_0, \dots, \xi z_d)$, where $\xi = \exp(2\pi i/n)$. The map h generates a cyclic group $\mathbb{Z}_n$ which acts freely on F. This free action gives rise to a regular n-fold covering $F \to F/\mathbb{Z}_n$.

Since the geometric monodromy has finite order n, the algebraic monodromy $h_* : H_q(F;\mathbb{C}) \to H_q(F;\mathbb{C})$ also has finite order, and is therefore diagonalizable, with eigenvalues among the n^{th} roots of unity. We denote the homology eigenspace of ξ^k by $H_q(F;\mathbb{C})_k$, and write $b_q(F)_k = \dim\bigl(H_q(F;\mathbb{C})_k\bigr)$.

Proposition 1.1. *Let ξ^k and ξ^l be two n^{th} roots of unity which generate the same cyclic subgroup of $\mathbb{Z}_n = \langle \xi \rangle$. Then, for each q, the homology eigenspaces $H_q(F;\mathbb{C})_k$ and $H_q(F;\mathbb{C})_l$ are isomorphic.*

Proof. Fix a positive integer m with $1 \le m < n$, so that $(m,n) = 1$ and $\xi^l = (\xi^k)^m$. Then $h^m : F \to F$ is another characteristic homeomorphism for the Milnor fibration. Hence $L := h_* : H_q(F;\mathbb{C}) \to H_q(F;\mathbb{C})$ and L^m are similar: there is an automorphism S so that $L^m = S^{-1}LS$. If $Lv = \xi^k v$, then $L^m v = (\xi^k)^m v = \xi^l v = S^{-1}LSv$, so $L(Sv) = \xi^l(Sv)$. This shows that S restricts to a homomorphism $\bar{S} : \ker(L - \xi^k \operatorname{Id}) \to \ker(L - \xi^l \operatorname{Id})$, which of course is injective, since S is.

Now let $w \in \ker(L - \xi^l \operatorname{Id})$. Since S is surjective, there exists v such that $Sv = w$. A computation as above shows $L^m v = \xi^l v$, and so $v \in \ker(L^m - (\xi^k)^m \operatorname{Id})$. Write $L^m - (\xi^k)^m \operatorname{Id} = A(L - \xi^k \operatorname{Id})$, where $A = L^{m-1} + \xi^k L^{m-2} + \dots + (\xi^k)^{m-2} L + (\xi^k)^{m-1} \operatorname{Id}$. It is readily checked that the eigenvalues of A are of the form $\frac{\xi^{rm} - \xi^{km}}{\xi^r - \xi^k}$, with $r \neq k$, or of the form $m\xi^{k(m-1)}$, and thus are all non-zero. Hence A is invertible. This implies that $v \in \ker(L - \xi^k \operatorname{Id})$, whence $\bar{S}$ is surjective. We have shown that $\bar{S} : H_q(F;\mathbb{C})_k \to H_q(F;\mathbb{C})_l$ is an isomorphism, finishing the proof. $\square$

Let $p : \mathbb{C}^{d+1} \setminus \{0\} \to \mathbb{CP}^d, p(z_0, \dots, z_d) = (z_0 : \dots : z_d)$, be the Hopf bundle map, with fiber $\mathbb{C}^*$. Let $p_M : M \to M^* := p(M)$ denote the restriction of the Hopf bundle to M, and note that M^* is the complement of the projective hypersurface defined by the homogeneous polynomial f. The restriction $p_F : F \to M^*$ of the Hopf bundle to the Milnor fiber is the orbit map of the free action of the geometric monodromy h on F, and we therefore have $F/\mathbb{Z}_n \cong M^*$. These spaces and maps fit together with the Milnor fibration in the following diagram.

$$\begin{array}{ccccc}
\mathbb{Z}_n & \longrightarrow & F & \xrightarrow{p_F} & F/\mathbb{Z}_n \\
\downarrow & & \downarrow & & \downarrow \simeq \\
\mathbb{C}^* & \longrightarrow & M & \xrightarrow{p_M} & M^* \\
\downarrow & & \downarrow f & & \\
\mathbb{C}^*/\mathbb{Z}_n & \xrightarrow{\simeq} & \mathbb{C}^* & &
\end{array}$$

It is readily seen that the pullback by p_M of the covering $F \to F/\mathbb{Z}_n$ coincides with the pullback by f of the covering $\mathbb{C}^* \to \mathbb{C}^*/\mathbb{Z}_n$. Thus, the homotopy sequences of the fibrations in the above diagram mesh into the following commuting diagram with exact rows and columns.

$$\begin{array}{ccccc}
 & & \pi_1(F) & = & \pi_1(F) \\
 & & \downarrow & & \downarrow{\scriptstyle p_{F\#}} \\
\pi_1(\mathbb{C}^*) & \longrightarrow & \pi_1(M) & \xrightarrow{p_{M\#}} & \pi_1(M^*) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle \Phi} \\
\mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_n
\end{array}$$

