

HOMOTOPY TYPES OF COMPLEMENTS OF 2-ARRANGEMENTS IN \mathbb{R}^4

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ABSTRACT. We study the homotopy types of complements of arrangements of n transverse planes in \mathbb{R}^4 , obtaining a complete classification for $n \leq 6$, and lower bounds for the number of homotopy types in general. Furthermore, we show that the homotopy type of a 2-arrangement in \mathbb{R}^4 is not determined by the cohomology ring, thereby answering a question of Ziegler. The invariants that we use are derived from the characteristic varieties of the complement. The nature of these varieties illustrates the difference between real and complex arrangements.

1. INTRODUCTION

In [13], Goresky and MacPherson introduced a generalization of the notion of complex hyperplane arrangement. A 2-arrangement in \mathbb{R}^{2d} is a finite collection \mathcal{A} of codimension 2 linear subspaces so that, for every subset $\mathcal{B} \subseteq \mathcal{A}$, the space $\bigcap_{H \in \mathcal{B}} H$ has even dimension. The main object of study is the complement of the arrangement, $X(\mathcal{A}) = \mathbb{R}^{2d} \setminus \bigcup_{H \in \mathcal{A}} H$. Goresky and MacPherson computed the cohomology groups of X . Björner and Ziegler [4] and Ziegler [30] determined the structure of the cohomology algebra $H^*(X; \mathbb{Z})$. These results generalize the classical work of Arnol'd, Brieskorn, and Orlik and Solomon on the cohomology ring of the complement of a complex hyperplane arrangement, see [24]. Unlike the situation obtaining for the Orlik-Solomon algebra, which is completely determined by the intersection lattice, there remained an ambiguity in the relations defining $H^*(X; \mathbb{Z})$. Even in the simplest case of 2-arrangements in \mathbb{R}^4 , a striking phenomenon occurs, showing that this ambiguity cannot be resolved, [30].

Each 2-arrangement \mathcal{A} in \mathbb{R}^4 is a realization of the uniform matroid $U_{2,n}$, where $n = |\mathcal{A}|$ is the cardinality of the arrangement. Thus, the intersection lattice of such an arrangement is uniquely determined by n . Furthermore, the homology groups of the complement, X , the lower central series quotients of the group $G = \pi_1(X)$, and the Chen groups of G also depend only on n .

On the other hand, the cohomology ring of X is a more subtle invariant. The relations in $H^*(X; \mathbb{Z})$ depend on, and are determined by the linking numbers of the associated link. Ziegler [30] found a pair of 2-arrangements of four planes which have non-isomorphic cohomology rings. His method, which uses an invariant derived from $H^*(X; \mathbb{Z})$, does not seem, however, to extend beyond $n = 4$.

In this paper, we introduce new homotopy-type invariants of complements of 2-arrangements. These invariants, derived from the Alexander module, work for

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arbitrary n . As a first step towards the homotopy classification of 2-arrangements, we prove the following (see Corollary 6.6).

Theorem 1.1. *For every integer $n \geq 1$, there exist at least $p(n-1) - \lfloor \frac{n-1}{2} \rfloor$ different homotopy types of complements of 2-arrangements of n planes in \mathbb{R}^4 , where $p(\cdot)$ is the partition function, and $\lfloor \cdot \rfloor$ is the integer part function.*

At the end of [30], Ziegler asks whether the cohomology ring determines the homotopy type of the complement of a 2-arrangement, proposing as a candidate for a negative answer the remarkable pair of arrangements of 6 planes found by Mazurovskii in [22]. Using successive cablings on Mazurovskii's pair, we answer Ziegler's question, as follows (see Theorem 8.4).

Theorem 1.2. *For every integer $n \geq 6$, there exists a pair of 2-arrangements of n planes in \mathbb{R}^4 , whose complements have isomorphic cohomology rings, but different homotopy types.*

Rigid isotopy of arrangements implies isotopy of their singularity links. The converse is not clear, though, since an isotopy may go outside the class of such links. On the other hand, the classification, up to rigid isotopy, of 2-arrangements in \mathbb{R}^4 is equivalent to the classification, also up to rigid isotopy, of configurations of skew lines in \mathbb{R}^3 . Such configurations were introduced by Viro in [28], and have been intensively studied since then, see the survey article by Crapo and Penne [8]. The rigid isotopy classification of configurations of n skew lines in \mathbb{R}^3 , was achieved by Viro [28] for $n \leq 5$, and by Mazurovskii [22] for $n = 6$.

It is readily seen that rigid isotopy of arrangements implies homotopy equivalence of their complements. The converse is not true. Indeed, as first noted by Viro, there exist configurations that are not isotopic to their mirror image. But clearly, the complements of mirror pairs are diffeomorphic, and thus homotopy equivalent. The next result shows that this is the only exception, for $n \leq 6$ (see Theorem 9.4).

Theorem 1.3. *For 2-arrangements of $n \leq 6$ planes in \mathbb{R}^4 , the homotopy types of complements are in one-to-one correspondence with the rigid isotopy types modulo mirror images.*

This theorem recovers Ziegler's classification of homotopy types of arrangements of $n = 4$ planes. The number of homotopy types from the classification in Theorem 1.3, together with the lower bound from Theorem 1.1, are tabulated below.[†]

n	1	2	3	4	5	6	7
Homotopy types	1	1	1	2	4	11	?
Lower bound	1	1	1	2	3	5	8

The homotopy-type invariants that we use in our classification of 2-arrangements are derived from the characteristic varieties of their complements. Given a space X with $H_1(X) \cong \mathbb{Z}^n$, the k^{th} determinantal ideal of the Alexander module of X defines a subvariety, $V_k(X)$, of the complex algebraic torus, $(\mathbb{C}^*)^n$, whose monomial isomorphism type depends only on the homotopy type of X —in fact, only

[†]Recently, Borobia and Mazurovskii [5] achieved the rigid isotopy classification of configurations of 7 lines. If the assertion of Theorem 1.3 were to hold for $n = 7$, it would give 37 distinct homotopy types of complements of arrangements of 7 planes. We have verified this in the particular case of horizontal arrangements, for which there are 24 distinct homotopy types.

on $\pi_1(X)$ —see [10, 17]. We call $V_k(X)$ the k^{th} characteristic variety of X . From this variety, we extract in Theorem 5.6 the following homotopy-type invariants for the space X : the list $\Sigma_k(X)$ of codimensions of irreducible components, and the number $\text{Tors}_{p,k}(X)$ of p -torsion points. These numerical invariants are readily computable by standard methods of geometric topology and commutative algebra, and are powerful enough to detect all the differences in homotopy types listed in the above theorems.

The characteristic varieties of complements of divisors in complex algebraic manifolds have been intensively studied recently, see [1, 18, 16, 7, 19, 20]. Deep results as to their qualitative nature have been obtained by Arapura [1], who showed that all the irreducible components of such characteristic varieties are (possibly translated) subtori of a complex algebraic torus. Building on this work, a more precise description of the characteristic varieties of complex hyperplane arrangements has emerged. In all known examples, if X is the complement of such an arrangement, all positive-dimensional subtori of $V_k(X)$ pass through the origin $\mathbf{1}$ of the torus.

On the other hand, if X is the complement of a 2-arrangement in \mathbb{R}^4 , we find that the characteristic varieties of X may contain positive-dimensional subtori that do not pass through $\mathbf{1}$. For the non-complex Ziegler arrangement, the variety V_2 contains three subtori of $(\mathbb{C}^*)^4$, one of which is translated by $(1, -1, 1, 1)$, see Example 5.10. But this is still a rather mild qualitative difference. For the indecomposable Mazurovskiĭ arrangements, the variety V_1 is not even a union of translated subtori, see §8. These phenomena may be thought of as manifestations of the non-complex nature of real arrangements.

The paper is organized as follows.

In §2, we review the basic facts about 2-arrangements in \mathbb{R}^4 , and their associated configurations of lines and singularity links. In §3, we look in detail at some special classes of arrangements: the decomposable ones, and the horizontal ones. In §4, we associate several braids to a 2-arrangement, and use these braids to compute the fundamental group of the complement. In §5, we review Alexander modules and define numerical homotopy-type invariants from the associated characteristic varieties. In §6, we study the bottom characteristic varieties V_{n-2} , obtaining a complete characterization for depth 2, completely decomposable arrangements. In §7, we study the top characteristic varieties V_1 , and their torsion points. In §8, we study in detail the Mazurovskiĭ arrangements, and their cablings. Using the results and techniques from §§6–8, we complete the homotopy-type classification of 2-arrangements of 6 planes or less in §9.

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2. ARRANGEMENTS, LINE CONFIGURATIONS, AND LINKS

In this section we collect some facts about arrangements of transverse planes in \mathbb{R}^4 , and the corresponding configurations of skew lines in \mathbb{R}^3 and links in \mathbb{S}^3 .

2.1. We start by defining our basic objects of study in a concrete way.

Definition 2.2. A 2-arrangement in \mathbb{R}^4 is a finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of pairwise transverse 2-dimensional vector subspaces of \mathbb{R}^4 . The *union* of the arrangement is $U(\mathcal{A}) = \bigcup_{H \in \mathcal{A}} H$. The *complement* of the arrangement is $X(\mathcal{A}) = \mathbb{R}^4 \setminus U(\mathcal{A})$. The *link* of the arrangement is $L(\mathcal{A}) = \mathbb{S}^3 \cap U(\mathcal{A})$.

Each plane $H_i \in \mathcal{A}$ can be written as $H_i = \ker \lambda_i \cap \ker \lambda'_i$, for some linear forms $\lambda_i, \lambda'_i : \mathbb{R}^4 \rightarrow \mathbb{R}$. The transversality condition means that $H_i \cap H_j = \{0\}$, for all $i \neq j$. That is, $\det(\lambda_i, \lambda'_i, \lambda_j, \lambda'_j) \neq 0$ for $i \neq j$.

Alternatively, identifying \mathbb{R}^4 with $\mathbb{C}^2 = \{(z, w)\}$, each plane in \mathcal{A} can be written as $H_i = \{f_i = 0\}$, where $f_i(z, w) = a_i z + b_i \bar{z} + c_i w + d_i \bar{w}$, for some $a_i, b_i, c_i, d_i \in \mathbb{C}$. In terms of real coordinates $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $u = \operatorname{Re} w$, $v = \operatorname{Im} w$, we have $\lambda_i(x, y, u, v) = \operatorname{Re} f_i(x + iy, u + iv)$ and $\lambda'_i(x, y, u, v) = \operatorname{Im} f_i(x + iy, u + iv)$, where $i = \sqrt{-1}$.

With notation as above, let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the polynomial map in z, \bar{z}, w, \bar{w} given by $f = f_1 \cdots f_n$. We say that f is a *defining polynomial* for the arrangement \mathcal{A} . Obviously, the union of the arrangement is the zero locus of the defining polynomial.

Example 2.3. The most basic example of a 2-arrangement is a *complex* arrangement. Such an arrangement consists of complex lines through the origin of \mathbb{C}^2 . Any two complex arrangements differ by an \mathbb{R} -linear change of variables, and thus have diffeomorphic complements. We denote the complex arrangement of n lines by \mathcal{A}_n , and take its defining polynomial to be $f_n(z, w) = (z - w) \cdots (z - nw)$. The link $L(\mathcal{A}_n)$ is the n -component Hopf link. The *trivial* arrangement is \mathcal{A}_1 .

2.4. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a 2-arrangement in \mathbb{R}^4 . Its link, $L = \{L_1, \dots, L_n\}$, consists of n unknotted circles in \mathbb{S}^3 . The complement of the arrangement, $X(\mathcal{A})$, is homotopy equivalent to the complement of the link, $Y(L) = \mathbb{S}^3 \setminus L$, via radial deformation. Using this observation, we can compute homotopy-type invariants of $X = X(\mathcal{A})$ by methods of knot theory.

The homology groups of X depend only on the number of planes in the arrangement: $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^n$, $H_2 = \mathbb{Z}^{n-1}$, $H_k = 0$ for $k > 2$. The cohomology ring of X , on the other hand, also depends on the linking numbers $l_{i,j} = \operatorname{lk}(L_i, L_j)$. Specifically,

$$H^*(X; \mathbb{Z}) = \bigwedge^* \mathbb{Z}^n / (l_{ij} e_i e_j + l_{jk} e_j e_k + l_{ki} e_k e_i = 0),$$

where $\bigwedge^* \mathbb{Z}^n$ is the exterior algebra on e_1, \dots, e_n . As noted by Ziegler [30], one can compute the linking numbers of $L(\mathcal{A})$ directly from the defining equations of \mathcal{A} . Indeed, if $H_i = \{\lambda_i = \lambda'_i = 0\}$, then $l_{i,j} = \operatorname{sgn}(\det(\lambda_i, \lambda'_i, \lambda_j, \lambda'_j))$.

