Resonance varieties and Dwyer–Fried invariants

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Abstract. The Dwyer–Fried invariants of a finite cell complex X are the subsets $\Omega^i_r(X)$ of the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$ which parametrize the regular $\mathbb{Z}^r$-covers of X having finite Betti numbers up to degree i. In previous work, we showed that each Ω-invariant is contained in the complement of a union of Schubert varieties associated to a certain subspace arrangement in $H^1(X, \mathbb{Q})$. Here, we identify a class of spaces for which this inclusion holds as equality. For such "straight" spaces X, all the data required to compute the Ω-invariants can be extracted from the resonance varieties associated to the cohomology ring $H^*(X, \mathbb{Q})$. In general, though, translated components in the characteristic varieties affect the answer.

Contents

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1. Introduction

One of the most fruitful ideas to arise from arrangement theory is that of turning the cohomology ring of a space into a family of cochain complexes, parametrized by the cohomology group in degree 1, and extracting certain varieties from these data, as the loci where the cohomology of those cochain complexes jumps. What makes these "resonance" varieties really useful is their close connection with a different kind of jumping loci: the "characteristic" varieties, which record the jumps in homology with coefficients in rank 1 local systems.

In this paper, we use the geometry of the cohomology jump loci to study a classical problem in topology: determining which infinite covers of a space have finite Betti numbers. Restricting our attention to regular, free abelian covers of a fixed rank allows us to state the problem in terms of a suitable filtration on the rational Grassmannian. Under favorable circumstances, the finiteness of the Betti numbers of such covers is exclusively controlled by the incidence varieties to the resonance varieties of our given space.

1.1. Cohomology jump loci and straightness. Let X be a connected CW-complex with finite skeleta. To such a space, we associate two types of jump loci. The first are the *resonance varieties* $\mathcal{R}^i(X)$. These are homogeneous subvarieties of the affine space $H^1(X,\mathbb{C})=\mathbb{C}^n$, where $n=b_1(X)$, and they are defined in terms of the cohomology algebra $A=H^*(X,\mathbb{C})$, as follows. For each $a\in A^1$, left-multiplication by a defines a cochain complex $(A,\cdot a)$. Then

$$\mathcal{R}^i(X)=\{a\in H^1(X,\mathbb{C})\mid H^j(A,\cdot a)\neq 0,\ \text{for some } j\leq i\}. \tag{1}$$

The second type of jump loci we consider here are the *characteristic varieties* $\mathcal{W}^i(X)$. These are Zariski closed subsets of the complex algebraic torus $H^1(X,\mathbb{C}^{\times})^0=\operatorname{Hom}(\pi_1(X),\mathbb{C}^{\times})^0=(\mathbb{C}^{\times})^n$, defined as follows. Each character $\rho\colon\pi_1(X)\to\mathbb{C}^{\times}$ gives rise to a rank 1 local system on X, call it $\mathcal{L}_\rho$. Then

$$\mathcal{W}^i(X)=\{\rho\in H^1(X,\mathbb{C}^{\times})^0\mid H_j(X,\mathcal{L}_\rho)\neq 0,\ \text{for some } j\leq i\}. \tag{2}$$

One of our goals in this paper is to isolate a class of spaces for which the resonance and characteristic varieties have a rather simple nature, and are intimately related to each other.

We say that X is *locally k-straight* if, for each $i \le k$, all components of $\mathcal{W}^i(X)$ passing through the origin 1 are algebraic subtori, and the tangent cone at 1 to $\mathcal{W}^i(X)$ equals $\mathcal{R}^i(X)$. If, moreover, all positive-dimensional components of $\mathcal{W}^i(X)$ contain the origin, we say X is *k-straight*. For locally straight spaces, the resonance varieties $\mathcal{R}^i(X)$ are finite unions of rationally defined linear subspaces.

Examples of straight spaces include Riemann surfaces, tori, and knot complements. Under some further assumptions, the straightness properties behave well with respect to finite direct products and wedges.

A related notion is Sullivan's k-formality. Using the tangent cone formula from [11], it is readily seen that 1-formal spaces are locally 1-straight. In general though, 1-formality does not imply 1-straightness, and the converse does not hold, either.

1.2. Dwyer–Fried invariants. The second goal of this paper is to analyze the homological finiteness properties of all regular, free abelian covers of a given space X, and relate these properties to the resonance varieties of X, under a straightness assumption.

The connected, regular $\mathbb{Z}^r$-covers $\hat{X} \to X$ are parametrized by the Grassmannian of r-planes in the vector space $H^1(X, \mathbb{Q})$. Moving about this variety, and recording when all the Betti numbers $b_1(\hat{X}), \dots, b_i(\hat{X})$ are finite defines subsets

$$\Omega^i_r(X) \subset \mathrm{Gr}_r(H^1(X, \mathbb{Q})), \tag{3}$$

which we call the *Dwyer-Fried invariants* of X. These sets depend only on the homotopy type of X. Consequently, if G is a finitely generated group, the sets $\Omega^i_r(G) := \Omega^i_r(K(G, 1))$ are well-defined.

In [13], Dwyer and Fried showed that the support varieties of the Alexander invariants of a finite cell complex X completely determine the Ω-sets of X. In [29] and [34], this foundational result was refined and reinterpreted in terms of the characteristic varieties of X, as follows. Let $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^\times)$ be the coefficient homomorphism induced by the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$. Then,

$$\Omega^i_r(X) = \big\{P \in \mathrm{Gr}_r(\mathbb{Q}^n) \;\big|\; \dim_{\mathbb{C}}(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)) = 0\big\}. \tag{4}$$

We pursue this study here, by investigating the relationship between the Dwyer–Fried sets and the resonance varieties. Given a homogeneous variety $V \subset \Bbbk^n$, let $\sigma_r(V) \subset \mathrm{Gr}_r(\Bbbk^n)$ be the variety of r-planes incident to V. Our main result reads as follows.

Theorem 1.1. *Let X be a connected CW-complex with finite k-skeleton.*

(1) *Suppose X is locally k-straight. Then, for all $i \leq k$ and $r \geq 1$,*
$$\Omega^i_r(X) \subseteq \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X,\mathbb{Q})).$$

(2) *Suppose X is k-straight. Then, for all $i \leq k$ and $r \geq 1$,*
$$\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X,\mathbb{Q})).$$

As a consequence, if X is k-straight, then each set $\Omega^i_r(X)$ with $i \leq k$ is the complement of a finite union of special Schubert varieties in the Grassmannian of r-planes in $\mathbb{Q}^n$, where $n = b_1(X)$. In particular, $\Omega^i_r(X)$ is a Zariski open subset of $\mathrm{Gr}_r(\mathbb{Q}^n)$.

1.3. Applications. We illustrate our techniques with a broad variety of examples, coming from low-dimensional topology, toric topology, algebraic geometry, and the theory of hyperplane arrangements.

One class of spaces for which things work out very well are the toric complexes. Every simplicial complex K on n vertices determines a subcomplex T_K of the n-torus, with fundamental group the right-angled Artin group associated to the 1-skeleton of K. It turns out that all toric complexes are straight (that is, k-straight, for all k). As shown by Papadima and Suciu in [26], the resonance varieties of a toric complex are unions of coordinate subspaces, which can be read off directly from the corresponding simplicial complex. This leads to some explicit formulas for the Ω-sets of toric complexes and right-angled Artin groups.

The characteristic varieties of (quasi-) Kähler manifolds are fairly well understood, due to work of Beauville, Green–Lazarsfeld, Simpson, Campana, and finally Arapura [1]. In particular, if X is a smooth, quasi-projective variety, then all the components of $\mathcal{W}^1(X)$ are torsion-translated subtori. Recent work of Dimca, Papadima, and Suciu [11] sheds light on the first resonance variety of such varieties: if X is also 1-formal (e.g., if X is compact), then all the components of $\mathcal{R}^1(X)$ are rationally defined linear subspaces. It follows that X is locally 1-straight, and $\Omega^1_r(X) \subseteq \sigma_r(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$. In general, though, the inclusion can be strict. For instance, we prove in Theorem 10.11 the following: if $\mathcal{W}^1(X)$ has a 1-dimensional component not passing through 1, and $\mathcal{R}^1(X)$ has no codimension-1 components, then $\Omega^1_2(X) \neq \sigma_2(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$.