We now identify the map Φ, the characteristic homomorphism of the covering $p_{_F} : F \to M^*$, in the case where the polynomial f is square-free. If f decomposes as the product $f_1 \cdots f_s$ of *distinct* irreducible factors, then the *reduced* hypersurface $\mathcal{S} = f^{-1}(0)$ decomposes into irreducible components $\mathcal{S}_i = f_i^{-1}(0)$. Choose as generators of $\pi_1(M)$, meridian circles a_i about $\mathcal{S}_i$, with orientations determined by the complex orientations of $\mathbb{C}^{d+1}$ and S_i. The map $f_\#$ factors through $f_* : H_1(M) \to \mathbb{Z}$, which is given by $f_*([a_i]) = 1$ (see [6] pp. 76–77). On the other hand, the map $p_{_{M*}} : H_1(M) \to H_1(M^*)$ can be identified with the canonical projection $\mathbb{Z}^s \to \mathbb{Z}^s/(n_1, \dots, n_s)$, where $n_i = \deg(f_i)$ (see [6] pp. 102–103). We thus obtain:

Proposition 1.2. *If the polynomial f is square-free, then the homomorphism Φ is the composition $\Phi = \alpha \circ \phi$, where $\phi : \pi_1(M^*) \to H_1(M^*) \cong \mathbb{Z}^s/(n_1, \dots, n_s)$ is the abelianization map, and $\alpha : H_1(M^*) \to \mathbb{Z}_n$ is induced by the composition $\mathbb{Z}^s \xrightarrow{[1,\dots,1]^T} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_n$ (where $[\bullet]^T$ denotes the transpose).* $\square$

Note that this result implies that the Milnor fiber of a square-free polynomial is connected.

Now fix a basepoint $x_0 \in M^*$. Applying the Leray-Serre spectral sequence to the fibration $p_{_F} : F \to M^*$, we obtain

$$E^2_{p,q} = H_p\big(M^*; H_q(p_{_F}^{-1}(x_0); \mathbb{C})\big) \implies H_{p+q}(F; \mathbb{C}).$$

Since $E^2_{p,q} = 0$ for $q \neq 0$, the homology groups of the Milnor fiber of f are given by $H_p(F;\mathbb{C}) = H_p\big(M^*; H_0(p_{_F}^{-1}(x_0); \mathbb{C})\big)$. In other words, $H_*(F;\mathbb{C}) = H_*(M^*;\mathcal{V})$, where $\mathcal{V}$ is a rank n local coefficient system on M^* with stalk $\mathcal{V}_x = H_0(p_{_F}^{-1}(x);\mathbb{C}) \cong \mathbb{C}^n$. This local system is induced by the representation $\tau : \pi_1(M^*, x_0) \to \mathrm{GL}(\mathbb{C}^n)$ determined by the action of the fundamental group of the base on the homology of the fiber over the basepoint x_0 in the cyclic n-fold covering $p_{_F} : F \to M^*$.

We now identify the representation τ. Notice that the fundamental group $\pi_1(M^*, x_0)$ is generated by the images, $\bar{a}_i := p_{_{M\#}}(a_i)$, of the meridians of M. Let $T \in \mathrm{GL}(\mathbb{C}^n)$ denote the cyclic permutation matrix of order n defined by $T(\vec{e}_1) = \vec{e}_2, \dots, T(\vec{e}_{n-1}) = \vec{e}_n, T(\vec{e}_n) = \vec{e}_1$, where $\{\vec{e}_i\}$ is the standard basis for $\mathbb{C}^n$.

Proposition 1.3. *The representation* $\tau : \pi_1(M^*, x_0) \to \mathrm{GL}(\mathbb{C}^n)$ *is given by* $\tau(\bar{a}_i) = T$.

Proof. The action of $\pi_1(M^*, x_0)$ on the fiber $p_{_F}^{-1}(x_0) = \{\tilde{x}_0, \xi\tilde{x}_0, \ldots, \xi^{n-1}\tilde{x}_0\}$ (where $\xi = \exp(2\pi i/n)$) is determined by the characteristic homomorphism $\Phi : \pi_1(M^*, x_0) \to \mathbb{Z}_n$ of the cover. Checking that $\Phi(\bar{a}_i) = 1$ for each i, we conclude that each of the "meridians" $\bar{a}_i$ of M^* acts on $p_{_F}^{-1}(x_0)$ by cyclically permuting the points (i.e. $\bar{a}_i \cdot \{\tilde{x}_0, \xi\tilde{x}_0, \ldots, \xi^{n-1}\tilde{x}_0\} = \{\xi\tilde{x}_0, \xi^2\tilde{x}_0, \ldots, \xi^n\tilde{x}_0 = \tilde{x}_0\}$). Passing to homology, we identify $H_0(p_{_F}^{-1}(x_0); \mathbb{C})$ with $\mathbb{C}^n$ via the map $[\tilde{x}_0] \mapsto \vec{e}_1, [\xi\tilde{x}_0] \mapsto \vec{e}_2, \ldots, [\xi^{n-1}\tilde{x}_0] \mapsto \vec{e}_n$. The action of each of the generators $\bar{a}_i$ of $\pi_1(M^*, x_0)$ in homology is then given by the cyclic permutation matrix T, that is $\tau(\bar{a}_i) = T$. $\square$

Corollary 1.4. *The representation* τ *decomposes into a direct sum,* $\tau = \bigoplus_{k=0}^{n-1} \tau_k$, *of rank one representations* $\tau_k : \pi_1(M^*, x_0) \to \mathbb{C}^*$, *where the representation* τ_k *is given by* $\tau_k(\bar{a}_i) = \xi^k$.