As shown by Ziegler [30], the complement X fibers over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, with fiber $\mathbb{C} \setminus \{n-1 \text{ points}\}$, and thus X is a $K(G, 1)$ space. Alternatively, since all the linking numbers are non-zero, the link L is non-split, and thus $Y(L)$ is aspherical, see [6]. It follows that the homotopy type of X is determined by the isomorphism class of its fundamental group G .

As we shall see in Proposition 4.4, the monodromy of the bundle $X \rightarrow \mathbb{C}^*$ is a certain (pure) braid automorphism $\check{\beta} \in P_{n-1}$, and so G is a semidirect product of free groups, $G = \mathbb{F}_{n-1} \rtimes_{\check{\beta}} \mathbb{F}_1$. Since $\check{\beta}$ acts trivially on homology, a result of Falk and Randell [12] implies that the lower central series quotients of G depend only on n , being equal to those of the product $\Gamma = \mathbb{F}_{n-1} \times \mathbb{F}_1$. In fact, since all the linking numbers of L are equal to ± 1 , a result of Massey and Traldi [21] shows that the

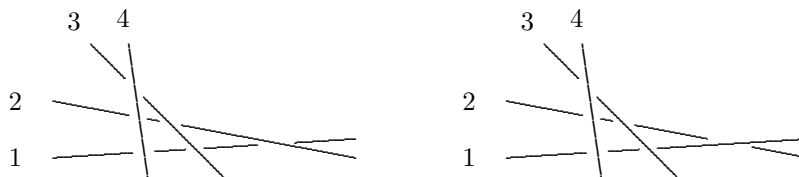


FIGURE 1. Ziegler's pair: The configurations \mathcal{C}^+ (left) and \mathcal{C}^- (right).

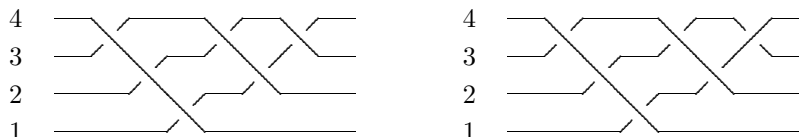


FIGURE 2. Ziegler's pair: The half-braids α^+ (left) and α^- (right).

lower central series quotients of both G and G/G''' are equal to the corresponding quotients of Γ and Γ/Γ''' .

2.5. Now let H be an affine hyperplane in \mathbb{R}^4 , generic with respect to \mathcal{A} . The *configuration* of \mathcal{A} corresponding to H is the configuration of skew lines in \mathbb{R}^3 defined as $\mathcal{C}_H(\mathcal{A}) = H \cap U(\mathcal{A})$. Conversely, given a configuration \mathcal{C} of skew lines, one obtains a 2-arrangement, $\mathcal{A}_p = \mathcal{A}(\mathcal{C})$, by coning at a generic point p , and translating p to 0.

Example 2.6. Let \mathcal{A}^+ and \mathcal{A}^- be the pair of 2-arrangements considered by Ziegler in [30]. The arrangement \mathcal{A}^+ is the complex arrangement \mathcal{A}_4 , with defining polynomial $f^+(z, w) = (z - w)(z - 2w)(z - 3w)(z - 4w)$. The arrangement \mathcal{A}^- has defining polynomial $f^-(z, w) = (z - \bar{w})(z - 2\bar{w})(z - 3\bar{w})(z - 4\bar{w})$. Projecting onto the hyperplane $\{v = 1\}$, we get configurations $\mathcal{C}^\pm = \{\ell_1^\pm, \ell_2^\pm, \ell_3, \ell_4\}$, with equations

$$\begin{aligned} \ell_1^\pm &= \{x - u = y \mp 1 = 0\}, & \ell_2^\pm &= \{x - 2u = y \mp 2 = 0\}, \\ \ell_3 &= \{x - 3u = y - 3 = 0\}, & \ell_4 &= \{x - 4u = y - 4 = 0\}. \end{aligned}$$

The two configurations are pictured in Figure 1.

2.7. Finally, let us consider the natural isotopy relation between arrangements, modeled on the similar notion for configurations.

Definition 2.8. Two arrangements \mathcal{A} and \mathcal{A}' are called *rigidly isotopic* if there is an isotopy of \mathbb{R}^4 connecting \mathcal{A} to \mathcal{A}' through arrangements.

The rigid isotopy class of $\mathcal{C}_H(\mathcal{A})$ does not depend on H , and the rigid isotopy class of $\mathcal{A}_p(\mathcal{C})$ does not depend on p . Therefore, we will denote them simply by $\mathcal{C}(\mathcal{A})$ and $\mathcal{A}(\mathcal{C})$, respectively. Moreover, rigid isotopy classes of configurations are in one-to-one correspondence with rigid isotopy classes of 2-arrangements. See Crapo and Penne [8] for details and references.

Remark 2.9. Given an arrangement, we can deform it by means of a rigid isotopy so that one of the planes has linking number +1 with all other planes. The analogous procedure for bringing one of the lines of a configuration on top of all others is explained in Penne [25].

3. DECOMPOSABLE AND HORIZONTAL ARRANGEMENTS

In this section we look at arrangements that can be obtained by a sequence of cabling operations from simpler arrangements, and also at arrangements whose corresponding configurations are “horizontal”. We consider in more detail the subclass of completely decomposable arrangements, and obtain a normal form for those of depth 2.

3.1. Let us start by recalling the following notion from knot theory (see [4]). Let $L = L_1 \cup \dots \cup L_n$ be a link in \mathbb{S}^3 . The (a, b) -cable of L about the k^{th} component is the link $L\{a, b\} = L \cup K(a, b)$, where $K(a, b)$ is an (a, b) -torus link contained in the boundary of a tubular neighborhood of L_k .

Now let \mathcal{A} be a 2-arrangement of n planes in \mathbb{R}^4 , with defining polynomial $f = f_1 \cdots f_n$. Fix an index $1 \leq k \leq n$, a positive integer r , and a number $\epsilon = \pm 1$. Given these data, we define the ϵr -cable about the k^{th} component of \mathcal{A} to be the arrangement $\mathcal{A}^k\{\epsilon r\}$ with defining polynomial

$$f(f_k + g_1) \cdots (f_k + g_r),$$

where each g_j is a linear form in z, \bar{z}, w, \bar{w} , whose coefficients are sufficiently small with respect to those of f , and such that $\text{sgn}(\det(f_k, g_j)) = \epsilon$, for $j = 1, \dots, r$.

The cabling operation is well-defined up to rigid isotopy of arrangements. The reverse operation is called decabbling. It is readily seen that the link of $\mathcal{A}^k\{\pm r\}$ is the $(r, \pm r)$ -cable about the k^{th} component of $L(\mathcal{A})$.

Definition 3.2. A 2-arrangement for which no decabbling is possible is called *indecomposable*; otherwise, it is called *decomposable*. If \mathcal{A} is connected to the trivial arrangement \mathcal{A}_1 by a finite sequence of cabling moves, then \mathcal{A} is called *completely decomposable*.

Example 3.3. The complex arrangement \mathcal{A}_n is the $(n - 1)$ -cable of \mathcal{A}_1 , and thus is completely decomposable. Its link is the corresponding cable about the unknot. The arrangement \mathcal{A}^- from Example 2.6 is the (-1) -cable of \mathcal{A}_3 , and thus is also completely decomposable.

3.4. We now define 2-arrangements in \mathbb{R}^4 corresponding to special collections of skew lines in \mathbb{R}^3 , variously called join configurations [28], horizontal configurations [22], or spindle configurations [8].

Definition 3.5. A configuration is called *horizontal* if it is rigidly isotopic to a configuration whose lines are stacked one over another in distinct planes, all parallel to a fixed (horizontal) plane. A 2-arrangement which admits an associated horizontal configuration is called *horizontal*.

A horizontal configuration \mathcal{C} of n lines determines a permutation $\tau = \tau(\mathcal{C})$ on $\{1, \dots, n\}$, as follows. Project perpendicularly all lines onto a fixed horizontal plane. Order these n lines in decreasing order of their (necessarily distinct) slopes. Order the n horizontal planes containing the lines in increasing order of their vertical heights. For every $i \in \{1, \dots, n\}$, put $\tau_i = k$ if the i^{th} line is contained in the k^{th} horizontal plane. This defines the permutation $\tau \in S_n$.

Conversely, every permutation $\tau \in S_n$ determines a horizontal configuration $\mathcal{C}(\tau)$ (see [9, 22]), and thereby a horizontal arrangement $\mathcal{A}(\tau)$. Explicitly, $\mathcal{A}(\tau)$ may be defined as follows.

Proposition 3.6. *Let $\tau \in S_n$. Choose real numbers a_i, b_i , $1 \leq i \leq n$, so that $a_1 < \dots < a_n$ and $b_{\tau_1} < \dots < b_{\tau_n}$. Then the polynomial*

$$(3.1) \quad f(z, w) = \prod_{i=1}^n \left(z - \frac{a_i + b_i}{2} w - \frac{a_i - b_i}{2} \bar{w} \right),$$

defines a horizontal 2-arrangement, whose associated permutation is τ .

For horizontal arrangements, the linking numbers have a particularly simple interpretation. Namely, if $\mathcal{A} = \mathcal{A}(\tau)$, then $l_{i,j} = \text{sgn}(\tau_i \tau_j)$.

Example 3.7. In Example 2.6, pick the vertical coordinate to be $y = \text{Im } z$. Then the lines of the configurations \mathcal{C}^\pm are contained in horizontal planes, parallel to the plane $y = 0$. In each case, the ordering given by the slopes is $(1, 2, 3, 4)$. The ordering given by the vertical heights is $(1, 2, 3, 4)$ for \mathcal{C}^+ , and $(2, 1, 3, 4)$ for \mathcal{C}^- . Thus $\mathcal{A}^+ = \mathcal{A}(1234)$ and $\mathcal{A}^- = \mathcal{A}(2134)$. The defining polynomials corresponding to the choices $a = (1, 2, 3, 4)$, $b = (\pm 1, \pm 2, 3, 4)$ in (3.1) are the polynomials f^\pm from Example 2.6. All linking numbers $l_{i,j}^\pm$ are equal to $+1$, except for $l_{1,2}^- = -1$.

The permutation τ associated to a horizontal arrangement $\mathcal{A} = \mathcal{A}(\tau)$ is not unique. The following result of Mazurovskii [23] lists various ways in which uniqueness is known to fail.

Proposition 3.8. *Two horizontal arrangements, defined by permutations τ and τ' in S_n , are rigidly isotopic if:*

- (a) $\tau' = \sigma\tau\sigma'$, where σ and σ' are circular permutations of $(1, \dots, n)$; or
- (b) $\tau' = \tau^{-1}$; or
- (c) $\tau' = (\tau_1, \dots, \tau_{i-1}, \tau'_i, \dots, \tau'_{i+s}, \tau_{i+s+1}, \dots, \tau_n)$, where $(\tau_i, \dots, \tau_{i+s})$ is a permutation of $\{m+1, \dots, m+s+1\}$, and $(\tau'_i - m, \dots, \tau'_{i+s} - m) = (s+1, \dots, 1)(\tau_i - m, \dots, \tau_{i+s} - m)(s+1, \dots, 1)$.

Remark 3.9. We do not know whether any two rigidly isotopic horizontal arrangements can be connected by a finite sequence of moves of type (a), (b), (c). There is another set of moves, introduced by Crapo and Penne, which is conjectured to be complete for horizontal configurations, see [8], p. 80. At any rate, the precise enumeration of the cosets of S_n modulo the equivalence relation generated by either set of moves seems to be a challenging combinatorial problem.

Example 3.10. We can use moves of type (a) to realize the rigid isotopy from Remark 2.9 in the case of horizontal arrangements. Indeed, if $\mathcal{A} = \mathcal{A}(\tau)$ for some $\tau \in S_n$ with $\tau_k = n$, then we can replace τ by $\tau' = \tau(k+1, \dots, n, 1, \dots, k)$. This yields a new arrangement, $\mathcal{A}' = \mathcal{A}(\tau')$, for which $\tau'_n = n$, and $l'_{i,n} = 1$ for all $i < n$.

Example 3.11. An important example of move (c) is as follows. Suppose the block $B = (\tau_i, \dots, \tau_j)$ is obtained by concatenation of two blocks, B_1 and B_2 , of consecutive integers, each block in either increasing order (a *positive block*), or in decreasing order (a *negative block*), and so that $\min B_2 = \max B_1 + 1$. Then the block $B' = (\tau'_i, \dots, \tau'_j)$ is also a concatenation of two blocks of consecutive integers, B'_1 and B'_2 . Moreover, B'_1 is B_2 shifted down by $|B_1|$ and B'_2 is B_1 shifted up by $|B_2|$. In essence, the move $(B_1 B_2) \rightarrow (B'_1 B'_2)$ allows us to flip-and-shift adjacent blocks of consecutive integers, provided that $\min B_2 = \max B_1 + 1$.