Hyperplane arrangements have been the main driving force behind the development of the theory of cohomology jump loci, and still provide a rich source of motivational examples for this theory. If $\mathcal{A}$ is an arrangement of hyperplanes in $\mathbb{C}^\ell$, then its complement, $X = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$, is a connected, smooth, quasi-projective variety. It turns out that X is also formal, locally straight, but not always straight. The first resonance variety of the arrangement is completely understood, owing to work of Falk [17], Cohen, Libgober, Suciu, Yuzvinsky, and others, with the state of the art being the recent work of Falk, Pereira, and Yuzvinsky [19, 30, 36]. This allows for explicit computations of the Dwyer–Fried invariants of various classes of arrangements. For the deleted B_3 arrangement, though, the computation is much more subtle, due to the presence of a translated component in the first characteristic variety.

1.4. Organization of the paper. The paper is divided in roughly two parts. In the first part (Sections 2–6), we recall some of the basic theory of cohomology jump loci, and develop the notion of straightness. In the second part (Sections 7–11), we develop the Dwyer–Fried theory in the straight context, and apply it in a variety of settings.

In §2 we define the Aomoto complex of a space X, and study it in more detail in the case when X admits a minimal cell structure. We use the Aomoto complex in §3 to define the resonance varieties, and establish some of the basic properties of these varieties.

In §4 we define two types of tangent cones to a subvariety of $(\mathbb{C}^{\times})^n$, and recall some of their features. In §5, we introduce the characteristic varieties, and review some pertinent facts about their tangent cones. Finally, in §6 we define and study the various notions of straightness, based on the geometry of the jump loci.

We start §7 with a review of the Dwyer–Fried invariants, and the way they relate to the characteristic varieties, after which we prove Theorem 1.1. In §8, we discuss the relevance of formality in this setting.

Finally, we show how our techniques work for three classes of spaces: toric complexes in §9, Kähler and quasi-Kähler manifolds in §10, and complements of hyperplane arrangements in §11. In each case, we explain what is known about the cohomology jump loci of those spaces, and use this knowledge to determine their straightness properties, and to compute some of their Ω-invariants.

2. The Aomoto complex

We start by recalling the definition of the (universal) Aomoto complex associated to the cohomology ring of a space X. When X admits a minimal cell structure, this cochain complex can be read off from the equivariant cellular chain complex of the universal abelian cover, X^{ab}.

2.1. A cochain complex from the cohomology ring. Let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$. Consider the cohomology algebra $A = H^*(X, \mathbb{C})$, with product operation given by the cup product of cohomology classes. For each $a \in A^1$, we have $a^2 = 0$, by graded-commutativity of the cup product.

Definition 2.1. The *Aomoto complex* of A (with respect to $a \in A^1$) is the cochain complex of finite-dimensional, complex vector spaces,

$$(A, a)\colon \ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots, \tag{5}$$

with differentials given by left-multiplication by a.

Here is an alternative interpretation. Pick a basis $\{e_1, \dots, e_n\}$ for the complex vector space $A^1 = H^1(X, \mathbb{C})$, and let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for $A_1 = H_1(X, \mathbb{C})$. Identify the symmetric algebra $\operatorname{Sym}(A_1)$ with the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$.

Definition 2.2. The *universal Aomoto complex* of A is the cochain complex of free S-modules,

$$\mathbf{A}\colon \ \cdots \longrightarrow A^i \otimes_{\mathbb{C}} S \xrightarrow{d^i} A^{i+1} \otimes_{\mathbb{C}} S \xrightarrow{d^{i+1}} A^{i+2} \otimes_{\mathbb{C}} S \longrightarrow \cdots, \tag{6}$$

where the differentials are defined by $d^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^i$, and then extended by S-linearity.

The fact that $\mathbf{A}$ is a cochain complex is verified as follows:

$$\begin{aligned} d^{i+1} d^i(u \otimes 1) &= \sum_{k=1}^n \sum_{j=1}^n e_k e_j u \otimes x_j x_k \\ &= \sum_{j<k} (e_k e_j + e_j e_k) u \otimes x_j x_k = 0. \end{aligned}$$

The relationship between the two definitions is given by the following well-known lemma.

Lemma 2.3. *The evaluation of the universal Aomoto complex* $\mathbf{A}$ *at an element* $a \in A^1$ *coincides with the Aomoto complex* (A, a).

Proof. Write $a = \sum_{j=1}^n a_j e_j \in A^1$, and let $\operatorname{ev}_a\colon S \to \mathbb{C}$ be the ring morphism given by $g \mapsto g(a_1, \dots, a_n)$. The resulting cochain complex, $\mathbf{A}(a) = \mathbf{A} \otimes_S \mathbb{C}$, has differentials $d^i(a) := d^i \otimes \mathrm{id}_{\mathbb{C}}$ given by

$$d^i(a)(u) = \sum_{j=1}^n e_j u \otimes \operatorname{ev}_a(x_j) = \sum_{j=1}^n e_j u \cdot a_j = a \cdot u. \tag{7}$$

Thus, $\mathbf{A}(a) = (A, a)$. □

2.2. Minimality and the Aomoto complex. As shown by Papadima and Suciu in [28], the universal Aomoto complex of $H^*(X, \mathbb{C})$ is functorially determined by the equivariant chain complex of the universal abelian cover X^{ab}, provided X admits a minimal cell structure.

More precisely, suppose X is a connected, finite-type CW-complex. We say the CW-structure on X is *minimal* if the number of i-cells of

X equals the Betti number $b_i(X)$, for every $i \geq 0$. Equivalently, the boundary maps in the cellular chain complex $C_\bullet(X,\mathbb{Z})$ are all zero maps. In particular, the homology groups $H_i(X,\mathbb{Z})$ are all torsion-free.

Theorem 2.4 ([28]). *Let X be a minimal CW-complex. Then the linearization of the cochain complex $C^\bullet(X^{\mathrm{ab}},\mathbb{C})$ coincides with the universal Aomoto complex of $H^*(X,\mathbb{C})$.*

Let us explain in more detail how this theorem works. Pick an isomorphism $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^n$, and identify $\mathbb{C}[\mathbb{Z}^n]$ with $\Lambda = \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$. Next, filter Λ by powers of the maximal ideal $I = (t_1-1,\dots,t_n-1)$, and identify the associated graded ring, $\operatorname{gr}(\Lambda)$, with the polynomial ring $S = \mathbb{C}[x_1,\dots,x_n]$, via the ring map $t_i - 1 \mapsto x_i$.

The minimality hypothesis allows us to identify $C_i(X^{\mathrm{ab}},\mathbb{C})$ with $\Lambda \otimes_{\mathbb{C}} H_i(X,\mathbb{C})$ and $C^i(X^{\mathrm{ab}},\mathbb{C})$ with $A^i \otimes_{\mathbb{C}} \Lambda$. Under these identifications, the boundary map $\partial^{\mathrm{ab}}_{i+1}\colon C_{i+1}(X^{\mathrm{ab}},\mathbb{C}) \to C_i(X^{\mathrm{ab}},\mathbb{C})$ dualizes to a map $\delta^i\colon A^i \otimes_{\mathbb{C}} \Lambda \to A^{i+1} \otimes_{\mathbb{C}} \Lambda$. Let $\operatorname{gr}(\delta^i)\colon A^i \otimes_{\mathbb{C}} S \to A^{i+1} \otimes_{\mathbb{C}} S$ be the associated graded of δ^i, and let $\operatorname{gr}(\delta^i)^{\,\mathrm{lin}}$ be its linear part. Theorem 2.4 then provides an identification

$$\operatorname{gr}(\delta^i)^{\,\mathrm{lin}} = d^i\colon A^i \otimes_{\mathbb{C}} S \to A^{i+1} \otimes_{\mathbb{C}} S, \tag{8}$$

for each $i \geq 0$.

Example 2.5. Let $X = T^n$ be the n-dimensional torus, with the standard product cell structure. Then X is a minimal cell complex, and $C_\bullet(X^{\mathrm{ab}},\mathbb{C})$ is the Koszul complex $K(t_1-1,\dots,t_n-1)$ over the ring Λ. The cohomology ring $H^*(T^n,\mathbb{C})$ is the exterior algebra E on variables $e_1,\dots,e_n$, and the universal Aomoto complex $E \otimes_{\mathbb{C}} S$ is the Koszul complex $K(x_1,\dots,x_n)$ over the ring S. In this case, Theorem 2.4 simply says that the substitution $t_i - 1 \mapsto x_i$ takes one Koszul complex to the other.