Proof. The matrix T is diagonalizable, with eigenvalues $1, \xi, \ldots, \xi^{n-1}$. $\square$

Corollary 1.5. *Let* $\mathcal{V}_k$ *denote the rank one local system on* M^* *induced by the representation* $\tau_k : \pi_1(M^*, x_0) \to \mathbb{C}^*$. *Then we have*

$$H_*(F; \mathbb{C}) = H_*(M^*; \mathcal{V}) = \bigoplus_{k=0}^{n-1} H_*(M^*; \mathcal{V}_k).$$

Theorem 1.6. *The two decompositions,*

$$H_*(F; \mathbb{C}) = \bigoplus_{k=0}^{n-1} H_*(F; \mathbb{C})_k \qquad \text{and} \qquad H_*(F; \mathbb{C}) = \bigoplus_{k=0}^{n-1} H_*(M^*; \mathcal{V}_k),$$

of the homology of the Milnor fiber coincide.

Proof. Via the natural identification $H_q(F; \mathbb{C}) = H_q(M^*; \mathcal{V})$, we view the algebraic monodromy as $h_* : H_q(M^*; \mathcal{V}) \to H_q(M^*; \mathcal{V})$. Since $p_{_F} \circ h = p_{_F}$, the geometric monodromy h acts as the identity on M^*, and restricts to a homeomorphism $\bar{h}$ of the fiber $p_{_F}^{-1}(x_0)$ that cyclically permutes the points. The algebraic monodromy is thus given by the action induced by $\bar{h}$ on the homology of the fiber $p_{_F}^{-1}(x_0)$. Under the identification $H_0(p_{_F}^{-1}(x_0); \mathbb{C}) \cong \mathbb{C}^n$ noted above, the induced automorphism $\bar{h}_*$ corresponds to the cyclic permutation matrix $T \in \mathrm{GL}(\mathbb{C}^n)$.

Now the homology eigenspace of ξ^k is given by $H_*(F; \mathbb{C})_k = H_*(M^*; \mathcal{W}_k)$, where $\mathcal{W}_k$ denotes the restriction of the local system $\mathcal{V}$ to the subspace $\ker(\bar{h}_* - \xi^k \mathrm{Id}) \subset H_0(p_{_F}^{-1}(x_0); \mathbb{C}) \cong \mathbb{C}^n$. Since $\bar{h}_* = T$, the local system $\mathcal{W}_k$ is naturally isomorphic to the local system $\mathcal{V}_k$ induced by the representation τ_k. Therefore we have

$$H_*(F; \mathbb{C})_k \cong H_*(M^*; \mathcal{W}_k) \cong H_*(M^*; \mathcal{V}_k). \quad \square$$

Remark 1.7. Since the rank one representation τ_0 is trivial, the local system $\mathcal{V}_0$ is also trivial and we have $H_*(F;\mathbb{C})_0 = H_*(M^*;\mathcal{V}_0) = H_*(M^*;\mathbb{C})$.

If f is a homogeneous polynomial with a linear factor, there is a choice of coordinates for which $f = z_0 \cdot g$. Since the restriction, $p : \mathbb{C}^{d+1} \setminus \{z_0 = 0\} \to \mathbb{CP}^d \setminus \mathbb{CP}^{d-1} \cong \mathbb{C}^d$, of the Hopf bundle is a trivial $\mathbb{C}^*$ bundle, the restriction $p_{_M} : M \to M^*$ is also trivial, and we have $M \cong \mathbb{C}^* \times M^*$. In this instance we may realize M^* as the (affine) hyperplane section $M^* = M \cap \{z_0 = 1\}$. Moreover, we may assume that the meridians a_i about the irreducible components of f other than $\{z_0 = 0\}$ lie entirely in M^*. Hence we have:

Proposition 1.8. *If $f = f_1 \cdots f_s$ is a homogeneous polynomial with a linear factor, say f_s, then the fundamental group of M^* is generated by the meridians $a_1, \dots, a_{s-1}$.* □

Remark 1.9. Identifying $\{z_0 = 1\} \subset \mathbb{C}^{d+1}$ with $\mathbb{C}^d$, we have $M^* = M(f^*) = \mathbb{C}^d \setminus f^{*-1}(0)$, where $f^*(z_1, \dots, z_d) = g(1, z_1, \dots, z_d)$. In other words, M^* is the complement of an affine hypersurface.