3.12. We come now to a special class of horizontal arrangements, that can be constructed inductively from \mathcal{A}_1 by a sequence of cabling moves. Let $\mathcal{A} = \mathcal{A}(\tau)$, where $\tau \in S_n$, and $k = \tau_\ell$. An ϵr -cabling move on the k^{th} component of \mathcal{A} yields a new horizontal arrangement, $\mathcal{A}(\tau')$, where $\tau' \in S_{n+r}$ is given by

$$\tau'_i = \begin{cases} \tau_i & \text{if } i \notin \{\ell, \dots, \ell + r\} \text{ and } \tau_i < k \\ \tau_i + r & \text{if } i \notin \{\ell, \dots, \ell + r\} \text{ and } \tau_i > k \\ k + \epsilon(\ell - i) + \frac{1-\epsilon}{2}r & \text{if } i \in \{\ell, \dots, \ell + r\}. \end{cases}$$

In other words, an ϵr -cabling move on k shifts all the numbers in τ greater than k by r and replaces k by $(k, \dots, k + r)$ if $\epsilon = 1$ or by $(k + r, \dots, k)$ if $\epsilon = -1$.

Definition 3.13. A horizontal arrangement is called *completely decomposable* if the associated permutation can be obtained from the identity permutation (1) by a finite sequence of cabling moves.

It is apparent from the definitions that the link of a completely decomposable arrangement is obtained from the unknot by successive $(1, \pm 1)$ -cablings.

Example 3.14. All 2-arrangements of up to 5 planes are completely decomposable, except for $\mathcal{A}(31425)$, which is indecomposable. Among arrangements of 6 planes, for example, $\mathcal{A}(K) = \mathcal{A}(341256)$ is completely decomposable, $\mathcal{A}(314256)$ is decomposable but not completely so, and $\mathcal{A}(241536)$ is indecomposable.

3.15. We now introduce a measure of the complexity of a completely decomposable arrangement \mathcal{A} . Pick a permutation τ such that $\mathcal{A} = \mathcal{A}(\tau)$. Construct a sequence of permutations connecting τ to the identity permutation, $\tau = \tau_0 \rightarrow \tau_1 \rightarrow \dots \rightarrow \tau_d = (1)$, as follows. At each step, partition the current permutation into blocks of consecutive integers, either in increasing order (positive cablings), or in decreasing order (negative cablings), and contract each block to a single number (via decabbling moves), renumbering accordingly. Let $d(\tau) = d$.

Definition 3.16. The *depth* of a completely decomposable arrangement \mathcal{A} is

$$\text{depth}(\mathcal{A}) = \min_{\tau} \{d(\tau) \mid \mathcal{A} \text{ is rigidly isotopic to } \mathcal{A}(\tau)\}.$$

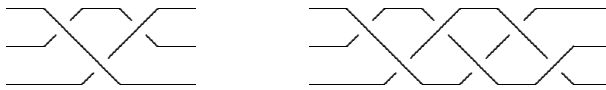
Example 3.17. The only arrangement of depth 0 is the trivial arrangement $\mathcal{A}_1 = \mathcal{A}(1)$. The arrangements of depth 1 are the complex arrangements $\mathcal{A}_n = \mathcal{A}(1 \dots n)$, with $n > 1$. The arrangement $\mathcal{A}(21435)$ is completely decomposable, via the sequence of permutations $(21435) \rightarrow (123) \rightarrow (1)$, and so has depth 2.

For arrangements of depth 2, we single out the following type.

Definition 3.18. Let $\mathcal{A}(\tau)$ be a depth 2, completely decomposable arrangement. We say that $\mathcal{A}(\tau)$ is in *normal form* if $\tau = (I_1, \dots, I_r, J)$, where I_1, \dots, I_r are negative blocks, J is a positive block, and the following conditions hold:

- (i) $I_1 < \dots < I_r < J$,
- (ii) $2 \leq |I_1| \leq \dots \leq |I_r|$,
- (iii) $|I_1| \leq |J|$ if $r = 1$.

Proposition 3.19. *Every arrangement of depth 2 is rigidly isotopic to a unique arrangement in normal form.*


 FIGURE 3. The braids α and β associated to $\mathcal{A}(213)$.

Proof. Let \mathcal{A} be a depth 2 arrangement of n planes. Up to rigid isotopy, we may assume that $\mathcal{A} = \mathcal{A}(\tau)$, where $d(\tau) = 2$. Applying the type (a) move of Example 3.10, we may further assume that n is fixed by τ . Then the permutation sequence of $\mathcal{A} = \mathcal{A}(\tau)$ has the following form: $\tau \rightarrow (1, \dots, r) \rightarrow (1)$. Applying repeatedly the type (c) move of Example 3.11, we can push all the positive blocks of τ (including singletons) to the right, packing all of them into a single positive block (that will contain n), and also arrange the negative blocks in increasing order of their sizes from left to the right. In this way, we arrive at the normal form $\mathcal{A}(I_1, \dots, I_r, J)$ for \mathcal{A} . The uniqueness is guaranteed by the conditions imposed on I_1, \dots, I_r and $|J|$. \square

Thus, we may refer to *the* normal form of an arrangement of depth 2. As we shall see in §6, the normal form is a complete homotopy type invariant for complements of such arrangements.

4. BRAIDS AND FUNDAMENTAL GROUPS

In this section, we associate to a 2-arrangement of n planes several braids on n strings, and use these braids to find presentations for the fundamental group of the complement.

4.1. Let B_n be Artin's braid group on n strings, with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and relations $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i - j| = 1$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$, see [2]. Also, let $\Delta_n = (\sigma_{n-1} \cdots \sigma_1)(\sigma_{n-1} \cdots \sigma_2) \cdots (\sigma_{n-1} \sigma_{n-2})(\sigma_{n-1}) \in B_n$ be “Garside's braid”—the half-twist on n strings.

Consider a configuration $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$ of n skew-lines in \mathbb{R}^3 . Associated to \mathcal{C} , there is a braid on n strings, $\alpha = \alpha(\mathcal{C}) \in B_n$, see Mazurovskii [22] and Crapo and Penne [8]. The procedure that takes \mathcal{C} to α is illustrated in Figures 1 and 2. Set $\beta = \alpha \Delta_n \alpha \Delta_n^{-1}$. We call α and β , the *half-braid*, respectively the *full-braid* of the configuration \mathcal{C} (or of the arrangement $\mathcal{A} = \mathcal{A}(\mathcal{C})$). As is well-known, conjugation by Δ_n is the involution $\sigma_i \mapsto \sigma_{n-i}$. Thus, the braid β is obtained by concatenating α with another copy of α , rotated by 180° , see Figure 3. Clearly, β is a pure braid in P_n .

The following result of Mazurovskii [22] and Crapo and Penne [8] establishes the direct connection between the link and the braid of an arrangement. First recall the classical theorem of Alexander, according to which every link in \mathbb{S}^3 is isotopic to the closure of a braid (see [2]).

Proposition 4.2. *Let \mathcal{A} be a 2-arrangement in \mathbb{R}^4 and $L = L(\mathcal{A})$ its link. Let $\mathcal{C} = \mathcal{C}(\mathcal{A})$ be the associated configuration of skew lines in \mathbb{R}^3 and $\beta = \beta(\mathcal{C})$ its full-braid. Then L is isotopic to the closure of β .*

Let X be the complement of the arrangement \mathcal{A} , and $G = \pi_1(X)$ its fundamental group. Recall that X is homotopy equivalent to the complement Y of the link L .

Since L is the closure of β , the group G has Artin presentation

$$(4.1) \quad G = \langle x_1, \dots, x_n \mid \beta(x_i) = x_i, i = 1, \dots, n \rangle,$$

see [2, 4].

4.3. As mentioned in Remark 2.9, we can bring one of the lines of \mathcal{C} , say ℓ_n , on top of all the other ones. Discarding ℓ_n , we get a configuration $\check{\mathcal{C}}$ of $n - 1$ skew lines, so that $\mathcal{C} = \mathcal{C}' \cup \{\ell_n\}$. It follows that $L = \check{L} \cup L_n$, where \check{L} is the closure of $\check{\beta} = \beta(\check{\mathcal{C}}) \in P_{n-1}$. Furthermore, it is readily seen that the half-braid of $\check{\mathcal{C}}$ is given by $\iota(\check{\alpha}) = \sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \alpha$, where $\iota : B_{n-1} \hookrightarrow B_n$ is the standard inclusion $\iota(\sigma_i) = \sigma_{i+1}$. We call $\check{\alpha}$ and $\check{\beta}$, the *reduced half-braid*, respectively the *reduced full-braid* of the arrangement $\mathcal{A} = \mathcal{A}(\mathcal{C})$.

It is now apparent that the complement of L in \mathbb{S}^3 is homotopy equivalent to the complement of \check{L} in the solid torus $\mathbb{S}^1 \times D^2 = \mathbb{S}^3 \setminus (L_n \times D^2)$. These geometric considerations lead to the following:

Proposition 4.4. *The complement $X(\mathcal{A})$ of a 2-arrangement of n planes in \mathbb{R}^4 is homotopy equivalent to the total space of a bundle over the circle, with fiber $D^2 \setminus \{n - 1 \text{ points}\}$, and monodromy the braid automorphism $\check{\beta}$.*

Thus, X is a $K(G, 1)$, with fundamental group a semidirect product of free groups, $G = \mathbb{F}_{n-1} \rtimes_{\check{\beta}} \mathbb{F}_1$. The Artin representation of $\check{\beta}$ provides a presentation for G , corresponding to this split extension:

$$G = \langle x_1, \dots, x_n \mid x_n^{-1} x_i x_n = \check{\beta}(x_i), i = 1, \dots, n - 1 \rangle.$$

Example 4.5. For the complex arrangement \mathcal{A}_n , the half-braid is the half-twist $\alpha = \Delta_n$, and the full-braid is the full-twist $\beta = \Delta_n^2$. Since $\check{\beta} = \Delta_{n-1}^2$ acts on \mathbb{F}_{n-1} by conjugation by $x_1 \cdots x_{n-1}$, the group G is isomorphic to $\mathbb{F}_{n-1} \times \mathbb{F}_1$, where $\mathbb{F}_1 = \langle x_1 \cdots x_n \rangle$.

For a non-complex 2-arrangement, the group G is in general *not* isomorphic to a direct product, as we shall later see. Nevertheless, we may still use the underlying idea of Example 4.5, and simplify the presentation of G , by cutting off a full twist from $\check{\beta}$.

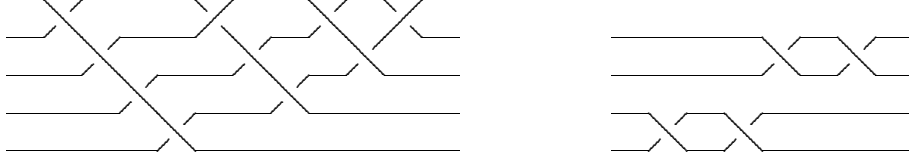
Proposition 4.6. *Let \mathcal{A} be a 2-arrangement of n planes, with reduced half-braid $\check{\alpha}$. Set $\xi = \Delta_{n-1} \check{\alpha}^{-1}$. Then, the fundamental group G of \mathcal{A} is isomorphic to $\mathbb{F}_{n-1} \rtimes_{\xi^2} \mathbb{F}_1$, and has presentation*

$$(4.2) \quad G = \langle x_1, \dots, x_n \mid x_n x_i x_n^{-1} = \xi^2(x_i), i = 1, \dots, n - 1 \rangle.$$

Proof. Recall that $G_1 \rtimes_{\phi} G_2 \cong G_1 \rtimes_{\phi'} G_2$ if $\phi' = \gamma \phi^{\pm 1}$, where $\gamma \in \text{Inn}(G_1)$. Thus, it suffices to show that \check{b} differs from ξ^{-2} by an inner automorphism of \mathbb{F}_{n-1} . This follows from the fact that $\Delta_{n-1}^2 \in \text{Center}(B_{n-1}) \cap \text{Inn}(\mathbb{F}_{n-1})$:

$$\check{b} = \check{\alpha} \Delta_{n-1} \check{\alpha} \Delta_{n-1} = \xi^{-1} \Delta_{n-1}^2 \xi^{-1} = \Delta_{n-1}^2 \xi^{-2}. \quad \square$$

Remark 4.7. Recall also that $G_1 \rtimes_{\phi} G_2 \cong G_1 \rtimes_{\phi'} G_2$ if $\phi' = \psi \phi \psi^{-1}$. We can use this observation to further simplify the above presentation, by conjugating $\xi \in P_{n-1}$ by a suitable automorphism of \mathbb{F}_{n-1} . In practice, this will be achieved by either changing the basis of \mathbb{F}_{n-1} , or by conjugating ξ by a suitable braid $\delta \in B_{n-1}$.