Example 2.6. Let $Y = S^1 \vee S^2$, and identify $\pi_1(Y) = \mathbb{Z}$, with generator t, and $\pi_2(Y) = \mathbb{Z}[t^{\pm 1}]$. Given a polynomial $f \in \mathbb{Z}[t]$, let $\varphi\colon S^2 \to Y$ be a map representing f, and attach a 3-cell to Y along φ to obtain a CW-complex X_f. For instance, if $f(t) = t-1$, then $X_f \simeq S^1 \times S^2$. More generally, X_f is minimal if and only if $f(1) = 0$, in which case $H_*(X_f,\mathbb{Z}) \cong H_*(S^1 \times S^2,\mathbb{Z})$.

Now identify $\pi_1(X_f) = \mathbb{Z}$ and $\mathbb{C}[\mathbb{Z}]$ with $\Lambda = \mathbb{C}[t^{\pm 1}]$. The chain complex $C_\bullet(X_f^{\mathrm{ab}},\mathbb{C})$ can then be written as

$$\Lambda \xrightarrow{f(t)} \Lambda \xrightarrow{0} \Lambda \xrightarrow{t-1} \Lambda\,. \tag{9}$$

Finally, identify $\operatorname{gr}(\Lambda)$ with $S = \mathbb{C}[x]$, and set $g(x) = f(1+x)$. Suppose $f(1) = 0$, so that X_f is minimal. Then, the linear term of $g(x)$

is $f'(1)\cdot x$, and so the universal Aomoto complex of $H^*(X_f,\mathbb{C})$ is

$$S \xrightarrow{\ x\ } S \xrightarrow{\ 0\ } S \xrightarrow{f'(1)x} S\ . \tag{10}$$

3. Resonance varieties

In this section, we review the definition and basic properties of the resonance varieties of a space X, which measure the deviation from exactness of the Aomoto complexes associated to the cohomology ring $H^*(X,\mathbb{C})$.

3.1. Jump loci for the Aomoto-Betti numbers. As usual, let X be a connected CW-complex with finite k-skeleton. Denote by A the cohomology algebra $H^*(X,\mathbb{C})$. Computing the homology of the Aomoto complexes (A,a) for various values of the parameter $a\in A^1$, and recording the resulting Betti numbers, carves out some very interesting subsets of the affine space $A^1=\mathbb{C}^n$, where $n=b_1(X)$.

Definition 3.1. The *resonance varieties* of X are the sets

$$\mathcal{R}^i_d(X)=\{a\in A^1 \mid \dim_{\mathbb{C}} H^i(A,\cdot a)\geq d\},$$

defined for all integers $0\leq i\leq k$ and $d>0$.

The degree-1, depth-1 resonance variety is especially easy to describe: $\mathcal{R}^1_1(X)$ consists of those elements $a\in A^1$ for which there exists an element $b\in A^1$, not proportional to a, and such that $b\cdot a=0$.

The terminology from Definition 3.1 is justified by the following well-known lemma. For completeness, we include a proof.

Lemma 3.2. *The sets $\mathcal{R}^i_d(X)$ are homogeneous algebraic subvarieties of the affine space $A^1=H^1(X,\mathbb{C})$.*

Proof. By definition, an element $a\in A^1$ belongs to $\mathcal{R}^i_d(X)$ if and only if $\operatorname{rank}\delta^{i-1}(a)+\operatorname{rank}\delta^i(a)\leq c_i-d$, where $c_i=c_i(X)$ is the number of i-cells of X. As a set, then, $\mathcal{R}^i_d(X)$ can be written as the intersection

$$\bigcap_{\substack{r+s=c_i-d+1\\ r,s\geq 0}} \{a\in A^1 \mid \operatorname{rank}\delta^{i-1}(a)\leq r-1 \text{ or } \operatorname{rank}\delta^i(a)\leq s-1\}.$$

Using this description, we may rewrite $\mathcal{R}^i_d(X)$ as the zero-set of a sum of products of determinantal ideals,

$$\mathcal{R}^i_d(X)=V\Bigg(\sum_{p+q=c_{i+1}+d-1} E_p(\delta^{i-1})\cdot E_q(\delta^i)\Bigg). \tag{11}$$

Clearly, $a\in\mathcal{R}^i_d(X)$ if and only if $\lambda a\in\mathcal{R}^i_d(X)$, for all $\lambda\in\mathbb{C}^\times$. Thus, $\mathcal{R}^i_d(X)$ is a homogeneous variety. $\square$

Sometimes it will be more convenient to consider the projectivization $\overline{\mathcal{R}}^i_d(X)$, viewed as a subvariety of $\mathbb{P}(A^1) = \mathbb{CP}^{n-1}$.

The resonance varieties $\mathcal{R}^i_d(X)$ are homotopy-type invariants of the space X. The following (folklore) result makes this more precise.

Lemma 3.3. *Suppose $X \simeq X'$. There is then a linear isomorphism $H^1(X',\mathbb{C}) \cong H^1(X,\mathbb{C})$ which restricts to isomorphisms $\mathcal{R}^i_d(X') \cong \mathcal{R}^i_d(X)$, for all $i \le k$ and $d > 0$.*

Proof. Let $f\colon X \to X'$ be a homotopy equivalence. The induced homomorphism in cohomology, $f^*\colon H^*(X',\mathbb{C}) \xrightarrow{\simeq} H^*(X,\mathbb{C})$, defines isomorphisms $(H^*(X',\mathbb{C}), a) \xrightarrow{\simeq} (H^*(X,\mathbb{C}), f^*(a))$ between the respective Aomoto complexes, for all $a \in H^1(X',\mathbb{C})$. Hence, $f^*\colon H^1(X',\mathbb{C}) \xrightarrow{\simeq} H^1(X,\mathbb{C})$ restricts to isomorphisms $\mathcal{R}^i_d(X') \xrightarrow{\simeq} \mathcal{R}^i_d(X)$. ☐

3.2. Discussion. In each degree $i \ge 0$, the resonance varieties provide a descending filtration,

$$H^1(X,\mathbb{C}) = \mathcal{R}^i_0(X) \supseteq \mathcal{R}^i_1(X) \supseteq \mathcal{R}^i_2(X) \supseteq \cdots . \tag{12}$$

Note that $0 \in \mathcal{R}^i_d(X)$, for $d \le b_i(X)$, but $\mathcal{R}^i_d(X) = \emptyset$, for $d > b_i(X)$. In degree 0, we have $\mathcal{R}^0_1(X) = \{0\}$, and $\mathcal{R}^0_d(X) = \emptyset$, for $d > 1$. In degree 1, the varieties $\mathcal{R}^1_d(X)$ depend only on the group $G = \pi_1(X, x_0)$—in fact, only on the cup-product map $\cup\colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$. Accordingly, we will sometimes write $\mathcal{R}^1_d(G)$ for $\mathcal{R}^1_d(X)$.

Example 3.4. Let T^n be the n-dimensional torus. Using the exactness of the Koszul complex from Example 2.5, we see that $\mathcal{R}^i_d(T^n)$ equals $\{0\}$ if $d \le \binom{n}{i}$, and is empty, otherwise.

Example 3.5. Let S_g be the compact, connected, orientable surface of genus $g > 1$. With suitable identifications $H^1(S_g,\mathbb{C}) = \mathbb{C}^{2g}$ and $H^2(S_g,\mathbb{C}) = \mathbb{C}$, the cup-product map $\cup\colon H^1(S_g,\mathbb{C}) \wedge H^1(S_g,\mathbb{C}) \to H^2(S_g,\mathbb{C})$ is the standard symplectic form. A computation then shows

$$\mathcal{R}^i_d(S_g) = \begin{cases} \mathbb{C}^{2g} & \text{if } i=1,\ d<2g-1, \\ \{0\} & \text{if } i=1,\ d\in\{2g-1,2g\}, \text{ or } i\in\{0,2\},\ d=1, \\ \emptyset & \text{otherwise.} \end{cases} \tag{13}$$

One may extend the definition of resonance varieties to arbitrary fields $\Bbbk$, provided $H_1(X,\mathbb{Z})$ is torsion-free, if $\operatorname{char}\Bbbk = 2$. The resulting varieties, $\mathcal{R}^i_d(X,\Bbbk)$, behave well under field extensions: if $\Bbbk \subseteq \mathbb{K}$, then $\mathcal{R}^i_d(X,\Bbbk) = \mathcal{R}^i_d(X,\mathbb{K}) \cap H^1(X,\Bbbk)$. In particular, $\mathcal{R}^i_d(X,\mathbb{Q})$ is just the set of rational points on the integrally defined variety $\mathcal{R}^i_d(X) = \mathcal{R}^i_d(X,\mathbb{C})$.