2. The Milnor Fiber of a Central Arrangement

A *hyperplane arrangement* (or merely an arrangement) is a finite collection of codimension one affine subspaces of Euclidean space. If all the hyperplanes of a given arrangement pass through the origin, the arrangement is said to be *central*, and is *affine* otherwise.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. For each hyperplane $H_i \in \mathcal{A}$, choose a linear form ℓ_i so that $H = \ker(\ell_i)$. Then

$$Q = Q(\mathcal{A}) = \prod_{i=1}^{n} \ell_i$$

is a defining polynomial of $\mathcal{A}$, and is homogeneous of degree $n = |\mathcal{A}|$. (If $d \geq 2$, the singularity of Q at the origin is, in general, not isolated.) Let $Q : M \to \mathbb{C}^*$ be the Milnor fibration, with total space

$$M = M(Q) = \mathbb{C}^{d+1} \setminus Q^{-1}(0) = \mathbb{C}^{d+1} \setminus \bigcup_{i=1}^{n} H_i,$$

the complement of $\mathcal{A}$, and fiber $F = Q^{-1}(1)$, the Milnor fiber of the arrangement $\mathcal{A}$.

For each hyperplane $H_i \in \mathcal{A}$, let a_i denote the homotopy class of the meridian about H_i. Choose coordinates so that, say $H_n = \ker(z_0)$. Then, $Q^* = Q(1, z_1, \dots, z_d)$ is the defining polynomial for an affine arrangement $\mathcal{A}^*$ in $\mathbb{C}^{d+1} \cap \{z_0 = 1\} = \mathbb{C}^d$, with complement M^*. By Proposition 1.8, the fundamental group $\pi_1(M^*)$ is generated by the meridians $a_1, \dots, a_{n-1}$.

Since M^* is the complement of the arrangement $\mathcal{A}^*$, we obtain a combinatorial description of the 1-eigenspace of the monodromy, $H_*(F;\mathbb{C})_0 = H_*(M^*;\mathcal{V}_0) = H_*(M^*;\mathbb{C})$, by using Remark 1.7 in conjunction with well-known results of Orlik and Solomon [20]. We continue studying the other eigenspaces by recalling a vanishing theorem due to Kohno (which we paraphrase for rank one local systems here).

Theorem 2.1 ([11]). *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^d$ and let ρ be a rank one representation of the fundamental group of the complement M of $\mathcal{A}$ which satisfies the following conditions:*

(i) *For each meridian $a_i \in \pi_1(M)$, we have $\rho(a_i) \neq 1$;*

(ii) *For each subarrangement $\{H_{i_1}, \dots, H_{i_r}\}$ such that $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_r}) < r$, we have $\prod_{j=1}^{r} \rho(a_{i_j}) \neq 1$.*

Then if $\mathcal{V}$ is the local coefficient system induced by the representation ρ, we have $H_q(M; \mathcal{V}) = 0$ for $q \neq d$, and $\dim(H_d(M;\mathcal{V})) = (-1)^d \cdot \chi(M)$.

In the context of the Milnor fiber problem, we obtain the following corollaries of this result.

Corollary 2.2. *If ξ^k is a primitive n^{th} root of unity, then $H_q(F;\mathbb{C})_k = H_q(M^*;\mathcal{V}_k) = 0$ for $q \neq d$, and $b_d(F)_k = (-1)^d \cdot \chi(M^*)$.*

Proof. If ξ^k is a primitive n^{th} root of unity, then the representation $\tau_k : \pi_1(M^*) \to \mathbb{C}^*$ satisfies the conditions of Theorem 2.1. □

Corollary 2.3. *If $n = |\mathcal{A}|$ is prime, then for $1 \leq k \leq d-1$ we have*

$$b_q(F)_k = \begin{cases} 0 & \text{if } q \neq d, \\ (-1)^d \cdot \chi(M^*) & \text{if } q = d, \end{cases}$$

and the Betti numbers of the Milnor fiber F of $\mathcal{A}$ are given by

$$b_q(F) = \begin{cases} b_q(M^*) & \text{if } q \neq d, \\ b_d(M^*) + (-1)^d \cdot (n-1) \cdot \chi(M^*) & \text{if } q = d. \end{cases}$$

3. The Milnor Fiber of a Generic Arrangement

In this section, we use the techniques developed in sections 1 and 2 to compute the Betti numbers of the Milnor fiber of a generic arrangement and the dimensions of the eigenspaces of the algebraic monodromy, recovering a result of Orlik and Randell ([19], see also [15]).

Definition 3.1. A central arrangement of n hyperplanes in $\mathbb{C}^{d+1}$ is called *generic* if $n > d+1$ and the intersection of every subset of $d+1$ distinct hyperplanes is the origin.

Theorem 3.2 ([19]). *Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. Then the Betti numbers of the Milnor fiber F of $\mathcal{A}$ are given by*

(i)
$$b_q(F) = \begin{cases} \binom{n-1}{q} & \text{if } q \leq d-1, \\ \binom{n-1}{d} + (n-1)\binom{n-2}{d} & \text{if } q = d, \end{cases}$$

and the dimensions of the homology eigenspaces of the monodromy are given by

$$\text{(ii)} \qquad b_q(F)_0 = \begin{cases} \binom{n-1}{q} & \text{if } q \le d, \\ 0 & \text{if } q > d, \end{cases}$$

and

$$\text{(iii)} \qquad b_q(F)_k = \begin{cases} 0 & \text{if } q \ne d, \\ \binom{n-2}{d} & \text{if } q = d, \end{cases}$$

for $1 \le k \le n-1$.