 FIGURE 4. The braids α and ξ associated to $\mathcal{A}(21435)$.

4.8. We now identify the braids $\alpha \in B_{n+1}$ and $\xi \in P_n$ associated to a horizontal arrangement of $n + 1$ planes in terms of the generators σ_i of the braid group and of the generators $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ of the pure braid group, respectively.

Proposition 4.9. *Let $\mathcal{A}(\tau)$ be a horizontal 2-arrangement of $n + 1$ planes. Then:*

(a) *The half-braid α has the form*

$$\alpha = (\sigma_n^{l_{n,n+1}} \cdots \sigma_1^{l_{1,n+1}})(\sigma_n^{l_{n-1,n}} \cdots \sigma_2^{l_{1,n}}) \cdots (\sigma_n^{l_{2,3}} \sigma_{n-1}^{l_{1,3}})(\sigma_n^{l_{1,2}}),$$

where $l_{i,j}$ is the sign of the permutation $(\tau_i \tau_j)$.

(b) *The pure braid ξ can be combed as $\xi = \xi_2 \cdots \xi_n$, where*

$$\xi_j = \prod_{i=1}^{j-1} A_{i,j}^{e_{i,j}} \quad \text{and} \quad e_{i,j} = \begin{cases} 1 & \text{if } \tau_i > \tau_j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Part (a) follows from the definitions of $\mathcal{C} = \mathcal{C}(\tau)$ and $\alpha = \alpha(\mathcal{C})$, and the fact that the linking numbers of $L(\mathcal{A}(\tau))$ are given by $l_{i,j} = \text{sgn}(\tau_i \tau_j)$.

For part (b), it is enough to show that, for all k with $1 \leq k \leq n - 1$,

$$(4.3) \quad \Delta_n (\sigma_{n-1}^{-l_{1,2}})(\sigma_{n-2}^{-l_{1,3}} \sigma_{n-1}^{-l_{2,3}}) \cdots (\sigma_{n-k}^{-l_{1,k+1}} \cdots \sigma_{n-1}^{-l_{k,k+1}}) = \\ (A_{1,2}^{e_{1,2}}) \cdots (A_{1,k+1}^{e_{1,k+1}} \cdots A_{k,k+1}^{e_{k,k+1}}) (\sigma_{k+1} \cdots \sigma_{n-1}) \cdots (\sigma_2 \cdots \sigma_{n-k}) \Delta_{n-k},$$

Indeed, the identity (4.3) for $k = n - 1$ yields the desired combed form of ξ .

The proof of (4.3) is by induction on k , using the braid relations. The step $k = 1$ is as follows:

$$\begin{aligned} \Delta_n \sigma_{n-1}^{\pm 1} &= (\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1 \sigma_2)(\sigma_1) \sigma_{n-1}^{\pm 1} \\ &= (\sigma_1)(\sigma_2 \sigma_1) \cdots (\sigma_{n-2} \sigma_{n-3})(\sigma_{n-1} \sigma_{n-2}) \sigma_{n-1}^{\pm 1} (\sigma_1 \cdots \sigma_{n-3}) \cdots (\sigma_1 \sigma_2)(\sigma_1) \\ &= (\sigma_1)(\sigma_2 \sigma_1) \cdots (\sigma_{n-2} \sigma_{n-3}) \sigma_{n-2}^{\pm 1} (\sigma_{n-1} \sigma_{n-2}) (\sigma_1 \cdots \sigma_{n-3}) \cdots (\sigma_1 \sigma_2)(\sigma_1) \\ &\quad \cdots \\ &= (\sigma_1) \sigma_1^{\pm 1} (\sigma_2 \sigma_1) \cdots (\sigma_{n-2} \sigma_{n-3})(\sigma_{n-1} \sigma_{n-2}) (\sigma_1 \cdots \sigma_{n-3}) \cdots (\sigma_1 \sigma_2)(\sigma_1) \\ &= A_{1,2}^e (\sigma_2 \cdots \sigma_{n-1}) \Delta_{n-1}, \end{aligned}$$

where $e = \frac{1 \pm 1}{2}$. The induction step is similar but tedious, and will be omitted. \square

Example 4.10. The complex arrangement \mathcal{A}_n is horizontal, with corresponding permutation the identity $\tau = (1 \cdots n)$. Thus $\alpha = \Delta_n$ and $\xi = 1$. The arrangement $\mathcal{A} = \mathcal{A}^-$ from Example 2.6 is also horizontal, with permutation $\tau = (2134)$. Thus $\alpha = \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3^{-1}$ and $\xi = A_{1,2}$. The braids $\alpha = \sigma_4 \sigma_3 \sigma_2 \sigma_1 \sigma_4^{-1} \sigma_3 \sigma_2 \sigma_4 \sigma_3 \sigma_4^{-1}$ and $\xi = A_{1,2} A_{3,4}$ associated to the horizontal arrangement $\mathcal{A}(21435)$ are illustrated in Figure 4.

5. DETERMINANTAL IDEALS AND CHARACTERISTIC VARIETIES

We start this section with a review of the determinantal ideals of the Alexander module of a space, following Hillman [15] and Turaev [26, 27]. From the varieties defined by these ideals, we extract numerical homotopy-type invariants, that will be used for the rest of this paper.

5.1. Let X be a connected, finite CW-complex, with basepoint $*$, and fundamental group $\pi_1(X, *)$. Let $p : \tilde{X} \rightarrow X$ be the universal abelian cover, corresponding to the abelianization homomorphism $\text{ab} : \pi_1(X, *) \rightarrow H_1(X; \mathbb{Z})$. The relative homology group $A(X) = H_1(\tilde{X}, p^{-1}(*) ; \mathbb{Z})$ has the structure of a (left) module over the group ring $\mathbb{Z}[H_1(X; \mathbb{Z})]$, and is known as the *Alexander module* of X .

Now assume that $H_1(X, \mathbb{Z})$ is isomorphic to \mathbb{Z}^n , the free abelian group on t_1, \dots, t_n . A choice of isomorphism, $\psi : H_1(X) \xrightarrow{\cong} \mathbb{Z}^n$, identifies $\mathbb{Z}[H_1(X)]$ with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the ring of Laurent polynomials in n variables, and defines a Λ -module structure on the Alexander module of X , which we will denote by $A(X, \psi)$. From a presentation of the fundamental group, $\pi_1(X) = \langle x_1, \dots, x_q \mid r_1, \dots, r_s \rangle$, one gets a presentation for the Alexander module,

$$\Lambda^s \xrightarrow{M} \Lambda^q \rightarrow A(X, \psi) \rightarrow 0,$$

where $M = \left(\partial r_i / \partial x_j \right)^{\text{ab}}$ is the abelianized Jacobian matrix of Fox derivatives.

Define the k^{th} *determinantal ideal* of $A_\psi(X)$ to be the ideal $E_k(X, \psi)$ generated by the codimension k minors of the Alexander matrix M . Clearly, $E_k(X, \psi) \subseteq E_\ell(X, \psi)$ if $k \leq \ell$. The determinantal ideals depend only on the homotopy type of X (in fact, only on its fundamental group), and on the identification $\psi : H_1(X) \rightarrow \mathbb{Z}^n$.

If $\pi_1(X)$ has positive deficiency (i.e., admits a presentation with more generators than relations), then $E_1(X, \psi)$ is of the form $I \cdot (\Delta_{X, \psi})$, where I is the augmentation ideal of Λ , and $\Delta_{X, \psi} \in \Lambda$ is the (multi-variable) *Alexander polynomial* of X , see [11].

5.2. We now associate to X subvarieties $V_k(X, \psi)$ of the algebraic torus $(\mathbb{C}^*)^n$, defined by the determinantal ideals $E_k(X, \psi)$, following [10, 17]. The coordinate ring of $(\mathbb{C}^*)^n$ is $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$, the ring of Laurent polynomials with complex coefficients. Then, for each $k \geq 0$, we set

$$V_k(X, \psi) = \{(t_1, \dots, t_n) \in (\mathbb{C}^*)^n \mid g(t_1, \dots, t_n) = 0, \text{ for all } g \in \sqrt{E_k(X, \psi) \otimes \mathbb{C}}\},$$

where $\sqrt{\mathfrak{a}}$ denotes the radical of an ideal \mathfrak{a} . Clearly, $V_k(X, \psi) \supseteq V_\ell(X, \psi)$ if $k \leq \ell$.

Definition 5.3. Two algebraic subvarieties V and V' of $(\mathbb{C}^*)^n$ are said to have the same *monomial isomorphism type* if there exists an automorphism $\phi_A : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ of the form

$$\phi_A(t_i) = t_1^{a_{i1}} \cdots t_n^{a_{in}}, \quad 1 \leq i \leq n,$$

for some matrix $A = (a_{i,j}) \in \text{GL}_n(\mathbb{Z})$, which maps V into V' .

Proposition 5.4. *The monomial isomorphism type of the subvariety $V_k(X, \psi)$ of the algebraic torus $(\mathbb{C}^*)^n$ depends only on the isomorphism type of $\pi_1(X)$, and not on the identification $\psi : H_1(X) \rightarrow \mathbb{Z}^n$. We call $V_k(X) = V_k(X, \psi)$ the k^{th} characteristic variety of X .*

Proof. Let X and Y be connected, finite CW-complexes, and let $h : \pi_1(X) \rightarrow \pi_1(Y)$ be an isomorphism. Let $h_* : H_1(X) \rightarrow H_1(Y)$ be the abelianization of h , and set $\bar{h} = \psi_Y h_* \psi_X^{-1} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$. The extension of \bar{h} to $\Lambda_{\mathbb{C}} = \mathbb{C}\mathbb{Z}^n$ restricts to an isomorphism $E_k(X, \psi_X) \otimes \mathbb{C} \rightarrow E_k(Y, \psi_Y) \otimes \mathbb{C}$, for each $k \geq 0$. Now let ϕ be the (monomial) automorphism of $(\mathbb{C}^*)^n$ induced by \bar{h} . Clearly, ϕ restricts to an isomorphism $V_k(X, \psi_X) \rightarrow V_k(Y, \psi_Y)$. \square

In other words, for each $k \geq 0$, the monomial isomorphism type of $V_k(X)$ is an isomorphism type of $\pi_1(X)$, and thus, a homotopy-type invariant of X . Furthermore, if $\pi_1(X)$ has positive deficiency, the Alexander polynomial $\Delta_X = \Delta_{X, \psi}$ is well-defined up to a monomial change of basis in $(\mathbb{C}^*)^n$, and up to multiplication by a unit $ct_1^{i_1} \cdots t_n^{i_n} \in \Lambda_{\mathbb{C}}$. Note that $V_1(X) = \mathbf{1} \cup \{\Delta_X = 0\}$, where $\mathbf{1} = (1, \dots, 1)$ is the origin of the complex torus $(\mathbb{C}^*)^n$.

5.5. By themselves, the characteristic varieties are not very practical homotopy-type invariants. We extract from them several numerical invariants that are powerful enough for our purposes. For each integer $p \geq 2$, let

$$\Omega_p^n = \{(\omega_1, \dots, \omega_n) \in (\mathbb{C}^*)^n \mid \omega_i \text{ is a } p^{\text{th}} \text{ root of unity}\}$$

be the set of p -torsion points of $(\mathbb{C}^*)^n$.

Theorem 5.6. *The following are isomorphism type invariants of $\pi_1(X)$:*

- (a) *The list $\Sigma_k(X)$ of codimensions of irreducible components of $V_k(X)$;*
- (b) *The list $\Sigma_{\mathbf{1}, k}(X)$ of codimensions of irreducible components of $V_k(X)$ passing through $\mathbf{1}$;*
- (c) *The number $\text{Tors}_{p, k}(X) = |\Omega_p^n \cap V_k(X)|$ of p -torsion points of $V_k(X)$.*

Proof. By Proposition 5.4, an isomorphism of fundamental groups determines a monomial isomorphism of the corresponding characteristic varieties. Part (a) follows from the fact that an isomorphism of algebraic varieties sends irreducible components to irreducible components of the same codimension. Part (b) follows from Part (a), and the fact that a monomial isomorphism fixes $\mathbf{1}$. Part (c) follows from the fact that a monomial isomorphism preserves the set of p -torsion points. \square

5.7. Now let $X = X(\mathcal{A})$ be the complement of a 2-arrangement of n planes in \mathbb{R}^4 . Recall that X has the homotopy type of a 2-complex (modeled on the Artin presentation of its fundamental group G), and that $H_1(X) = \mathbb{Z}^n$. Thus, we can define the k^{th} characteristic variety of \mathcal{A} to be $V_k(\mathcal{A}) = V_k(X)$. As we shall see, the descending tower of characteristic varieties has the form $(\mathbb{C}^*)^n = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{n-2} \supseteq V_{n-1} \supseteq V_n = \emptyset$, with V_1 being a hypersurface in $(\mathbb{C}^*)^n$, if $n \geq 3$, and V_{n-1} consisting of the single point $\mathbf{1}$, if $n \geq 2$. We will focus on the nontrivial ends of the tower, namely V_1 and V_{n-2} , which we shall call the *top*, respectively the *bottom* characteristic variety of \mathcal{A} .