3.3. Depth one resonance varieties. Most important for us are the depth-1 resonance varieties, $\mathcal{R}^i_1(X)$, and their unions up to a fixed degree, $\mathcal{R}^i(X) = \bigcup_{j=0}^{i} \mathcal{R}^j_1(X)$. The latter varieties can be written as

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^j(A, \cdot a) \neq 0, \text{ for some } j \leq i\}. \tag{14}$$

These sets provide an ascending filtration of the first cohomology group,

$$\{0\} = \mathcal{R}^0(X) \subseteq \mathcal{R}^1(X) \subseteq \cdots \subseteq \mathcal{R}^k(X) \subseteq H^1(X, \mathbb{C}) = \mathbb{C}^n. \tag{15}$$

For low values of $n = b_1(X)$, the variety $\mathcal{R}^1(X)$ is easy to describe.

Proposition 3.6. *If $n \leq 1$, then $\mathcal{R}^1(X) = \{0\}$. If $n = 2$, then $\mathcal{R}^1(X) = \mathbb{C}^2$ or $\{0\}$, according to whether the cup product vanishes on $H^1(X, \mathbb{C})$ or not.*

For $n \geq 3$, the resonance variety $\mathcal{R}^1(X)$ can be much more complicated; in particular, it may have irreducible components which are not linear subspaces. The following example (a particular case of a more general construction described in [11]) illustrates this phenomenon.

Example 3.7. Let $X = F(T^2, 3)$ be the configuration space of 3 labeled points on the torus. The cohomology ring of X is the exterior algebra on generators $a_1, a_2, a_3, b_1, b_2, b_3$ in degree 1, modulo the ideal $\langle (a_1 - a_2)(b_1 - b_2), (a_1 - a_3)(b_1 - b_3), (a_2 - a_3)(b_2 - b_3) \rangle$. A calculation reveals that

$$\mathcal{R}^1(X) = \{(a, b) \in \mathbb{C}^6 \mid a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = a_1 b_2 - a_2 b_1 = 0\}.$$

Hence, $\mathcal{R}^1(X)$ is isomorphic to $Q = V(a_1 b_2 - a_2 b_1)$, a smooth quadric hypersurface in $\mathbb{C}^4$. (The projectivization of Q is the image of the Segré embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $\mathbb{CP}^3$.)

The depth-1 resonance varieties of a product or a wedge of two spaces can be expressed in terms of the resonance varieties of the factors. Start with a product $X = X_1 \times X_2$, where both X_1 and X_2 have finite k-skeleton, and identify $H^1(X, \mathbb{C}) = H^1(X_1, \mathbb{C}) \times H^1(X_2, \mathbb{C})$.

Proposition 3.8 ([29])**.** *For all $i \leq k$,*

$$\mathcal{R}^i_1(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{R}^p_1(X_1) \times \mathcal{R}^q_1(X_2).$$

Corollary 3.9. *We have $\mathcal{R}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{R}^p(X_1) \times \mathcal{R}^q(X_2)$, for all $i \leq k$.*

Example 3.10. For a Riemann surface of genus $g > 1$, formula (13) yields $\mathcal{R}^0(S_g) = \{0\}$ and $\mathcal{R}^i(S_g) = \mathbb{C}^{2g}$, for all $i \geq 1$. For a product of two such surfaces, Corollary 3.9 now gives

$$\mathcal{R}^i(S_g \times S_h) = \begin{cases} \{0\} & \text{if } i = 0, \\ \mathbb{C}^{2g} \times \{0\} \cup \{0\} \times \mathbb{C}^{2h} & \text{if } i = 1, \\ \mathbb{C}^{2(g+h)} & \text{if } i \geq 2. \end{cases} \tag{16}$$

Next, consider a wedge $X = X_1 \vee X_2$, and identify $H^1(X, \mathbb{C}) = H^1(X_1, \mathbb{C}) \times H^1(X_2, \mathbb{C})$.

Proposition 3.11 ([29]). *Suppose X_1 and X_2 have positive first Betti numbers. Then*

$$\mathcal{R}^i_1(X) = \begin{cases} H^1(X, \mathbb{C}) & \text{if } i = 1, \\ \mathcal{R}^i_1(X_1) \times H^1(X_2, \mathbb{C}) \cup H^1(X_1, \mathbb{C}) \times \mathcal{R}^i_1(X_2) & \text{if } 1 < i \leq k. \end{cases}$$

Corollary 3.12. *Let $X = X_1 \vee X_2$, where X_1 and X_2 have positive first Betti numbers. Then $\mathcal{R}^i(X) = H^1(X, \mathbb{C})$, for all $1 \leq i \leq k$.*

4. Tangent cones to affine varieties

We now discuss two versions of the tangent cone to a subvariety of the complex algebraic torus $(\mathbb{C}^\times)^n$.

4.1. The tangent cone. We start by reviewing a well-known notion in algebraic geometry (see [20, pp. 251–256]). Let $W \subset (\mathbb{C}^\times)^n$ be a Zariski closed subset, defined by an ideal I in the Laurent polynomial ring $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Picking a finite generating set for I, and multiplying these generators with suitable monomials if necessary, we see that W may also be defined by the ideal $I \cap R$ in the polynomial ring $R = \mathbb{C}[t_1, \dots, t_n]$.

Now consider the polynomial ring $S = \mathbb{C}[z_1, \dots, z_n]$, and let J be the ideal generated by the polynomials

$$g(z_1, \dots, z_n) = f(z_1 + 1, \dots, z_n + 1), \tag{17}$$

for all $f \in I \cap R$. Finally, let $\operatorname{in}(J)$ be the ideal in S generated by the initial forms of all non-zero elements from J.

Definition 4.1. The *tangent cone* of W at 1 is the algebraic subset $\operatorname{TC}_1(W) \subset \mathbb{C}^n$ defined by the initial ideal $\operatorname{in}(J) \subset S$.

The tangent cone $\operatorname{TC}_1(W)$ is a homogeneous subvariety of $\mathbb{C}^n$, depending only on the analytic germ of W at the identity. In particular, $\operatorname{TC}_1(W) \neq \emptyset$ if and only if $1 \in W$. Moreover, TC_1 commutes with finite unions, but not necessarily with intersections. Explicit equations for the tangent cone to a variety can be found using the Gröbner basis algorithm described in [14, Proposition 15.28].

4.2. The exponential tangent cone. A competing notion of tangent cone was introduced by Dimca, Papadima and Suciu in [11], and further studied in [29] and [34].

Definition 4.2. The *exponential tangent cone* of W at 1 is the homogeneous subvariety $\tau_1(W)$ of $\mathbb{C}^n$, defined by

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

The exponential tangent cone $\tau_1(W)$ depends only on the analytic germ of W at the identity. In particular, $\tau_1(W) \neq \emptyset$ if and only if $1 \in W$. Moreover, τ_1 commutes with finite unions, as well as arbitrary intersections. The most important property of this construction is given in the following result from [11] (see [34] for full details).

Theorem 4.3 ([11])**.** *The exponential tangent cone $\tau_1(W)$ is a finite union of rationally defined linear subspaces of $\mathbb{C}^n$.*

These subspaces can be described explicitly. By the above remarks, we may assume $W = V(f)$, where f is a non-zero Laurent polynomial with $f(1) = 0$. Write $f = \sum_{a\in S} c_a t_1^{a_1} \cdots t_n^{a_n}$, where S is a finite subset of $\mathbb{Z}^n$, and $c_a \in \mathbb{C} \setminus \{0\}$ for each $a = (a_1, \dots, a_n) \in S$. We say a partition $\mathsf{p} = (\mathsf{p}_1 \,|\, \cdots \,|\, \mathsf{p}_q)$ of the support S is *admissible* if $\sum_{a\in \mathsf{p}_i} c_a = 0$, for all $1 \leq i \leq q$. For each such partition, let $L(\mathsf{p})$ be the rational linear subspace consisting of all points $x \in \mathbb{Q}^n$ for which the dot product $(a - b) \cdot x$ vanishes, for all $a, b \in \mathsf{p}_i$ and all $1 \leq i \leq q$. Then

$$\tau_1(W) = \bigcup_{\mathsf{p}} L(\mathsf{p}) \otimes \mathbb{C}, \tag{18}$$

where the union is taken over all (maximal) admissible partitions of S.