This result is an immediate consequence of a vanishing theorem due to Hattori [9] and Cohen [4] which we state below. First we establish some notation.

Definition 3.3. An affine arrangement of m hyperplanes in $\mathbb{C}^d$ is called a *general position* arrangement if $m > d$, the intersection of every subset of d distinct hyperplanes is a point, and the intersection of every subset of $d+1$ hyperplanes is empty.

Proposition 3.4 ([19])**.** *Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. Let M be the complement of $\mathcal{A}$ and let $p_{_M} : M \to M^*$ be the restriction of the Hopf bundle. Then M^* is the complement of a general position arrangement of $n-1$ hyperplanes in $\mathbb{C}^d$.*

Theorem 3.5 ([9], [4])**.** *Let $\mathcal{V}$ be a non-trivial rank one local system on the complement M^* of a general position arrangement of $n-1$ hyperplanes in $\mathbb{C}^d$. Then $H_q(M^*;\mathcal{V}) = 0$ for $q \ne d$, and $\dim\big(H_d(M^*;\mathcal{V})\big) = \binom{n-2}{d}$.*

Proof of Theorem 3.2. Recall (Theorem 1.6) that the homology eigenspaces of the monodromy are given by $H_*(F;\mathbb{C})_k = H_*(M^*;\mathcal{V}_k)$ for $k = 0, 1, \dots, n-1$. The Betti numbers of the complement M^* of a general position arrangement of $n-1$ hyperplanes in $\mathbb{C}^d$ are given by

$$b_q(M^*;\mathbb{C}) = \begin{cases} \binom{n-1}{q} & \text{if } q \le d, \\ 0 & \text{if } q > d \end{cases}$$

(see [19]). This result, together with Remark 1.7, proves part (ii) of Theorem 3.2.

Since each of the rank one local systems $\mathcal{V}_k$ on M^* is non-trivial for $1 \le k \le d-1$, part (iii) follows directly from Theorem 3.5. Part (i) is obtained by combining the results of parts (ii) and (iii). $\square$

Remark 3.6. The analogue of Theorem 3.5 holds for arrangements which have only *normal crossings* ([24]). Thus the analogue of Theorem 3.2 holds for the Milnor fiber of any arrangement $\mathcal{A}$ for which the associated affine arrangement $\mathcal{A}^*$ has only normal crossings. However, the combinatorial formulas for the Betti numbers of the Milnor fiber and the dimensions of the eigenspaces of the monodromy need not be so simple.

4. The Milnor Fiber of a Complexified 3-Arrangement

We now use the techniques of sections 1 and 2 to obtain an algorithm for computing the homology of the Milnor fiber of a complexified real arrangement in $\mathbb{C}^3$. This algorithm gives a direct sum decomposition of the homology groups $H_q(F;\mathbb{C})$ into the eigenspaces of the algebraic monodromy $h_* : H_q(F;\mathbb{C}) \to H_q(F;\mathbb{C})$.

Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^3$ defined by real forms, and let $\mathcal{A}^* = \{H_1, \dots, H_m\}$ denote the associated affine 2-arrangement, with complement M^*. R. Randell has shown that a presentation for the fundamental group of M^* may be obtained from the underlying real arrangement $\mathcal{A}_{\mathbb{R}}$ (see also Salvetti [23], and see [8] for a good description of Randell's algorithm).

Theorem 4.1 ([22])**.** *The group $\pi = \pi_1(M^*) = \pi_1(M^*, x_0)$ has a presentation given by*

$$\pi = \big(a_1, \dots, a_m \mid \bigcup_{p\in P} R_p\big),$$

where P is the set of multiple points of $\mathcal{A}_{\mathbb{R}}$, and each R_p is a set of relators on suitable conjugates of the generators associated to those hyperplanes which pass through the multiple point p.

Remark 4.2. In group presentations as above, each of the relator sets R_p is of the form

$$R_p = \left[a_{i_1}^{w_1}, \dots, a_{i_s}^{w_s}\right],$$

where each of the hyperplanes $\{H_{i_j}\}$ passes through the multiple point p, and the w_j are words in the generating set $\{a_i\}$. Conjugation is defined by $u^v = v^{-1}uv$, and the bracket denotes the set of relators

$$[u_1, \dots, u_s] = \{u_1 \dots u_s = u_{\sigma(1)} \dots u_{\sigma(s)} \mid \sigma = (1, 2, \dots, s)^k, k = 1, \dots, s-1\}.$$

Note that the bracket of two words is the usual commutator.

Let K denote the canonical 2-complex associated with Randell's presentation $\pi = (a_1, \dots, a_m \mid r_1, \dots, r_l)$ of the fundamental group of M^*. Note that, since M^* is a Stein space of complex dimension two, M^* has the homotopy type of a two-dimensional complex. In fact, the following holds.

Theorem 4.3 ([8])**.** *M^* and K are homotopy equivalent.*

We therefore suppress any further mention of the complex K, and work solely with the complement M^*.