In order to find explicit equations for the characteristic varieties, we need to choose a particular presentation for $G = \pi_1(X)$. Unless otherwise specified, we shall use the presentation (4.2) associated to the semidirect product structure $G = \mathbb{F}_{n-1} \rtimes_{\xi^2} \mathbb{F}_1$ from Proposition 4.6. This presentation yields an identification $\psi_{\xi} : H_1(X) \rightarrow \mathbb{Z}^n$. Let $A = A(X, \psi_{\xi})$ be the corresponding Alexander module. A presentation matrix for A is the $(n-1) \times n$ (Alexander) matrix

$$M = \begin{pmatrix} t_n \cdot \text{id} - \Theta(\xi^2) & d_1 \end{pmatrix},$$

where $d_1 = (1 - t_1 \cdots 1 - t_{n-1})^\top$ and $\Theta : P_{n-1} \rightarrow \mathrm{GL}_{n-1}(\Lambda)$ is the Gassner representation of the pure braid group, see Birman [2].

The $k \times k$ minors of M generate the determinantal ideal E_k , whose radical, $\sqrt{E_k}$, defines the k^{th} characteristic variety $V_k = V_k(\mathcal{A})$. Note that $E_{n-1} = I$ and $E_n = \Lambda$, and so $V_{n-1} = \mathbf{1}$ and $V_n = \emptyset$.

Now recall that a link group has deficiency 1, see e.g. [4]. Thus we may define the *Alexander polynomial of \mathcal{A}* to be $\Delta_{\mathcal{A}} = \Delta_{X, \psi_\xi}$. For $n = 1$, we have $\Delta_{\mathcal{A}} = 1$. For $n > 1$, we have

$$\Delta_{\mathcal{A}}(t_1, \dots, t_n) = \frac{1}{t_n - 1} \det(t_n \cdot \mathrm{id} - \Theta(\xi^2)),$$

see Penne [25]. Thus $\Delta_{\mathcal{A}} = 1$ for $n = 2$. For $n \geq 3$, the triviality of the Gassner representation evaluated at $\mathbf{1}$ implies that $\mathbf{1} \in V_1(\mathcal{A})$ and so $V_1(\mathcal{A}) = \{\Delta_{\mathcal{A}}(t_1, \dots, t_n) = 0\}$.

Remark 5.8. For certain purposes, it is more natural to start from the Artin presentation (4.1) associated to the semidirect product structure $G = \mathbb{F}_{n-1} \rtimes_{\beta} \mathbb{F}_1$. The resulting presentation, $A(X, \psi_\beta)$, for the Alexander module coincides with the usual presentation of the Alexander module of the link $L(\mathcal{A})$. We will denote the associated Alexander polynomial by $\Delta_{L(\mathcal{A})} = \Delta_{X, \psi_\beta}$.

Example 5.9. Let \mathcal{A}_n be the arrangement of $n \geq 3$ complex lines through the origin of \mathbb{C}^2 . Recall that $\xi = 1$ and $\beta = \Delta_n^2$ in this case. It is readily seen that $E_k = I \cdot (t_n - 1)^{n-k-1}$. Thus $V_1 = \cdots = V_{n-2} = \{t_n - 1 = 0\}$, and $\Delta_{\mathcal{A}_n} = (t_n - 1)^{n-2}$, whereas $\Delta_{L(\mathcal{A}_n)} = (t_1 \cdots t_n - 1)^{n-2}$.

Example 5.10. Let \mathcal{A} be the arrangement $\mathcal{A}^- = \mathcal{A}(2134)$. Recall that $\xi = A_{1,2}$. The Artin representation of $\xi : \mathbb{F}_3 \rightarrow \mathbb{F}_3$ is given by $\xi(x_1) = x_1 x_2 x_1 x_2^{-1} x_1^{-1}$, $\xi(x_2) = x_1 x_2 x_1^{-1}$, $\xi(x_3) = x_3$. Consider the new basis $y_1 = x_1$, $y_2 = x_1 x_2$, $y_3 = x_3$ for \mathbb{F}_3 . In this basis, $\xi(y_1) = y_2 y_1 y_2^{-1}$, $\xi(y_2) = y_2$, $\xi(y_3) = y_3$, and so the Alexander matrix is:

$$M = \begin{pmatrix} t_4 - t_2^2 & (t_2 + 1)(t_1 - 1) & 0 & 1 - t_1 \\ 0 & t_4 - 1 & 0 & 1 - t_2 \\ 0 & 0 & t_4 - 1 & 1 - t_3 \end{pmatrix}.$$

The determinantal ideals are $E_1 = I \cdot (t_4 - 1)(t_4 - t_2^2)$, and $E_2 = I \cdot (t_4 - 1, t_2^2 - 1) + \Lambda \cdot (t_2 + 1)(t_1 - 1)(t_3 - 1)$. The characteristic varieties are

$$\begin{aligned} V_1 &= \{t_4 - 1 = 0\} \cup \{t_4 - t_2^2 = 0\}, \\ V_2 &= \{t_4 - 1 = t_2 + 1 = 0\} \cup \{t_4 - 1 = t_2 - 1 = t_1 - 1 = 0\} \\ &\quad \cup \{t_4 - 1 = t_2 - 1 = t_3 - 1 = 0\}. \end{aligned}$$

Now let \mathcal{A} be the arrangement $\mathcal{A}^+ = \mathcal{A}(1234)$. We know from Example 5.9 that its characteristic varieties are $V_1 = V_2 = \{t_4 - 1 = 0\}$. We plainly see that the characteristic varieties of \mathcal{A}^+ and \mathcal{A}^- have a different number of components, and so are not isomorphic. Thus, $X^+ \not\cong X^-$. In fact, $H^*(X^+) \not\cong H^*(X^-)$, as was shown by Ziegler [30].

6. BOTTOM CHARACTERISTIC VARIETIES

In this section we study the bottom characteristic varieties $V_{n-2}(\mathcal{A})$ of arrangements of n planes that are obtained from the trivial arrangement by a sequence of

cablings operations. We obtain a complete characterization of these varieties when the sequence has length 2.

6.1. Let \mathcal{A} be a depth 2 arrangement, and $\mathcal{A}(I_1, \dots, I_r, J)$ its normal form, as introduced in Definition 3.18. By Proposition 4.9, the pure braid ξ can be taken to be $\xi = A_{I_1} \cdots A_{I_r}$, where A_I is the full twist on the strings I , with the convention that whenever a negative block appears as a subindex of a braid generator, it will be understood as a set of integers in increasing order.

Let us first consider a full twist A_I corresponding to the indexing set $I = \{j, j+1, \dots, k\}$. The Artin representation of A_I is given by $A_I(x_i) = x_I x_i x_I^{-1}$, $i \in I$ and $A_I(x_i) = x_i$ otherwise, where $x_I = x_j \cdots x_k$. The change of basis $y_k = x_I$ and $y_i = x_i$ for $i \neq k$ simplifies A_I to $A_I(y_i) = y_k y_i y_k^{-1}$, $i \in I$ and $A_I(y_i) = y_i$ otherwise. The Gassner representation of A_I^2 in this new basis is given by:

$$(6.1) \quad \Theta(A_I^2) = \begin{pmatrix} t_k^2 & \cdots & 0 & (t_k+1)(1-t_j) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_k^2 & (t_k+1)(1-t_{k-1}) \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Now notice that the factors A_{I_1}, \dots, A_{I_r} of ξ braid on mutually disjoint groups of strings. Therefore, we can change the basis in the free group for each block separately, as above. Hence, the Gassner representation $\Theta(\xi^2)$ is a block-diagonal matrix, with blocks as in (6.1). The Alexander matrix is:

$$(6.2) \quad M = \begin{pmatrix} t_n - \Theta(A_{I_1}^2) & \cdots & 0 & 0 & d_1(I_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & t_n - \Theta(A_{I_r}^2) & 0 & d_1(I_r) \\ 0 & \cdots & 0 & t_n - \text{id}_{|J|-1} & d_1(J) \end{pmatrix}$$

where $d_1(I)$ is the column vector whose entries are $t_i - 1$, for $i \in I$.

The radical $\sqrt{E_{n-2}}$ of the ideal of 2×2 -minors of M is generated by

$$t_n - 1; \quad t_{k_p}^2 - 1; \quad (t_{k_p} + 1)(t_{i_p} - 1)(t_{j_p} - 1),$$

where $k_p = \max I_p$, for all $i_p \in I_p \setminus \{k_p\}$, $j_p \notin I_p$, and $1 \leq p \leq r$.

The ideal $\sqrt{E_{n-2}}$ defines the characteristic variety V_{n-2} . In order to describe this subvariety of $(\mathbb{C}^*)^n$, we need some notation. Given a subset I of $[1, n] = \{1, \dots, n\}$, let $\bar{I} = [1, n] \setminus I$, and $\check{I} = \bar{I} \cup \{\max I\}$. Also, let $T(I)$ be the subtorus of $(\mathbb{C}^*)^n$ given by $T(I) = \{t_i - 1 = 0 \mid i \in I\}$, and $\bar{T}(I)$ the translated subtorus given by $\bar{T}(I) = \{t_i + 1 = 0 \mid i \in I\}$.

Proposition 6.2. *Let \mathcal{A} be a depth 2, completely decomposable arrangement of n planes, with normal form $\mathcal{A}(I_1, \dots, I_r, J)$. The bottom characteristic variety $V_{n-2}(\mathcal{A})$ has the following irreducible components:*

(a) *Subtori passing through $\mathbf{1}$:*

$$T(\check{I}_p), \text{ for } p \in [1, r] \quad \text{and} \quad T(\check{J}), \text{ if } |J| > 1.$$

(b) *Translated subtori:*

$$T(\cup_{p \notin P} I_p \cup \{n\}) \cap \overline{T}(\cup_{p \in P} \{\max I_p\}), \text{ for } \emptyset \neq P \subseteq [1, r].$$

Example 6.3. The arrangement $\mathcal{A} = \mathcal{A}(214356)$ from Example 3.17 is in normal form, with $I_1 = \{2, 1\}$, $I_2 = \{4, 3\}$, and $J = \{5, 6\}$. The components of $V_4(\mathcal{A})$ are:

$$\begin{aligned} &\{t_6 - 1 = t_4 + 1 = t_2 + 1 = 0\}, \\ &\{t_6 - 1 = t_4 + 1 = t_2 - 1 = t_1 - 1 = 0\}, \\ &\{t_6 - 1 = t_4 - 1 = t_3 - 1 = t_2 + 1 = 0\}, \\ &\{t_6 - 1 = t_5 - 1 = t_4 - 1 = t_3 - 1 = t_2 - 1 = 0\}, \\ &\{t_6 - 1 = t_5 - 1 = t_4 - 1 = t_2 - 1 = t_1 - 1 = 0\}, \\ &\{t_6 - 1 = t_4 - 1 = t_3 - 1 = t_2 - 1 = t_1 - 1 = 0\}. \end{aligned}$$

6.4. Let \mathcal{A} be a completely decomposable arrangement of depth 2, with normal form $\mathcal{A}(I_1, \dots, I_r, J)$. Recall that $|I_1| \leq \dots \leq |I_r|$. Define

$$S(\mathcal{A}) = \{|I_1|, \dots, |I_r|\}.$$

Since we also have $I_1 \leq \dots \leq I_r < J$, the ordered list $S(\mathcal{A})$, together with the number of planes, $n = |J| + \sum_{k=1}^r |I_k|$, determines the normal form.

Let $\Sigma = \Sigma_{n-2}(\mathcal{A})$ be the list of codimensions of irreducible components of $V_{n-2}(\mathcal{A})$, and $\Sigma_1 = \Sigma_{1, n-2}(\mathcal{A})$ be the sublist corresponding to components passing through $\mathbf{1}$. From Proposition 6.2, we see that

$$(6.3a) \quad \Sigma_1 = \{n + 1 - |I_p|\}_{p=1, \dots, r} \quad \text{if } |J| = 1$$

$$(6.3b) \quad \Sigma_1 = \{n + 1 - |I_p|\}_{p=1, \dots, r} \cup \{n + 1 - |J|\} \quad \text{if } |J| > 1$$

$$(6.3c) \quad \Sigma \setminus \Sigma_1 = \{r + 1 + \sum_{p \notin P} (|I_p| - 1)\}_{\emptyset \subsetneq P \subseteq [1, r]}$$

The lists Σ_1 and Σ have lengths

$$(6.4) \quad d_1 = r + \epsilon_J + 1, \quad d = 2^r + r + \epsilon_J,$$

where $\epsilon_J = 0$ if $|J| > 1$ and $\epsilon_J = -1$ if $|J| = 1$.