4.3. Relating the two tangent cones. The next lemma (first noted in [11]) records a general relationship between the two kinds of tangent cones discussed here. For completeness, we include a proof, along the lines of the proof given in [29], though slightly modified to fit the definition of TC_1 given here.

Lemma 4.4 ([11, 29])**.** *For every Zariski closed subset $W \subset (\mathbb{C}^\times)^n$, we have $\tau_1(W) \subseteq \mathrm{TC}_1(W)$.*

Proof. Without loss of generality, we may assume $1 \in W$. Let $f \in R$ be a non-zero polynomial in $I = I(W)$, let $g \in S$ be the polynomial defined by (17), and let $g_0 = \mathrm{in}(g)$. We then have

$$\begin{aligned} f(e^{\lambda z_1}, \dots, e^{\lambda z_n}) &= f(1 + \lambda z_1 + O(\lambda^2), \dots, 1 + \lambda z_n + O(\lambda^2)) \\ &= g(\lambda z_1 + O(\lambda^2), \dots, \lambda z_n + O(\lambda^2)) \\ &= g_0(\lambda z_1, \dots, \lambda z_n) + \text{higher order terms}. \end{aligned}$$

Now suppose $z \in \tau_1(W)$, that it to say, $f(e^{\lambda z_1}, \dots, e^{\lambda z_n}) = 0$, for all $\lambda \in \mathbb{C}$. The above calculation shows that $g_0(\lambda z) = 0$, for all λ; in particular, $g_0(z) = 0$. We conclude that $z \in \mathrm{TC}_1(W)$. □

If W is an algebraic subtorus of $(\mathbb{C}^\times)^n$, then $\tau_1(W) = \mathrm{TC}_1(W)$, and both coincide with the tangent space at the origin, $T_1(W)$. In general, though, the inclusion from Lemma 4.4 can be strict, even when 1 is a smooth point of W. Here is an example illustrating this phenomenon.

Example 4.5. Let W be the hypersurface in $(\mathbb{C}^\times)^3$ with equation $t_1 + t_2 + t_3 - t_1t_2 - t_1t_3 - t_2t_3 = 0$. Then $\tau_1(W)$ is a union of 3 lines in $\mathbb{C}^3$, given by the equations $z_1 = z_2 + z_3 = 0$, $z_2 = z_1 + z_3 = 0$, and $z_3 = z_1 + z_2 = 0$. On the other hand, $\mathrm{TC}_1(W)$ is a plane in $\mathbb{C}^3$, with equation $z_1 + z_2 + z_3 = 0$. Hence, $\mathrm{TC}_1(W)$ strictly contains $\tau_1(W)$.

5. Characteristic varieties

In this section, we briefly review the characteristic varieties of a space, and their relation to the resonance varieties.

5.1. Jump loci for twisted homology. Let X be a connected CW-complex with finite k-skeleton, $k \geq 1$. Without loss of generality, we may assume X has a single 0-cell, call it x_0. Let $G = \pi_1(X, x_0)$ be the fundamental group of X, and let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ be the group of complex characters of G. Clearly, $\widehat{G} = \widehat{G}_{\mathrm{ab}}$, where $G_{\mathrm{ab}} = H_1(X, \mathbb{Z})$ is the abelianization of G. Thus, the universal coefficient theorem allows us to identify

$$\widehat{G} = H^1(X, \mathbb{C}^\times). \tag{19}$$

Each character $\rho\colon G \to \mathbb{C}^\times$ determines a rank 1 local system $\mathcal{L}_\rho$ on our space X. Computing the homology groups of X with coefficients in such local systems leads to a natural filtration of the character group.

Definition 5.1. The *characteristic varieties* of X are the sets

$$\mathcal{V}^i_d(X) = \{ \rho \in H^1(X, \mathbb{C}^\times) \mid \dim_{\mathbb{C}} H_i(X, \mathcal{L}_\rho) \geq d \},$$

defined for all $0 \leq i \leq k$ and all $d > 0$.

Clearly, $1 \in \mathcal{V}^i_d(X)$ if and only if $d \leq b_i(X)$. In degree 0, we have $\mathcal{V}^0_1(X) = \{1\}$ and $\mathcal{V}^0_d(X) = \emptyset$, for $d > 1$. In degree 1, the sets $\mathcal{V}^i_d(X)$ depend only on the group $G = \pi_1(X, x_0)$—in fact, only on its maximal metabelian quotient, G/G''.

The jump loci $\mathcal{V}^i_d(X)$ are Zariski closed subsets of the algebraic group $H^1(X, \mathbb{C}^\times)$; moreover, these varieties are homotopy-type invariants of X. For details and further references, see [34].

5.2. Depth one characteristic varieties. Most important for us are the depth one characteristic varieties, $\mathcal{V}^i_1(X)$, and their unions up to a fixed degree, $\mathcal{V}^i(X) = \bigcup_{j=0}^{i} \mathcal{V}^j_1(X)$. These varieties yield an ascending filtration of the character group,

$$\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \mathcal{V}^k(X) \subseteq H^1(X, \mathbb{C}^\times). \tag{20}$$

Let $\widehat{G}^0 = H^1(X, \mathbb{C}^\times)^0$ be the identity component of the character group $\widehat{G}$. Writing $n = b_1(X)$, we may identify $\widehat{G}^0$ with the complex algebraic torus $(\mathbb{C}^\times)^n$. Set $\mathcal{W}^i_d(X) = \mathcal{V}^i_d(X) \cap \widehat{G}^0$, and $\mathcal{W}^i(X) = \bigcup_{j=0}^{i} \mathcal{W}^j_1(X)$. We then have

$$\mathcal{W}^i(X) = \mathcal{V}^i(X) \cap \widehat{G}^0. \tag{21}$$

Example 5.2. Let L be an n-component link in S^3, with complement X. Using a basis for $H_1(X, \mathbb{Z}) = \mathbb{Z}^n$ given by (oriented) meridians, we may identify $H^1(X, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then

$$\mathcal{W}^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\}, \tag{22}$$

where $\Delta_L = \Delta_L(t_1, \dots, t_n)$ is the Alexander polynomial of the link. For details and references, see [33].

As shown in [29] (see also [34]), the characteristic varieties satisfy product and wedge formulas similar to those satisfied by the resonance varieties.

Proposition 5.3. *Let X_1 and X_2 be connected CW-complexes with finite k-skeleton, and fix an integer $1 \leq i \leq k$. Then*

(1) $\mathcal{W}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{W}^p(X_1) \times \mathcal{W}^q(X_2)$.
(2) *Suppose, moreover, that $b_1(X_1) > 0$ and $b_1(X_2) > 0$. Then* $\mathcal{W}^i(X_1 \vee X_2) = H^1(X_1 \vee X_2, \mathbb{C}^\times)^0$.

5.3. Characteristic subspace arrangements. As before, let X be a connected CW-complex with finite k-skeleton. Set $n = b_1(X)$, and identify $H^1(X, \mathbb{C}) = \mathbb{C}^n$ and $H^1(X, \mathbb{C}^\times)^0 = (\mathbb{C}^\times)^n$. Applying Theorem 4.3 to the characteristic varieties $\mathcal{W}^i(X) \subseteq (\mathbb{C}^\times)^n$ leads to the following definition.

Definition 5.4. For each $i \leq k$, the *i-th characteristic arrangement* of X, denoted $\mathcal{C}_i(X)$, is the subspace arrangement in $H^1(X, \mathbb{Q})$ whose complexified union is the exponential tangent cone to $\mathcal{W}^i(X)$:

$$\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}.$$

We thus have a sequence $\mathcal{C}_0(X)$, $\mathcal{C}_1(X), \dots, \mathcal{C}_k(X)$ of rational subspace arrangements, all lying in the same affine space $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$. As noted in [34, Lemma 6.7], these subspace arrangements depend only on the homotopy type of X.

5.4. Tangent cone and resonance. Of great importance in the theory of cohomology jumping loci is the relationship between characteristic and resonance varieties, based on the tangent cone construction. A foundational result in this direction is the following theorem of Libgober [21].

Theorem 5.5 ([21]). *The tangent cone at 1 to $\mathcal{W}^i_d(X)$ is included in $\mathcal{R}^i_d(X)$, for all $i \le k$ and $d > 0$.*

Since tangent cones commute with finite unions, we get:

$$\mathrm{TC}_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X), \quad \text{for all } i \le k. \tag{23}$$

For many spaces of interest, inclusion (23) holds as an equality. In general, though, the inclusion is strict.