The homology of the universal cover of M^* is canonically isomorphic to the homology of the chain complex $\mathbf{C}_\bullet$:

$$\big(\mathbb{Z}\pi\big)^l \xrightarrow{\partial_2} \big(\mathbb{Z}\pi\big)^m \xrightarrow{\partial_1} \mathbb{Z}\pi \to 0,$$

where $\mathbb{Z}\pi$ denotes the group ring of π, the homomorphism $\partial_2 = \left(\dfrac{\partial r_i}{\partial a_j}\right)$ is the Jacobian matrix of Fox derivatives, and the homomorphism ∂_1 is defined by $\partial_1 = (a_j - 1_\pi)$ (see e.g. [3] pp. 45–46).

Let $\rho : \pi \to \mathrm{Aut}(V)$ be a complex representation of the fundamental group of M^*, and let $\mathcal{V}$ denote the local coefficient system on M^* induced by ρ. The homology of M^* with coefficients in the local system $\mathcal{V}$ is canonically isomorphic to the homology of the complex $\mathbf{C}_\bullet \otimes_{\mathbb{Z}\pi} V$, where the original chain complex $\mathbf{C}_\bullet$ is considered as a right $\mathbb{Z}\pi$-module, and $V = \mathcal{V}_{x_0}$ (the stalk of the local system at the point x_0) is a left $\mathbb{Z}\pi$-module via the representation ρ (see [3]). The maps in the resulting complex

$$\begin{array}{ccccc}
(\mathbb{Z}\pi)^l \otimes_{\mathbb{Z}\pi} V & \xrightarrow{\partial_2 \otimes \mathrm{Id}_V} & (\mathbb{Z}\pi)^m \otimes_{\mathbb{Z}\pi} V & \xrightarrow{\partial_1 \otimes \mathrm{Id}_V} & \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} V \\
\| & & \| & & \| \\
V^l & \xrightarrow{\Delta_2} & V^m & \xrightarrow{\Delta_1} & V
\end{array}$$

are obtained using the representation ρ. For instance,

$$\Delta_1 = \partial_1 \otimes \mathrm{Id}_V = (\rho(a_j) - \rho(1_\pi)) = (\rho(a_j) - \mathrm{Id}_V)\,.$$

Remark 4.4. The above construction provides an algorithm for computing $H_*(M^*; \mathcal{V})$, the homology of M^* with coefficients in an arbitrary local system $\mathcal{V}$.

The main result of this section now follows immediately.

Theorem 4.5. *Let $\mathcal{A}$ be a complexified real central arrangement in $\mathbb{C}^3$, let $\mathcal{A}^*$ be the associated affine 2-arrangement with complement M^*, and let $\mathcal{V}$ be the local system on M^* induced by the representation τ of $\pi = \pi_1(M^*)$. Then the homology of the complex $\mathbf{C}_\bullet \otimes_{\mathbb{Z}\pi} V$, where V is the stalk of $\mathcal{V}$, is canonically isomorphic to $H_*(F;\mathbb{C})$, the homology of the Milnor fiber of $\mathcal{A}$. Furthermore, for each k, the eigenspace, $H_q(F;\mathbb{C})_k$, of the algebraic monodromy is naturally isomorphic to the q^{th} homology group of the complex $\mathbf{C}_\bullet \otimes_{\mathbb{Z}\pi} \mathbb{C}_k$, where $\mathbb{C}_k$ denotes the stalk of the rank one local system $\mathcal{V}_k$ induced by the rank one representation τ_k.*

Remark 4.6. Theorem 4.5 provides an algorithm for computing the homology of the Milnor fiber of any (central) arrangement in $\mathbb{C}^3$ that is defined by real forms. This algorithm gives a direct sum decomposition of the homology groups $H_q(F;\mathbb{C})$ into the eigenspaces of the algebraic monodromy. We give several examples in the next section.

Remark 4.7. It has been observed ([19], [21]) that the homology of the Milnor fiber of an arrangement $\mathcal{A}$ is a "more subtle" invariant of $\mathcal{A}$ than the (co-)homology of the complement, which is combinatorially determined by the intersection lattice of $\mathcal{A}$ [20]. It is possible that the algorithm presented here may be used to distinguish among arrangements with isomorphic lattices. However, we know of no such example at this time.

Remark 4.8. It should be possible to extend the results of this section to an arbitrary central arrangement in $\mathbb{C}^3$. Indeed, W. Arvola [2] has given a presentation for $\pi_1(M^*)$ in this more general setting, but it is not known whether the associated 2-complex, K, is homotopy equivalent to M^*. On the other hand, if $\mathcal{C}$ is a curve in $\mathbb{C}^2$ that is transverse to the line at infinity in $\mathbb{CP}^2$, A. Libgober [13] exhibits a presentation for $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ for which the associated 2-complex is homotopy equivalent to $\mathbb{C}^2 \setminus \mathcal{C}$. Thus the methods presented here may be applied in this situation.