Theorem 6.5. *The list $S(\mathcal{A})$ is a complete homotopy-type invariant for depth 2, completely decomposable arrangements \mathcal{A} of n planes.*

Proof. Let \mathcal{A}' be another completely decomposable arrangement of depth 2, with normal form $\mathcal{A}(I'_1, \dots, I'_r, J')$. Assume $X(\mathcal{A}) \simeq X(\mathcal{A}')$. Then, by Theorem 5.6, $\Sigma(\mathcal{A}) = \Sigma(\mathcal{A}')$ and $\Sigma_1(\mathcal{A}) = \Sigma_1(\mathcal{A}')$. We want to show that $S(\mathcal{A}) = S(\mathcal{A}')$. There are four cases to consider, according to the sizes of J and J' :

- $|J| = 1, |J'| > 1$. Then, by (6.4), the system of equations $d = d', d_1 = d'_1$ has no solution.
- $|J| = |J'| = 1$. Then, by (6.3a), $\Sigma_1(\mathcal{A}) = \Sigma_1(\mathcal{A}')$ implies $S(\mathcal{A}) = S(\mathcal{A}')$.
- $|J| = |J'| > 1$. Then, (6.3b), $\Sigma_1(\mathcal{A}) = \Sigma_1(\mathcal{A}')$ implies $S(\mathcal{A}) = S(\mathcal{A}')$.
- $|J| > |J'| > 1$. Then, by (6.4), $r = r'$. If $r = 1$, equation (6.3b) implies that $\{|I_1|, |J|\} = \{|I'_1|, |J'|\}$ as *unordered* lists. But condition (iii) from Definition 3.18 and the fact that $|J| > |J'|$ rule out this possibility. If $r > 1$, equation (6.3c) implies $|J| = n + r + 2 - \max(\Sigma \setminus \Sigma_1) - \min(\Sigma \setminus \Sigma_1)$. Hence $|J| - |J'| = r - r' = 0$, which again is impossible.

Conversely, assume $S(\mathcal{A}) = S(\mathcal{A}')$. Then, as noted above, the normal forms of \mathcal{A} and \mathcal{A}' coincide. By Proposition 3.19, \mathcal{A} and \mathcal{A}' are rigidly isotopic, and thus $X(\mathcal{A}) \simeq X(\mathcal{A}')$. \square

Corollary 6.6. *The number of homotopy classes of 2-arrangements of n planes which are completely decomposable of depth at most 2 equals $p(n-1) - \lfloor (n-1)/2 \rfloor$, where $p(\cdot)$ is the partition function, and $\lfloor \cdot \rfloor$ is the integer part function.*

Proof. Follows from the Theorem by an elementary counting argument. \square

7. TOP CHARACTERISTIC VARIETIES

In this section we study the top characteristic variety $V_1(\mathcal{A})$ of a 2-arrangement \mathcal{A} , and the number $\text{Tors}_{p,1}(\mathcal{A})$ of its p -torsion points, for p a prime number.

7.1. Let us start with a completely decomposable arrangement, $\mathcal{A} = \mathcal{A}(\tau)$. Let $\tau = \tau_0 \rightarrow \tau_1 \rightarrow \dots \rightarrow \tau_d = (1)$ be the decomposition sequence for τ . Recall that each permutation in the sequence is partitioned into blocks of consecutive integers. We call such a block B *essential* if either $|B| \geq 2$, or $B = \tau_{d-1}$ and $|B| > 2$.

Theorem 7.2. *The top characteristic variety $V_1(\mathcal{A})$ of a completely decomposable arrangement of n planes is the union of an arrangement $\mathcal{V}(\mathcal{A})$ of codimension 1 subtori of $(\mathbb{C}^*)^n$, all passing through $\mathbf{1}$.*

Proof. Choose $\tau \in S_n$ so that $\mathcal{A} = \mathcal{A}(\tau)$ and $\text{depth}(\mathcal{A}) = d(\tau)$. Let m be the number of essential blocks of τ . Recall from §3.12 that $L = L(\mathcal{A})$ is an iterated torus link, obtained by $(1, \pm 1)$ -cablings on the unknot. Thus, it is a spliced link in the sense of Eisenbud and Neumann [11]. The decomposition sequence of τ corresponds to a minimal splice diagram of L : The (signed) essential blocks B_1, \dots, B_m of the permutations in the sequence correspond to the (signed) nodes v_{n+1}, \dots, v_{n+m} of the diagram, and the integers $1, \dots, n$ to the arrowheads v_1, \dots, v_n . Then, according to [11], Theorem 12.1, the Alexander polynomial of L is given by:

$$\Delta_L(t_1, \dots, t_n) = \prod_{j=n+1}^{n+m} (t_1^{l_{1,j}} t_2^{l_{2,j}} \dots t_n^{l_{n,j}} - 1)^{\delta_j - 2},$$

where $l_{i,j} = \pm 1$ is the linking number of L_i with the “virtual component” corresponding to v_j , and δ_j is the valency of v_j . Thus, each irreducible component of $V_1(\mathcal{A}) = \{\Delta_L = 0\}$ is a codimension 1 subtorus. (It actually can be shown that $V_1(\mathcal{A})$ has precisely m components.) \square

To compute the number of torsion points on $V_1(\mathcal{A})$, we may now use a result of Björner and Ekedahl [3]. Indeed, an arrangement \mathcal{V} of codimension 1 subtori in $(\mathbb{C}^*)^n$ defines an arrangement \mathcal{V}_p of hyperplanes in $(\mathbb{Z}_p)^n$: To a subtorus $t_1^{a_1} \dots t_n^{a_n} - 1 = 0$ corresponds the hyperplane $a_1 x_1 + \dots + a_n x_n = 0 \pmod p$. Proposition 3.2 of [3] then implies the following.

Proposition 7.3. *The number of p -torsion points on the union $U = U(\mathcal{V})$ of an arrangement of subtori in $(\mathbb{C}^*)^n$ is given by:*

$$\text{Tors}_p(U) = - \sum_{x \in L \setminus \{\hat{0}\}} \mu(\hat{0}, x) p^{\dim(x)},$$

where L is the intersection lattice of the arrangement \mathcal{V}_p , with minimal element $\hat{0} = (\mathbb{Z}_p)^n$, and Möbius function μ .

Example 7.4. The arrangement $\mathcal{A} = \mathcal{A}(312546)$ is completely decomposable, of depth 3. Its decomposition sequence is $(312546) \rightarrow (2134) \rightarrow (12) \rightarrow (1)$. Proposition 4.9 gives $\xi = A_{1,3}A_{2,3}A_{4,5}$. The Artin representation of ξ in the basis $y_1 = x_1$, $y_2 = x_1x_2$, $y_3 = x_1x_2x_3$, $y_4 = x_4$, $y_5 = x_4x_5$ is given by $\xi(y_1) = y_3y_2^{-1}y_1y_2y_3^{-1}$, $\xi(y_2) = y_3y_2y_3^{-1}$, $\xi(y_3) = y_3$, $\xi(y_4) = y_5y_4y_5^{-1}$, and $\xi(y_5) = y_5$. The Alexander polynomial is $\Delta_{\mathcal{A}}(t_1, \dots, t_6) = (t_6 - 1)(t_6 - t_5^2)(t_6 - t_3^2)(t_6 - t_3^2t_2^{-2})$. Proposition 7.3 yields $\text{Tors}_{2,1}(\mathcal{A}) = 32$ and $\text{Tors}_{3,1}(\mathcal{A}) = 585$.

For an arrangement of depth 2, we can give a more precise description of the top characteristic variety, based on formula (6.2) for the Alexander matrix.

Proposition 7.5. *Let \mathcal{A} be a depth 2 arrangement of n planes, with normal form $\mathcal{A}(I_1, \dots, I_r, J)$, and let $k_q = \max I_q$, for $1 \leq q \leq r$. Then:*

- (a) $\Delta_{\mathcal{A}}(t_1, \dots, t_n) = (t_n - 1)^{|J|+r-2} \prod_{q=1}^r (t_n - t_{k_q}^2)^{|I_q|-1}$;
- (b) $V_1(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid (t_n - 1) \prod_{q=1}^r (t_n - t_{k_q}^2) = 0\}$;
- (c) $\text{Tors}_{2,1}(\mathcal{A}) = 2^{n-1}$ and $\text{Tors}_{p,1}(\mathcal{A}) = p^{n-r-1}(p^{r+1} - (p-1)^{r+1})$, for $p \geq 3$.

7.6. Let L be a link in \mathbb{S}^3 . The Alexander polynomial of a sublink of L , and that of an (a, b) -cable about L , can be computed from the Alexander polynomial of L , via the following well-known formulae of Torres and Sumners–Woods, see [11, 15, 26].

Theorem 7.7. *Let $L = L_1 \cup \dots \cup L_n$ be a link in \mathbb{S}^3 . Set $T = t_1^{l_1} \dots t_{n-1}^{l_{n-1}}$, where $l_i = \text{lk}(L_i, L_n)$. Then:*

$$(7.1a) \quad \Delta_L(t_1, \dots, t_{n-1}, 1) = (T - 1)\Delta_{L \setminus L_n}(t_1, \dots, t_{n-1}),$$

Moreover, if $L' = L\{a, b\}$, with $\gcd(a, b) = 1$, then:

$$(7.1b) \quad \Delta_{L'}(t_1, \dots, t_n, t_{n+1}) = (T^a t_n^b t_{n+1}^b - 1)\Delta_L(t_1, \dots, t_{n-1}, t_n^a t_{n+1}).$$

Corollary 7.8. *Let \mathcal{A} be an arrangement of n planes, and let $\mathcal{A}^k\{r\}$ be an r -cable about it. Then:*

- (a) $V_1(\mathcal{A}\{r\}) = \{\mathbf{t} \in (\mathbb{C}^*)^{n+r} \mid t_1 \dots t_n - 1 = 0 \text{ or } (t_1, \dots, t_n) \in V_1(\mathcal{A})\}$;
- (b) $\text{Tors}_{p,1}(\mathcal{A}\{r\}) = p^{r-1} \text{Tors}_{p,1}(\mathcal{A}\{1\})$.

Proof. Let $L = L(\mathcal{A})$. Recall from §3.1 that $L(\mathcal{A}\{1\}) = L\{1, 1\}$. By the Sumners–Woods formula (7.1b), we have

$$\Delta_{L(\mathcal{A}\{1\})}(t_1, \dots, t_n, t_{n+1}) = (t_1 \dots t_{n+1} - 1)\Delta_L(t_1, \dots, t_{n-1}, t_n t_{n+1}).$$

After a monomial change of basis, this implies part (a) for $r = 1$. The general case follows from the same formula, by induction on r . Part (b) follows immediately from part (a). \square

7.9. We conclude this section with a recursion formula for $\text{Tors}_{2,1}(\mathcal{A})$. Let $\Delta_{L(\mathcal{A})}$ be the Alexander polynomial of the link $L(\mathcal{A})$. Define the *single-variable Alexander polynomial* of \mathcal{A} to be $\Delta_{\mathcal{A}}(t) := (t - 1)\Delta_{L(\mathcal{A})}(t, \dots, t)$. Furthermore, set

$$\delta(\mathcal{A}) = \begin{cases} 1 & \text{if } \Delta_{\mathcal{A}}(-1) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.10. The single-variable Alexander polynomial of the complex arrangement \mathcal{A}_n is $\Delta_{\mathcal{A}_n}(t) = (t - 1)(t^n - 1)^{n-2}$. Thus, $\delta(\mathcal{A}_n) = \frac{1+(-1)^n}{2}$.

Theorem 7.11. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a 2-arrangement in \mathbb{R}^4 . The number of 2-torsion points of the top characteristic variety $V_1(\mathcal{A})$ is given by the following formula:*

$$(7.2) \quad \text{Tors}_{2,1}(\mathcal{A}) = 2^{n-1} - \frac{1 + (-1)^n}{2} + \delta(\mathcal{A}) + \sum_{\mathcal{B} \in \Gamma(\mathcal{A})} \delta(\mathcal{B}),$$

where $\Gamma(\mathcal{A})$ is the set of all indecomposable, proper sub-arrangements of \mathcal{A} with an odd number of planes.