Example 5.6. Let M be the 3-dimensional Heisenberg nilmanifold. It is easily seen that $\mathcal{W}^1(M) = \{1\}$; hence, $\mathrm{TC}_1(\mathcal{W}^1(M)) = \{0\}$. On the other hand, $\mathcal{R}^1(M) = \mathbb{C}^2$, since the cup product vanishes on $H^1(M, \mathbb{C})$.

6. Straight spaces

We now delineate a class of spaces for which the characteristic and resonance varieties have a simple nature, and are intimately related to each other via the tangent cone constructions discussed in §4.

6.1. Straightness. As before, let X be a connected CW-complex with finite k-skeleton. For each $i \le k$, consider the following conditions on the varieties $\mathcal{W}^i(X)$ and $\mathcal{R}^i(X)$:

- (a) All components of $\mathcal{W}^i(X)$ passing through the origin are algebraic subtori.
- (b) $\mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.
- (c) All components of $\mathcal{W}^i(X)$ not passing through the origin are 0-dimensional.

Definition 6.1. We say X is *locally k-straight* if conditions (a) and (b) hold, for each $i \le k$. If these conditions hold for all $k \ge 1$, we say X is a *locally straight space.*

Definition 6.2. We say X is *k-straight* if conditions (a), (b), and (c) hold, for each $i \le k$. If these conditions hold for all $k \ge 1$, we say X is a *straight space.*

Clearly, the k-straightness property depends only on the homotopy type of a given space. In view of this observation, we may declare a group G to be k-straight if there is a classifying space $K(G,1)$ which is k-straight; in particular, such a group G must be of type F_k, i.e., have a $K(G,1)$ with finite k-skeleton. Note that a space is 1-straight if and only if its fundamental group is 1-straight. Similar considerations apply to local straightness.

The straightness conditions are quite stringent, and thus easily violated. Here are a few examples when this happens, all for $k=1$.

Example 6.3. The closed three-link chain L (the link 6^3_1 from Rolfsen's tables) has Alexander polynomial $\Delta_L = t_1+t_2+t_3-t_1t_2-t_1t_3-t_2t_3$. Let $X=S^3\setminus L$ be the link complement. By (22), the characteristic variety $W=\mathcal{W}^1(X)$ has equation $\Delta_L=0$. On the other hand, we know from Example 4.5 that $\tau_1(W)\neq \mathrm{TC}_1(W)$. Thus, W is not an algebraic torus, and so X is not locally 1-straight: condition (a) fails.

Example 6.4. The Heisenberg nilmanifold M from Example 5.6 is not locally 1-straight: condition (a) is met, but not condition (b).

Example 6.5. Let $G=\langle x_1,x_2 \mid x_1^2x_2 = x_2x_1^2\rangle$ be the group from [**29**, Example 6.4]. Then $\mathcal{W}^1(G)=\{1\}\cup\{(t_1,t_2)\in(\mathbb{C}^\times)^2 \mid t_1=-1\}$, and $\mathcal{R}^1(G)=\{0\}$. Thus, G is locally 1-straight, but not 1-straight: conditions (a) and (b) are met, but not condition (c).

Nevertheless, as we will see below, there is an abundance of interesting spaces which are, to a degree on another, straight.

6.2. Examples and discussion. To start with, let us consider the case when the first characteristic variety $\mathcal{W}^1(X)$ is as big as it can be.

Lemma 6.6. *Let X be a finite-type CW-complex. Suppose that $\mathcal{W}^1(X) = H^1(X,\mathbb{C}^\times)^0$. Then X is straight.*

Proof. Our assumption forces $\mathcal{W}^i(X)=H^1(X,\mathbb{C}^\times)^0$, for all $i\geq 1$. Straightness conditions (a) and (c) are automatically met, while condition (b) follows from (23). □

Examples of spaces satisfying the hypothesis of the above lemma include the Riemann surfaces S_g with $g\geq 2$.

Lemma 6.7. *Let X be a CW-complex with finite k-skeleton, $k\geq 1$. Suppose that $\mathcal{W}^k(X)$ is finite, and $\mathcal{R}^k(X)=\{0\}$. Then X is k-straight.*

Proof. Since $\mathcal{W}^k(X)$ is finite, conditions (a) and (c) are trivially satisfied. By formula (23), we have $\mathrm{TC}_1(\mathcal{W}^i(X))\subseteq \mathcal{R}^i(X)=\{0\}$, for all $i\leq k$. Since $1\in\mathcal{W}^i(X)$, equality must hold, i.e., condition (b) is satisfied. □

Examples of spaces satisfying the hypothesis of the above lemma include the tori T^n; these spaces are, in fact, straight.

Evidently, all spaces X with $b_1(X) = 0$ are straight. Next, we deal with the case $b_1(X) = 1$. The following example shows that not all finite CW-complexes with first Betti number 1 are (locally) straight.

Example 6.8. Let f be a polynomial in $\mathbb{Z}[t]$ with $f(1) = 0$, and let $X_f = (S^1 \vee S^2) \cup_\varphi e^3$ be the corresponding minimal CW-complex constructed in Example 2.6. A calculation with the equivariant chain complex (9) shows that $\mathcal{W}^1(X_f) = \{1\}$ and $\mathcal{W}^2(X_f) = V(f)$; clearly, both these sets are finite subsets of $H^1(X, \mathbb{C}^\times) = \mathbb{C}^\times$.

An analogous calculation with the universal Aomoto cochain complex (10) shows that $\mathcal{R}^1(X_f) = \{0\}$, yet $\mathcal{R}^2(X_f) = \{0\}$ only if $f'(1) \neq 0$, but $\mathcal{R}^2(X_f) = \mathbb{C}$, otherwise. Therefore, X_f is always 1-straight, but

$$X_f \text{ is locally 2-straight} \iff f'(1) \neq 0. \tag{24}$$

Applying a similar reasoning to cell complexes of the form $X_f = (S^1 \vee S^k) \cup_\varphi e^{k+1}$, with attaching maps corresponding to polynomials $f \in \mathbb{Z}[t]$ with $f(1) = f'(1) = 0$, we obtain the following result.

Proposition 6.9. *For each $k \geq 2$, there is a minimal CW-complex which has the integral homology of $S^1 \times S^k$ and which is $(k-1)$-straight, but not locally k-straight.*

Nevertheless, we have the following positive result, guaranteeing straightness in a certain range for spaces with first Betti number 1.

Proposition 6.10. *Let X be a CW-complex with finite k-skeleton. Assume $b_1(X) = 1$. Then,*

1. *X is 1-straight.*
2. *If, moreover, $b_i(X) = 0$ for $1 < i \leq k$, then X is k-straight.*

Proof. As noted in Proposition 3.6, the fact that $b_1(X) = 1$ implies $\mathcal{R}^1(X) = \{0\}$. Identify $H^1(X, \mathbb{C}^\times) = \mathbb{C}^\times$. By formula (23), we have $\operatorname{TC}_1(\mathcal{W}^1(X)) = \{0\}$. Hence, $\mathcal{W}^1(X)$ is a proper subvariety of $\mathbb{C}^\times$; consequently, it is a finite set. Part (1) now follows from Lemma 6.7.

For Part (2), note that $\mathcal{R}^i(X) = \{0\}$, for all $i \leq k$. As above, we conclude that $\mathcal{W}^k(X)$ is finite, and hence X is k-straight. □

Corollary 6.11. *Let K be a smoothly embedded d-sphere in S^{d+2}, and let $X = S^{d+2} \setminus K$ be its complement. Then*

1. *$\mathcal{R}^k(X) = \{0\}$ and $\mathcal{W}^k(X)$ is finite, for all $k \geq 1$.*
2. *X is straight.*

Proof. The knot complement has the homotopy type of a $(d+1)$-dimensional CW-complex, with the integral homology of S^1. The desired conclusions follow from Proposition 6.10 and its proof. □

For a knot K in S^3, the variety $\mathcal{W}^1(X) \subset \mathbb{C}^\times$ consists of 1, together with all the roots of the Alexander polynomial, Δ_K (these roots are always different from 1).

6.3. Products and wedges. We now look at how the straightness properties behave with respect to (finite) products and wedges of spaces.

Proposition 6.12. *Let X_1 and X_2 be CW-complexes with finite k-skeleta and positive first Betti numbers. Then the space $X = X_1 \vee X_2$ is k-straight.*

Proof. Follows from Proposition 5.3(2) and Lemma 6.6. □

In particular, a wedge of knot complements is straight.