5. Examples

In this section, we use the algorithm developed in section 4 to compute the homology of the Milnor fibers of several arrangements in $\mathbb{C}^3$. In each example, we give

(i) a defining polynomial $Q = Q(x, y, z)$ of the arrangement $\mathcal{A}$ of n hyperplanes;
(ii) a projective picture of the underlying real arrangement $\mathcal{A}_{\mathbb{R}}$ with the hyperplane $\{z = 0\}$ taken to infinity in the projective plane (see [18]);
(iii) a presentation of the fundamental group, $\pi = \pi_1(M^*)$, of the complement M^* of the affine 2-arrangement $\mathcal{A}^*$ associated with $\mathcal{A}$;
(iv) the dimensions, $b_q(F)_k$, of the eigenspaces of the algebraic monodromy, and the Betti numbers, $b_q(F)$, of the Milnor fiber of $\mathcal{A}$. (Of course, $b_q(F)_k = 0$ for $q > 2$, so we suppress those numbers.)

The ordering of the hyperplanes indicated in (ii) and (iii) is the slope-intercept lexicographic ordering obtained by rotating the figure in (ii) slightly in the counter-clockwise direction. The presentation given in (iii) is obtained by applying Randell's algorithm (see [22] and section 4) to this rotated figure (or equivalently, by sweeping a line of very large positive slope across the figure from right to left). The hyperplane at infinity is labelled "0" and plays no role in the presentation. Note that the ordering of the hyperplanes given in (ii) and (iii) need not correspond to the ordering of the linear factors in the polynomial Q given in (i).

The computations in (iv) were carried out using *Mathematica*. Using Proposition 1.1 and Corollary 2.2, we need only compute $b_q(F)_k$ for those k such that $(k, n) \neq 1$ and $k \leq n/2$. In the first example, we also provide the chain complex $\mathbf{C}_\bullet \otimes_{\mathbb{Z}\pi} \mathbb{C}_k$, which is used to compute the dimensions of the ξ^k-eigenspaces.

Note that in Examples 5.1 and 5.4 we have $b_1(F) > b_1(F)_0 = b_1(M^*)$. (See [1], [7] and [14] for conditions that insure that $b_1(F) = b_1(M^*)$.)

Example 5.1. Here $\mathcal{A}$ is the complexification of the A_3 ($= \mathrm{D}_3$) arrangement (see also [1]). Note that the fundamental group of the complement of $\mathcal{A}$ is the pure braid group on four strands. The group π we present below is the quotient of the pure braid group by its center.

(i) $Q = xyz(x - y)(x - z)(y - z)$

(ii)

Figure 1. The A_3 arrangement

(iii) $\pi = \left(a_1, a_2, a_3, a_4, a_5 \mid [a_5, a_3, a_1], [a_4^{a_2}, a_1], [a_3, a_2], [a_5, a_4, a_2]\right)$

$\mathbf{C}_\bullet \otimes_{\mathbb{Z}\pi} \mathbb{C}_k : \qquad \mathbb{C}^6 \xrightarrow{\Delta_2} \mathbb{C}^5 \xrightarrow{\Delta_1} \mathbb{C}$

where $\lambda = \xi^k$, $\quad \Delta_1 = \left[\lambda-1 \ \lambda-1 \ \lambda-1 \ \lambda-1 \ \lambda-1\right]^T$, $\quad$ and

$$\Delta_2 = \begin{bmatrix} \lambda^2-\lambda & 0 & \lambda-1 & 0 & 1-\lambda^2 \\ \lambda^2-1 & 0 & \lambda-\lambda^2 & 0 & 1-\lambda \\ \lambda-1 & 2-\lambda-\lambda^{-1} & 0 & \lambda^{-1}-1 & 0 \\ 0 & \lambda-1 & 1-\lambda & 0 & 0 \\ 0 & \lambda^2-\lambda & 0 & \lambda-1 & 1-\lambda^2 \\ 0 & \lambda^2-1 & 0 & \lambda-\lambda^2 & 1-\lambda \end{bmatrix}$$

(iv)

	$b_0(F)_k$	$b_1(F)_k$	$b_2(F)_k$
$k=0$	1	5	6
$k=1,3,5$	0	0	2
$k=2,4$	0	1	3
$\sum$	$b_0(F)=1$	$b_1(F)=7$	$b_2(F)=18$

Example 5.2. Here $\mathcal{A}$ is the complexification of the B_3 arrangement.

(i) $\quad Q = xyz(x-y)(x-z)(y-z)(x+y)(x+z)(y+z)$

(ii)

Figure 2. The B_3 arrangement

(iii)

$$\pi = \big(a_1,\dots,a_8 \ \big| \ [a_8,a_5,a_1], [a_7^{a_3}, a_4^{a_3a_2}, a_1], [a_6^{a_4^{a_3}a_2}, a_1], [a_8,a_6,a_4^{a_3},a_2], [a_7^{a_3},a_2], [a_5,a_2], [a_5,a_4,a_3], [a_6,a_3], [a_8,a_7,a_3]\big)$$

(iv)

	$b_0(F)_k$	$b_1(F)_k$	$b_2(F)_k$
$k=0$	1	8	15
$k\neq 0$	0	0	8
$\sum$	$b_0(F)=1$	$b_1(F)=8$	$b_2(F)=79$

Example 5.3. Here $\mathcal{A}$ is the complexification of the H_3 arrangement.