Proof. By definition,

$$(7.3) \quad \text{Tors}_{2,1}(\mathcal{A}) = \sum_{\omega \in \Omega_2^n} c_\omega,$$

where $\Omega_2^n = \{(\omega_1, \dots, \omega_n) \in (\mathbb{C}^*)^n \mid \omega_i = \pm 1\}$, and $c_\omega = 1$ if $\Delta_{L(\mathcal{A})}(\omega) = 0$, and $c_\omega = 0$ otherwise. For $\omega \in \Omega_2^n$, let $\mathcal{A}_\omega = \{H_i \in \mathcal{A} \mid \omega_i = -1\}$. There are several cases to consider:

- If $\omega = (-1, \dots, -1)$, then $\mathcal{A}_\omega = \mathcal{A}$, and so $c_\omega = \delta(\mathcal{A})$.
- If $\omega \neq (-1, \dots, -1)$, then \mathcal{A}_ω is a proper sub-arrangement of \mathcal{A} , and so, by repeated application of Torres's formula (7.1a), we have

$$(7.4) \quad \Delta_{L(\mathcal{A})}(\omega) = ((-1)^{|\mathcal{A}_\omega|} - 1) \Delta_{L(\mathcal{A}_\omega)}(-1, \dots, -1).$$

- If $|\mathcal{A}_\omega|$ is even, this formula says that $\Delta_{L(\mathcal{A})}(\omega) = 0$, and so $c_\omega = 1$. There are $2^{n-1} - \frac{1+(-1)^n}{2}$ such contributions to the sum (7.3).
- If $|\mathcal{A}_\omega|$ is odd, and \mathcal{A}_ω is decomposable, we may write $\mathcal{A}_\omega = \mathcal{A}'_\omega \{\pm 1\}$, with the cabling done about the last component of \mathcal{A}'_ω . Let \mathcal{A}''_ω be the sub-arrangement obtained by deleting the last component of \mathcal{A}'_ω . Clearly, $|\mathcal{A}''_\omega| = |\mathcal{A}_\omega| - 2$. Formulas (7.1b), and (7.1a) give

$$(7.5) \quad \Delta_{L(\mathcal{A}_\omega)}(-1, \dots, -1) = -2\Delta_{L(\mathcal{A}'_\omega)}(-1, \dots, -1, 1) = 4\Delta_{L(\mathcal{A}''_\omega)}(-1, \dots, -1).$$

Hence, $\delta(\mathcal{A}_\omega) = \delta(\mathcal{A}''_\omega)$. Iterating this decabling-deletion procedure, we eventually reach an arrangement \mathcal{B} for which the procedure must stop. There are two possibilities:

- * One is $\mathcal{B} = \mathcal{A}_3$, in which case $c_\omega = \delta(\mathcal{A}_3) = 0$.
- * The other is $\mathcal{B} \in \Gamma(\mathcal{A})$, in which case $c_\omega = \delta(\mathcal{B})$. Clearly, any element of $\Gamma(\mathcal{A})$ can be reached by the above procedure; thus there are $|\Gamma(\mathcal{A})|$ such contributions to the sum (7.3).

This completes the proof. \square

Remark 7.12. Note that $2^{n-1} - 1 \leq \text{Tors}_{2,1}(\mathcal{A}) \leq 2^n$. If $\text{Tors}_{2,1}(\mathcal{A}) = 2^{n-1} - 1$, and $n \geq 3$, then the top characteristic variety $V_1(\mathcal{A})$ is *not* the union of translated subtori of $(\mathbb{C}^*)^n$. For, otherwise, at least one of the subtori must be of the form $T = \{t_1^{a_1} \cdots t_n^{a_n} - 1 = 0\}$, since $\Delta_{L(\mathcal{A})}(1, \dots, 1) = 0$. But the torus T has 2^{n-1} torsion points of order 2.

Corollary 7.13. *If all the proper subarrangements of \mathcal{A} are completely decomposable, then $\text{Tors}_{2,1}(\mathcal{A}) = 2^{n-1} - \frac{1+(-1)^n}{2} + \delta(\mathcal{A})$.*

Corollary 7.14. *If \mathcal{A} is completely decomposable, then $\text{Tors}_{2,1}(\mathcal{A}) = 2^{n-1}$.*

Proof. The recursion formula (7.5), together with Example 7.10 imply that $\delta(\mathcal{A}) = \frac{1+(-1)^n}{2}$, and the conclusion follows from the previous corollary. \square

Example 7.15. The arrangement $\mathcal{A} = \mathcal{A}(31425)$ is horizontal, indecomposable, and all its subarrangements are completely decomposable. The (single variable) Alexander polynomial is $\Delta_{\mathcal{A}}(t) = (t-1)^4(4t^2 - t + 4)$, and so $\delta(\mathcal{A}) = 0$. From Corollary 7.13, we get $\text{Tors}_{2,1}(\mathcal{A}) = 16$.

Example 7.16. The arrangement $\mathcal{A} = \mathcal{A}(314256)$ is decomposable, but not completely decomposable, since it has $\mathcal{A}(31425)$ as a subarrangement. We have $\Delta_{\mathcal{A}}(t) = (t^6 - 1)(t-1)^3(t+1)(3t^2 - 2t + 3)$, and so $\delta(\mathcal{A}) = 1$. From Theorem 7.11, we get $\text{Tors}_{2,1}(\mathcal{A}) = 32$.

Example 7.17. The arrangement $\mathcal{A} = \mathcal{A}(241536)$ is horizontal, indecomposable, and all its proper subarrangements are completely decomposable. We have $\Delta_{\mathcal{A}}(t) = (t-1)^5(5t^4 + 6t^2 + 5)$, and so $\delta(\mathcal{A}) = 0$. From Corollary 7.13, we get $\text{Tors}_{2,1}(\mathcal{A}) = 31$. Hence, $V_1(\mathcal{A})$ is not a union of translated subtori of $(\mathbb{C}^*)^6$.

8. MAZUROVSKIĬ'S ARRANGEMENTS

In this section, we study the 2-arrangements associated to Mazurovskii's configurations. Using their associated cablings, we find infinitely many pairs of arrangements whose complements are cohomologically equivalent, but not homotopy equivalent.

8.1. In [22], Mazurovskii introduced a remarkable pair of configurations of skew lines, K and L , which have the same linking numbers, but are not rigidly isotopic. Let $\mathcal{K} = \mathcal{A}(K)$ and $\mathcal{L} = \mathcal{A}(L)$ be the corresponding arrangements of planes. The arrangement \mathcal{K} is horizontal, with associated permutation $\tau = (341256)$. Moreover, \mathcal{K} is completely decomposable, of depth 3; a minimal decomposition sequence is $(341256) \rightarrow (213) \rightarrow (12) \rightarrow (1)$. The arrangement \mathcal{L} is neither horizontal, nor decomposable. Defining polynomials for \mathcal{K} and \mathcal{L} are given by

$$\begin{aligned} f_{\mathcal{K}}(z, w) &= f(z, w) \cdot (z - 7w), \\ f_{\mathcal{L}}(z, w) &= f(z, w) \cdot \left(z - \frac{6-7i}{2}w - \frac{3+14i}{2}\bar{w} \right), \end{aligned}$$

where f is the following defining polynomial for $\mathcal{A}(34125)$:

$$\begin{aligned} f(z, w) &= \left(z - \frac{5-5i}{2}w + \frac{3-5i}{2}\bar{w} \right) \left(z - \frac{7-10i}{2}w + \frac{3-10i}{2}\bar{w} \right) \\ &\quad \times \left(z - \frac{5-14i}{2}w - \frac{3+14i}{2}\bar{w} \right) \left(z - \frac{7-9i}{2}w - \frac{3+9i}{2}\bar{w} \right) (z - 6w). \end{aligned}$$

The half-braids associated to \mathcal{K} and \mathcal{L} are pictured in Figure 5. We see that the linking numbers of $L(\mathcal{K})$ are $l_{1,4} = l_{2,4} = l_{1,3} = l_{2,3} = -1$, and all other $l_{i,j} = 1$, whereas the linking numbers of $L(\mathcal{L})$ are $l_{1,5} = l_{2,5} = l_{1,4} = l_{2,4} = -1$, and all other $l_{i,j} = 1$. The reordering of the components of $L(\mathcal{L})$ that fixes 1, 2, 6 and permutes 3, 4, 5 to 4, 5, 3 identifies the linking numbers of $L(\mathcal{K})$ and $L(\mathcal{L})$. Thus, $H^*(X(\mathcal{K}); \mathbb{Z}) \cong H^*(X(\mathcal{L}); \mathbb{Z})$.

8.2. In order to distinguish between the cohomologically equivalent arrangements \mathcal{K} and \mathcal{L} , we turn to their characteristic varieties. From Figure 5, we see that the reduced half-braids of \mathcal{K} and \mathcal{L} are:

$$\check{\alpha}_{\mathcal{K}} = \sigma_4 \sigma_3 \sigma_2 \sigma_1 \sigma_4 \sigma_3^{-1} \sigma_2^{-1} \sigma_4^{-1} \sigma_3^{-1} \sigma_4, \quad \check{\alpha}_{\mathcal{L}} = \sigma_4 \sigma_2 \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_4^{-1} \sigma_3^{-1} \sigma_2 \sigma_4.$$

The braids $\xi = \Delta_5 \check{\alpha}^{-1} \in P_5$ are expressed in terms of the pure braid generators, as follows. For \mathcal{K} , which is horizontal, Proposition 4.9 yields $\xi_{\mathcal{K}} = A_{1,3} A_{2,3} A_{1,4} A_{2,4}$.

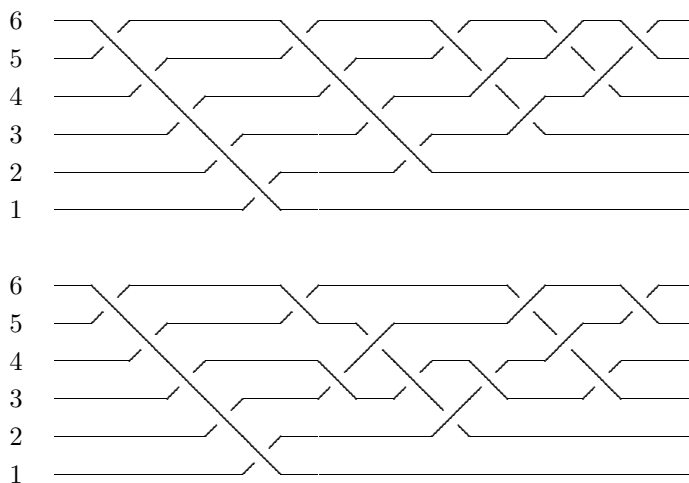


FIGURE 5. Mazurovskii's pair: The half-braids $\alpha_{\mathcal{K}}$ (top) and $\alpha_{\mathcal{L}}$ (bottom).

For \mathcal{L} , it is more convenient to work with the conjugate $\xi'_{\mathcal{L}} = \delta^{-1}\xi_{\mathcal{L}}\delta$, where $\delta = \sigma_1\sigma_3\sigma_4^{-1}$. Routine combing of the braid yields $\xi'_{\mathcal{L}} = A_{1,3}A_{2,3}A_{4,5}A_{1,4}A_{4,5}^{-1}A_{2,4}$.

The Artin representation of $\xi = \xi_{\mathcal{K}}$ in the basis $y_1 = x_1$, $y_2 = x_3$, $y_3 = x_1x_2$, $y_4 = x_1x_2x_3x_4$, $y_5 = x_5$ is given by $\xi(y_1) = y_4y_3^{-1}y_1y_3y_4^{-1}$, $\xi(y_2) = y_3y_2y_3^{-1}$, $\xi(y_3) = y_4y_3y_4^{-1}$, $\xi(y_4) = y_4$, and $\xi(y_5) = y_5$. The Alexander matrix of \mathcal{K} is:

$$\begin{pmatrix} t_6 - t_4^2t_3^{-2} & 0 & t_4^2t_3^{-2}(t_3 + 1)(1 - t_1) & (t_4 + 1)(t_1 - 1) & 0 & t_1 - 1 \\ 0 & t_6 - t_3^2 & (t_4 + t_3)(t_2 - 1) & (1 - t_3)(t_2 - 1) & 0 & t_2 - 1 \\ 0 & 0 & t_6 - t_4^2 & (t_4 + 1)(t_3 - 1) & 0 & t_3 - 1 \\ 0 & 0 & 0 & t_6 - 1 & 0 & t_4 - 1 \\ 0 & 0 & 0 & 0 & t_6 - 1 & t_5 - 1 \end{pmatrix}$$

An elementary computation shows that the bottom variety $V_4(\mathcal{K})$ has 6 irreducible components—3 codimension 4 translated subtori of $(\mathbb{C}^*)^6$, and 3 codimension 5 subtori passing through $\mathbf{1}$ —given by the following equations:

$$\begin{aligned} & \{t_6 - 1 = t_4 + 1 = t_3 + 1 = t_2 - 1 = 0\}, \\ & \{t_6 - 1 = t_4 + 1 = t_3 - 1 = t_1 - 1 = 0\}, \\ & \{t_6 - 1 = t_5 - 1 = t_4 - 1 = t_3 + 1 = 0\}, \\ & \{t_6 - 1 = t_5 - 1 = t_4 - 1 = t_3 - 1 = t_1 - 1 = 0\}, \\ & \{t_6 - 1 = t_5 - 1 = t_4 - 1 = t_3 - 1 = t_2 - 1 = 0\}, \\ & \{t_6 - 1 = t_4 - 1 = t_3 - 1 = t_2 - 1 = t_1 - 1 = 0\}. \end{aligned}$$

The primary decomposition of the ideal $E_4(\mathcal{L})$ is much harder to find. The implementation in *Macaulay 2* [14] of the Eisenbud, Huneke, and Vasconcelos algorithm yields such a decomposition, and the result is that $V_4(\mathcal{L}) = V_4(\mathcal{K})$. Thus, the bottom varieties fail to distinguish between the \mathcal{K} and \mathcal{L} arrangements.