Proposition 6.13. *Let X_1 and X_2 be two CW-complexes.*

1. *If X_1 and X_2 are locally k-straight, then so is $X_1 \times X_2$.*
2. *If X_1 and X_2 are 1-straight, then so is $X_1 \times X_2$.*

Proof. Both assertions follow from the product formulas for resonance and characteristic varieties (Proposition 5.3(1) and Corollary 3.9, respectively). □

Proposition 6.14. *Let X_1 and X_2 be CW-complexes with finite k-skeleton. Suppose $\mathcal{R}^k(X_j) = \{0\}$ and $\mathcal{W}^k(X_j)$ is finite, for $j = 1, 2$. Then $X_1 \times X_2$ is k-straight.*

Proof. From Lemma 6.7, we know that both X_1 and X_2 are k-straight. Now, Corollary 3.9 gives that $\mathcal{R}^k(X_1 \times X_2) = \{0\}$, while Proposition 5.3(1) gives that $\mathcal{W}^k(X_1 \times X_2)$ is finite. Hence, again by Lemma 6.7, $X_1 \times X_2$ is k-straight. □

In particular, a product of knot complements is straight. In general, though, a product of straight spaces need not be straight.

Example 6.15. Let K be a knot in S^3, with Alexander polynomial Δ_K not equal to 1 (for instance, the trefoil knot), and let X be its complement. Let $Y = \bigvee^n S^1$ be a wedge of n circles, with $n \geq 2$. By Corollary 6.11 and Proposition 6.12, both X and Y are straight.

By Proposition 6.13(2), the product $X \times Y$ is 1-straight. Nevertheless, $X \times Y$ is not 2-straight. Indeed, the variety $\mathcal{W}^2(X \times Y)$ has irreducible components of the form $\{\rho\} \times (\mathbb{C}^\times)^n$, where ρ runs through the roots of Δ_K. As these components do not pass through the origin, straightness condition (c) fails for $X \times Y$ in degree $i = 2$.

6.4. Rationality properties. Before proceeding, let us give an alternative characterization of straightness. As usual, let X be a connected CW-complex with finite k-skeleton, and let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^\times)^0$ be the exponential map (or rather, its corestriction to its image).

Theorem 6.16. *A space X as above is locally k-straight if and only if the following equalities hold, for all $i \le k$:*

$$(\alpha) \qquad \mathcal{W}^i(X) = \Big(\bigcup_{L\in\mathcal{C}_i(X)} \exp(L\otimes\mathbb{C}) \Big) \cup Z_i$$

$$(\beta) \qquad \mathcal{R}^i(X) = \bigcup_{L\in\mathcal{C}_i(X)} L\otimes\mathbb{C},$$

for some algebraic subsets $Z_i \subset H^1(X,\mathbb{C}^\times)^0$ not containing the origin.

The space X is k-straight if and only if, in addition, the sets Z_i are finite.

Proof. Suppose X is locally k-straight, and fix an index $i \le k$. By hypothesis (a) from Definition 6.2, the variety $\mathcal{W}^i(X)$ admits a decomposition into irreducible components of the form

$$\mathcal{W}^i(X) = \bigcup_{\alpha\in\Lambda} T_\alpha \cup Z_i, \tag{25}$$

for some algebraic subtori $T_\alpha = \exp(P_\alpha\otimes\mathbb{C})$ and some algebraic sets Z_i with $1 \notin Z_i$. Hence,

$$\tau_1(\mathcal{W}^i(X)) = \bigcup_{\alpha\in\Lambda} P_\alpha\otimes\mathbb{C}. \tag{26}$$

On the other hand, by the definition of characteristic subspace arrangements, we also have $\tau_1(\mathcal{W}^i(X)) = \bigcup_{L\in\mathcal{C}_i(X)} L\otimes\mathbb{C}$. By uniqueness of decomposition into irreducible components, the arrangement $\{P_\alpha\}_{\alpha\in\Lambda}$ must coincide with $\mathcal{C}_i(X)$. Consequently, equation (25) yields (α).

Equation (25) also implies $\tau_1(\mathcal{W}^i(X)) = \mathrm{TC}_1(\mathcal{W}^i(X))$. Hypothesis (b) then gives $\tau_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$, which is precisely condition (β).

Conversely, condition (α) implies (a) and conditions (α) and (β) together imply (b).

Finally, hypothesis (c) is satisfied if and only if the sets $Z_1,\dots,Z_k$ are all finite. ∎

Corollary 6.17. *Let X be a locally k-straight space. Then, for all $i \le k$,*

1. $\tau_1(\mathcal{W}^i(X)) = \mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.
2. $\mathcal{R}^i(X,\mathbb{Q}) = \bigcup_{L\in\mathcal{C}_i(X)} L$.

In particular, the resonance varieties $\mathcal{R}^i(X)$ are unions of rationally defined subspaces.

The next example (adapted from [11]) illustrates how this rationality property may be used to detect non-straightness.

Example 6.18. Consider the group G with generators x_1, x_2, x_3, x_4 and relators $r_1 = [x_1, x_2]$, $r_2 = [x_1, x_4][x_2^{-2}, x_3]$, $r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Direct computation shows that

$$\mathcal{R}^1(G) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\}.$$

Evidently, this variety splits into two linear subspaces defined over $\mathbb{R}$, but not over $\mathbb{Q}$. Thus, G is not (locally) 1-straight.

7. The Dwyer–Fried invariants

In this section, we recall the definition of the Dwyer–Fried sets, and the way these sets relate to the characteristic varieties of a space.

7.1. Betti numbers of free abelian covers.

As before, let X be a connected CW-complex with finite k-skeleton, and let $G = \pi_1(X, x_0)$. Denote by $n = b_1(X)$ the first Betti number of X. (We may as well assume $n > 0$, otherwise the whole theory is empty of content.) Fix an integer r between 1 and n, and consider the regular covers of X, with group of deck-transformations $\mathbb{Z}^r$.

Each such cover, $X^\nu \to X$, is determined by an epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$. The induced homomorphism in cohomology, $\nu^*\colon H^1(\mathbb{Z}^r, \mathbb{Q}) \hookrightarrow H^1(G, \mathbb{Q})$, defines an r-dimensional subspace, $P_\nu = \operatorname{im}(\nu^*)$, in the rational vector space $H^1(G, \mathbb{Q}) = \mathbb{Q}^n$. Conversely, each r-dimensional subspace $P \subset \mathbb{Q}^n$ can be written as $P = P_\nu$, for some epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$, and thus defines a regular $\mathbb{Z}^r$-cover of X.

To recap, the regular $\mathbb{Z}^r$-covers of X are parametrized by the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$, via the correspondence

$$\big\{\mathbb{Z}^r\text{-covers } X^\nu \to X\big\} \longleftrightarrow \big\{r\text{-planes } P_\nu := \operatorname{im}(\nu^*) \text{ in } H^1(X, \mathbb{Q})\big\}.$$

Moving about the rational Grassmannian and recording how the Betti numbers of the corresponding covers vary leads to the following definition.

Definition 7.1. The *Dwyer–Fried invariants* of X are the subsets

$$\Omega_r^i(X) = \big\{P_\nu \in \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \,\big|\, b_j(X^\nu) < \infty \text{ for } j \le i\big\}. \tag{27}$$

Set $n = b_1(X)$. For a fixed r between 1 and n, these sets form a descending filtration of the Grassmannian of r-planes in $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$,

$$\operatorname{Gr}_r(\mathbb{Q}^n) = \Omega_r^0(X) \supseteq \Omega_r^1(X) \supseteq \Omega_r^2(X) \supseteq \cdots. \tag{28}$$

If $r > n$, we adopt the convention that $\operatorname{Gr}_r(\mathbb{Q}^n) = \emptyset$ and define $\Omega^i_r(X) = \emptyset$ in this range.

As noted in [34], the Ω-sets are homotopy-type invariants. More precisely, if $f\colon X \to Y$ is a homotopy equivalence, the induced isomorphism in cohomology, $f^*\colon H^1(Y,\mathbb{Q}) \to H^1(X,\mathbb{Q})$, defines isomorphisms $f^*_r\colon \operatorname{Gr}_r(H^1(Y,\mathbb{Q})) \to \operatorname{Gr}_r(H^1(X,\mathbb{Q}))$, which send each subset $\Omega^i_r(Y)$ bijectively onto $\Omega^i_r(X)$.