(i)

$$\begin{aligned} Q = xyz&(\gamma x+y+\eta z)(\gamma x-y-\eta z)(\gamma x-y+\eta z)(\gamma x+y-\eta z)\cdot \\ &(x+\eta y+\gamma z)(x-\eta y-\gamma z)(x-\eta y+\gamma z)(x+\eta y-\gamma z)\cdot \\ &(\eta x+\gamma y+z)(\eta x-\gamma y-z)(\eta x-\gamma y+z)(\eta x+\gamma y-z) \end{aligned}$$

where γ denotes the golden ratio and $\eta = \gamma - 1$.

(ii)

Figure 3. The H_3 arrangement

(iii)

$$\begin{aligned}\pi = \big(a_1,\dots,a_{14} \,\big|\, & [a_{11},a_2,a_1],[a_{12},a_9^{a_8 v},a_6^{a_4},a_3,a_1],[a_8^{v a_4},a_1],[a_{12},a_2^{a_1}],\\ & [a_{13},a_{10}^{a_5},a_7^{a_5},a_4,a_1],[a_{14},a_5,a_1],[a_{13},a_9^{a_8 v w a_1},a_2^{a_1}],\\ & [a_{14},a_{10}^{v a_4 a_1},a_8^{v a_4 a_3^{a_1}},a_6^{a_4 a_3 a_1},a_2^{a_1}],[a_7^{a_5 a_4 a_3^{a_1}},a_2^{a_1}],\\ & [a_{11},a_3],[a_{13},a_8^{v a_4},a_3^{a_1}],[a_{10}^{v a_4 a_1},a_3^{a_1}],[a_{14},,a_7^{a_5 a_4},a_3^{a_1}],\\ & [a_{11},a_6,a_4],[a_9^{a_8 v},a_4],[a_{12},a_8^{v},a_4],[a_{14},a_4^{a_1}],[a_6,a_5],\\ & [a_{11},a_9,a_8,a_7,a_5],[a_{12},a_{10},a_5],[a_{13},a_5],[a_{13},a_6^{a_4 a_3 a_1}],\\ & [a_{12},a_7^{a_5}],[a_{14},a_9^{a_8 a_7 w a_2 a_1}],[a_{11},a_{10}]\big)\end{aligned}$$

where $v = a_7 a_5$ and $w = a_6^{a_4} a_3$.

(iv)

	$b_0(F)_k$	$b_1(F)_k$	$b_2(F)_k$
$k=0$	1	14	45
$k \neq 0$	0	0	32
$\sum$	$b_0(F)=1$	$b_1(F)=14$	$b_2(F)=493$

The following examples are two (distinct) realizations of the classical 9_3 configurations [10]. These arrangements appeared in [1] as examples with "similar" combinatorics whose Milnor fibers differ in homology.

Example 5.4. Here $\mathcal{A}$ is a realization of the Pappus configuration $(9_3)_1$.

(i) $Q = xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(-2x+5y-z)$

(ii)

Figure 4. A realization of the $(9_3)_1$ configuration

(iii)

$$\begin{aligned}\pi = \big(a_1,\dots,a_8 \,\big|\, & [a_8,a_4,a_1],[a_7,a_2,a_1],[a_6^{a_5 a_3},a_1],[a_5,a_3,a_1],[a_8,a_2^{a_1}],\\ & [a_6^{a_5 a_3},a_4^{a_1},a_2^{a_1}],[a_5,a_2^{a_1}],[a_8,a_6^{a_5},a_3],[a_7,a_3],[a_4,a_3],\\ & [a_7,a_6,a_5],[a_7,a_4],[a_8,a_5]\big)\end{aligned}$$

(iv)

	$b_0(F)_k$	$b_1(F)_k$	$b_2(F)_k$
$k=0$	1	8	19
$k=3,6$	0	1	13
$k \neq 0,3,6$	0	0	12
$\sum$	$b_0(F)=1$	$b_1(F)=10$	$b_2(F)=117$

Example 5.5. Here $\mathcal{A}$ is a realization of the configuration $(9_3)_2$.

(i) $Q = xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(4x+6y+6z)$

(ii)

Figure 4. A realization of the $(9_3)_2$ configuration

(iii)

$$\pi = \big(a_1,\dots,a_8 \;\big|\; [a_8,a_5,a_1],[a_7,a_1],[a_6^{a_3},a_4^{a_3a_2},a_1],[a_3^{a_2},a_1],[a_8,a_3,a_2],\\ [a_7,a_4^{a_3},a_2],[a_6^{a_3},a_2],[a_5,a_2],[a_7,a_6,a_3],[a_5,a_4,a_3],\\ [a_8,a_4^{a_3a_2}],[a_7,a_5],[a_8,a_6^{a_3}]\big)$$

(iv)

	$b_0(F)_k$	$b_1(F)_k$	$b_2(F)_k$
$k=0$	1	8	19
$k\neq 0$	0	0	12
$\sum$	$b_0(F)=1$	$b_1(F)=8$	$b_2(F)=115$

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