Let us then consider the top varieties. It is readily seen that the Alexander polynomial of \mathcal{K} is $\Delta_{\mathcal{K}}(t_1, \dots, t_6) = (t_6 - 1)(t_6 - t_3^2)(t_6 - t_4^2)(t_6 - t_4^2t_3^{-2})$, and so

$V_1(\mathcal{K})$ is the union of 4 codimension 1 subtori of $(\mathbb{C}^*)^6$. Since \mathcal{K} is completely decomposable, Corollary 7.14 implies that $\text{Tors}_{2,1}(\mathcal{K}) = 32$.

The Alexander polynomial of \mathcal{L} may be computed using *Mathematica* [29]. The result is too long to be displayed here, but suffices to say that it is an irreducible polynomial over \mathbb{Z} , consisting of 667 monomials. Direct computation shows that the single variable Alexander polynomial is $\Delta_{\mathcal{L}}(t) = 3(t-1)^5(3t^2 - 2t + 3)^2$. Hence $\delta(\mathcal{L}) = 0$. Since, as is readily checked, all proper subarrangements of \mathcal{L} are completely decomposable, Corollary 7.13 implies that $\text{Tors}_{2,1}(\mathcal{L}) = 31$.

Thus, $\text{Tors}_{2,1}(\mathcal{K}) \neq \text{Tors}_{2,1}(\mathcal{L})$. (As noted in Remark 7.12, this arithmetic difference translates into a geometric difference: $V_1(\mathcal{K})$ is a union of subtori, whereas $V_1(\mathcal{L})$ is not even the union of translated subtori.) Appealing now to Theorem 5.6, we conclude that the complements of \mathcal{K} and \mathcal{L} are not homotopy equivalent, although, as mentioned previously, they are cohomologically isomorphic. This answers Ziegler's question from [30].

8.3. We now use cablings of \mathcal{K} and \mathcal{L} to show that the above phenomenon happens for arrangements of n planes, for any $n \geq 6$.

Theorem 8.4. *Let \mathcal{K} and \mathcal{L} be Mazurovskii's arrangements of 6 transverse planes in \mathbb{R}^4 . Let $\mathcal{K}\{r\}$ and $\mathcal{L}\{r\}$ be their r -cables. Then, for each $r \geq 0$,*

- (a) $H^*(X(\mathcal{K}\{r\}); \mathbb{Z}) \cong H^*(X(\mathcal{L}\{r\}); \mathbb{Z})$;
- (b) $X(\mathcal{K}\{r\}) \not\cong X(\mathcal{L}\{r\})$.

Proof. As noted above, the links of \mathcal{K} and \mathcal{L} have the same linking numbers. Hence, the links of $\mathcal{K}\{r\}$ and $\mathcal{L}\{r\}$ have the same linking numbers. This implies that the complements of $\mathcal{K}\{r\}$ and $\mathcal{L}\{r\}$ are cohomologically equivalent.

Although \mathcal{K} and \mathcal{L} are distinguished by their 2-torsion points, their cables are not. Indeed, for $r \geq 1$, $\text{Tors}_{2,1}(\mathcal{K}\{r\}) = \text{Tors}_{2,1}(\mathcal{L}\{r\}) = 2^{r+5}$, as can be deduced from Theorem 7.11 for $r = 1$, and from Corollary 7.8 for $r > 1$. Hence, we turn to 3-torsion points. A *Mathematica* computation shows that $\text{Tors}_{3,1}(\mathcal{K}\{1\}) = 3^5 \cdot 7$ and $\text{Tors}_{3,1}(\mathcal{L}\{1\}) = 3^3 \cdot 61$. From Corollary 7.8, we get

$$\text{Tors}_{3,1}(\mathcal{K}\{r\}) = 3^{r+4} \cdot 7, \quad \text{and} \quad \text{Tors}_{3,1}(\mathcal{L}\{r\}) = 3^{r+2} \cdot 61,$$

showing that the respective complements are indeed not homotopy equivalent. \square

8.5. Mazurovskii introduced in [22] another interesting configuration of 6 lines, which he called M . Like the L configuration, the M configuration is non-horizontal and indecomposable (they are the only two such configurations of 6 lines, up to rigid isotopy and mirror images). But, unlike L , the M configuration does not have the linking numbers of any horizontal configuration. Let $\mathcal{M} = \mathcal{A}(M)$ be the corresponding arrangement. A defining polynomial for it is:

$$\begin{aligned} f_{\mathcal{M}}(z, w) &= (z - (10 - i)w + (9 - 4i)\bar{w})(z - (3 - 4i)w - (1 + 4i)\bar{w}) \\ &\quad \times \left(z - \frac{5 - 10i}{2}w + \frac{1 - 10i}{2}\bar{w}\right) \left(z - (6 - 5i)w + \frac{1 - 10i}{2}\bar{w}\right) \\ &\quad \times \left(z - \frac{21 - 29i}{4}w + \frac{1 - 9i}{4}\bar{w}\right) (z - 6w). \end{aligned}$$

The reduced braids associated to \mathcal{M} are:

$$\tilde{\alpha}_{\mathcal{M}} = \sigma_2 \sigma_3^{-1} \sigma_1 \sigma_2 \sigma_3^{-1} \sigma_4^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \quad \text{and} \quad \xi_{\mathcal{M}} = A_{2,4} A_{1,2} A_{3,4} A_{1,5} A_{3,5}.$$

A *Macaulay 2* computation shows that $V_4(\mathcal{M})$ consists of sixteen 2-torsion points. A *Mathematica* computation reveals that the Alexander polynomial of \mathcal{M} is an irreducible polynomial over \mathbb{Z} , consisting of 317 monomials. The single variable Alexander polynomial is $\Delta_{\mathcal{M}}(t) = (t-1)^5(t^2-t+1)(t^6-5t^5-t^4-6t^3-t^2-5t+1)$, and so $\delta(\mathcal{M}) = 0$. Theorem 7.11 gives $\text{Tors}_{2,1}(\mathcal{M}) = 31$, the same as for \mathcal{L} . (Thus, the top variety of \mathcal{M} is not the union of translated subtori.) On the other hand, a computation yields $\text{Tors}_{3,1}(\mathcal{L}) = 527$ and $\text{Tors}_{3,1}(\mathcal{M}) = 421$, showing that $X(\mathcal{L}) \not\cong X(\mathcal{M})$.

9. CLASSIFICATION OF 2-ARRANGEMENTS OF $n \leq 6$ PLANES

We start by reviewing the rigid isotopy classification of arrangements of up to 6 planes. We then show that the invariants introduced in §5 are powerful enough to classify up to homotopy the complements of such arrangements.

9.1. As noted in §2.7, rigid isotopy types of 2-arrangements in \mathbb{R}^4 are in one-to-one correspondence with rigid isotopy types of skew-lines configurations in \mathbb{R}^3 . An important concept introduced by Viro [28] was that of a mirror image of a configuration. We now translate this notion to arrangements.

Definition 9.2. An arrangement \mathcal{A}' is called a *mirror image* of \mathcal{A} if there is a reflection of \mathbb{R}^4 sending \mathcal{A} to \mathcal{A}' . The mirror image of \mathcal{A} is unique up to rigid isotopy; we denote it by $\overline{\mathcal{A}}$. An arrangement \mathcal{A} which is not isotopic to $\overline{\mathcal{A}}$ is called *non-mirror*.

As shown by Viro, there exist many non-mirror arrangements. For example, the complex arrangement, \mathcal{A}_n , and its mirror image under complex conjugation, $\overline{\mathcal{A}}_n$, are not rigidly isotopic provided $n \geq 3$. Also, an arbitrary arrangement of n lines is non-mirror, provided $n \equiv 3 \pmod{4}$.

Viro [28] and Mazurovskii [22] classified, up to rigid isotopy, all configurations 6 lines or less. For up to 5 lines, linking numbers invariants were used. For 6 lines, those invariants cannot tell apart the K and L configurations. For that, the Morton trace of the reduced full-braid is used in [22]. Translated to arrangements, the complete list of the 33 rigid isotopy types is as follows:

- $n = 1$: $\mathcal{A}(1)$
- $n = 2$: $\mathcal{A}(12)$
- $n = 3$: $\mathcal{A}(123)^*$
- $n = 4$: $\mathcal{A}(1234)^*$, $\mathcal{A}(2134)$
- $n = 5$: $\mathcal{A}(12345)^*$, $\mathcal{A}(21345)^*$, $\mathcal{A}(21435)^*$, $\mathcal{A}(31425)$
- $n = 6$: $\mathcal{A}(123456)^*$, $\mathcal{A}(213456)^*$, $\mathcal{A}(321456)$, $\mathcal{A}(214356)^*$, $\mathcal{A}(215436)^*$,
 $\mathcal{A}(312546)$, $\mathcal{A}(341256)^*$, $\mathcal{A}(314256)^*$, $\mathcal{A}(241536)$, $\mathcal{A}(L)^*$, $\mathcal{A}(M)^*$

where \mathcal{A}^* stands for a non-mirror arrangement \mathcal{A} and its mirror image $\overline{\mathcal{A}}$.

9.3. We now turn to the homotopy classification of complements of arrangements. Clearly, arrangements that are either rigidly isotopic, or mirror images of one another, have diffeomorphic (and thus, homotopy equivalent) complements. Thus, if we delete from the above list the mirror image $\overline{\mathcal{A}}$ from each pair $\mathcal{A}^* = (\mathcal{A}, \overline{\mathcal{A}})$, we are left with a list 20 arrangements, such that, the complement of any arrangement

n	\mathcal{A}	depth	Σ_{n-2}	Tors _{2,1}	Tors _{3,1}
1	$\mathcal{A}(1)$	0	0	0	0
2	$\mathcal{A}(12)$	1	0	1	1
3	$\mathcal{A}(123)$	1	1	4	9
4	$\mathcal{A}(1234)$	1	1	8	27
	$\mathcal{A}(2134)$	2	$2, 3_2$	8	45
5	$\mathcal{A}(12345)$	1	1	16	81
	$\mathcal{A}(21345)$	2	$2, 3, 4$	16	135
	$\mathcal{A}(21435)$	2	$3, 4_4$	16	171
	$\mathcal{A}(31425)$	—	5_{11}	16	141
6	$\mathcal{A}(123456)$	1	1	32	243
	$\mathcal{A}(213456)$	2	$2, 3, 5$	32	405
	$\mathcal{A}(321456)$	2	$2, 4_2$	32	405
	$\mathcal{A}(215436)$	2	$3, 4_2, 5_2$	32	513
	$\mathcal{A}(214356)$	2	$3, 4_2, 5_3$	32	513
	$\mathcal{K} = \mathcal{A}(341256)$	3	$4_3, 5_3$	32	567
	$\mathcal{A}(312546)$	3	$4, 5_6, 6$	32	585
	$\mathcal{A}(314256)$	—	$5_6, 6_5$	32	495
	$\mathcal{A}(241536)$	—	$5_2, 6_{13}$	31	513
	\mathcal{L}	—	$4_3, 5_3$	31	527
\mathcal{M}	—	6_{16}	31	421	

TABLE 1. Arrangements of $n \leq 6$ planes: Sequence of codimensions of components of V_{n-2} —where i_k stands for i repeated k times—and number of 2- and 3-torsion points on V_1 .

of $n \leq 6$ planes is homotopy equivalent to the complement of one of the arrangements in this shorter list. Table 1 shows that there are no repetitions among the homotopy types of these 20 arrangements. Hence, we have the following.

Theorem 9.4. *For 2-arrangements of $n \leq 6$ planes in \mathbb{R}^4 , the homotopy types of complements are in one-to-one correspondence with the rigid isotopy types modulo mirror images.*

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