Example 7.2. Let T^n be the n-dimensional torus. Since every connected cover of T^n is homotopy equivalent to a k-torus, for some $0 \le k \le n$, we conclude that $\Omega^i_r(T^n) = \operatorname{Gr}_r(\mathbb{Q}^n)$, for all $i \ge 0$ and $r \ge 1$.

7.2. Dwyer–Fried invariants and characteristic varieties. The next theorem reduces the computation of the Ω-sets to a more standard computation in algebraic geometry. The theorem was proved by Dwyer and Fried in [13], using the support loci for the Alexander invariants, and was recast in a slightly more general context by Papadima and Suciu in [29], using the characteristic varieties. We state this result in the form most convenient for our purposes, namely, the one established in [34].

Theorem 7.3 ([13, 29, 34])**.** *For all $i \le k$ and $1 \le r \le n$,*

$$\Omega^i_r(X) = \big\{P \in \operatorname{Gr}_r(\mathbb{Q}^n) \bigm| \#\big(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)\big) < \infty\big\}.$$

In other words, an r-plane $P \subset \mathbb{Q}^n$ belongs to $\Omega^i_r(X)$ if and only if the algebraic torus $T = \exp(P \otimes \mathbb{C})$ intersects the characteristic variety $W = \mathcal{W}^i(X)$ only in finitely many points. When this happens, the exponential tangent cone $\tau_1(T \cap W)$ equals $\{0\}$, forcing $P \cap L = \{0\}$, for every subspace $L \subset \mathbb{Q}^n$ in the characteristic subspace arrangement $\mathcal{C}_i(X)$. As in [34, Theorem 7.1], we obtain the following "upper bound" for the Dwyer–Fried invariants of X:

$$\Omega^i_r(X) \subseteq \Big(\bigcup_{L \in \mathcal{C}_i(X)} \big\{P \in \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \bigm| P \cap L \ne \{0\}\big\} \Big)^{\complement}. \tag{29}$$

7.3. The incidence correspondence. The right side of (29) may be reinterpreted in terms of the classical incidence correspondence from algebraic geometry.

Let V be a homogeneous variety in $\Bbbk^n$. Consider the locus of r-planes in $\Bbbk^n$ meeting V,

$$\sigma_r(V) = \big\{P \in \operatorname{Gr}_r(\Bbbk^n) \bigm| P \cap V \ne \{0\}\big\}. \tag{30}$$

This set is a Zariski closed subset of the Grassmannian $\operatorname{Gr}_r(\Bbbk^n)$, called the *variety of incident r-planes* to V.

Particularly manageable is the case when V is a non-zero linear subspace $L \subset \Bbbk^n$. The corresponding incidence variety, $\sigma_r(L)$, is known as the *special Schubert variety* defined by L. If L has codimension d in $\Bbbk^n$, then $\sigma_r(L)$ has codimension $d-r+1$ in $\operatorname{Gr}_r(\Bbbk^n)$.

Theorem 7.4 ([34]). *Let X be a CW-complex with finite k-skeleton. Then*

$$\Omega^i_r(X) \subseteq \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \setminus \bigcup_{L\in \mathcal{C}_i(X)} \sigma_r(L),$$

for all $i \le k$ and $r \ge 1$.

In other words, each Dwyer–Fried set $\Omega^i_r(X)$ is contained in the complement to the variety of incident r-planes to the i-th characteristic arrangement of X.

7.4. Straightness and the Dwyer–Fried invariants. Under a (local) straightness assumption, the bound from Theorem 7.4 can be expressed in terms of simpler, purely cohomological data. The next result proves Theorem 1.1, part (1) from the Introduction.

Corollary 7.5. *Suppose X is locally k-straight. Then, for all $i \le k$ and $r \ge 1$,*

$$\Omega^i_r(X) \subseteq \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X,\mathbb{Q})).$$

Proof. By Corollary 6.17, we have $\tau_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$. Hence, $\bigcup_{L\in\mathcal{C}_i(X)} L = \mathcal{R}^i(X,\mathbb{Q})$, and the desired conclusion follows from Theorem 7.4. ☐

Under a more stringent straightness assumption, the above inclusion holds as an equality. The next result proves Theorem 1.1, part (2) from the Introduction.

Theorem 7.6. *Suppose X is k-straight. Then, for all $i \le k$ and $r \ge 1$,*

$$\Omega^i_r(X) = \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X,\mathbb{Q})).$$

Proof. By Theorem 7.3, we have

$$\Omega^i_r(X) = \big\{P \bigm| \exp(P\otimes\mathbb{C}) \cap \mathcal{W}^i(X) \text{ is finite}\big\}. \tag{31}$$

Since X is k-straight, Theorem 6.16, part (α) yields

$$\Omega^i_r(X) = \big\{P \bigm| P\cap L = \{0\}, \text{ for all } L\in\mathcal{C}_i(X)\big\}. \tag{32}$$

By Theorem 6.16, part (β), the right side of (32) is the complement to $\sigma_r(\mathcal{R}^i(X,\mathbb{Q}))$, and we are done. ☐

Particularly interesting is the case when all the components of $\mathcal{R}^i(X)$ have the same codimension, say, r. In this situation, $\Omega^i_r(X)$ is the complement of the rational Chow divisor of $\mathcal{R}^i(X,\mathbb{Q})$.

The previous theorem yields a noteworthy qualitative result about the Dwyer–Fried sets of straight spaces, in arbitrary ranks $r \geq 1$.

Corollary 7.7. *Let X be a k-straight space. Then each set $\Omega^i_r(X)$ is the complement of a finite union of special Schubert varieties in the Grassmannian of r-planes in $H^1(X,\mathbb{Q})$. In particular, $\Omega^i_r(X)$ is a Zariski open set in $\operatorname{Gr}_r(H^1(X,\mathbb{Q}))$.*

The straightness hypothesis is crucial for Theorem 7.6 to hold. The next example shows the necessity of condition (c) from Definition 6.2.

Example 7.8. Let G be the group from Example 6.5. Recall we have $\mathcal{W}^1(G) = \{1\} \cup \{t \in (\mathbb{C}^\times)^2 \mid t_1 = -1\}$, but $\mathcal{R}^1(G) = \{0\}$. Thus, $\Omega^1_2(G) = \emptyset$, yet $\sigma_2(\mathcal{R}^1(G,\mathbb{Q}))^{\complement} = \{\text{pt}\}$.

8. The influence of formality

An important property that bridges the gap between the tangent cone to a characteristic variety and the corresponding resonance variety is that of formality.

8.1. Formality. As before, let X be connected CW-complex with finite 1-skeleton. In [**35**], Sullivan constructs an algebra $A_{\text{PL}}(X,\mathbb{Q})$ of polynomial differential forms on X with coefficients in $\mathbb{Q}$, and provides it with a natural commutative differential graded algebra (cdga) structure.

Let $H^*(X,\mathbb{Q})$ be the rational cohomology algebra of X, endowed with the zero differential. The space X is said to be *formal* if there is a zig-zag of cdga morphisms connecting $A_{\text{PL}}(X,\mathbb{Q})$ to $H^*(X,\mathbb{Q})$, with each such morphism inducing an isomorphism in cohomology. The space X is merely *k-formal* (for some $k \geq 1$) if each of these morphisms induces an isomorphism in degrees up to k, and a monomorphism in degree $k+1$.

Examples of formal spaces include rational cohomology tori, surfaces, compact connected Lie groups, as well as their classifying spaces. On the other hand, the only nilmanifolds which are formal are tori. Formality is preserved under wedges and products of spaces, and connected sums of manifolds.

The 1-minimality property of a space X depends only on its fundamental group, $G = \pi_1(X, x_0)$. Alternatively, a finitely generated group G is 1-formal if and only if its Malcev Lie algebra admits a quadratic presentation. Examples of 1-formal groups include free groups and free abelian groups of finite rank, surface groups, and groups with first Betti number equal to 0 or 1. The 1-formality property is preserved under free products and direct products.

8.2. The tangent cone formula. The main connection between the formality property and the cohomology jump loci is provided by the following theorem from [11]. For more details and references, we refer to the recent survey [27].

Theorem 8.1 ([11]). *Let X be a 1-formal space. For each $d > 0$, the exponential map* $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^\times)$ *restricts to an isomorphism of analytic germs,* $\exp\colon (\mathcal{R}^1_d(X),0) \xrightarrow{\simeq} (\mathcal{V}^1_d(X),1)$. *Thus, the following "tangent cone formula" holds:*

$$\tau_1(\mathcal{V}^1_d(X)) = \mathrm{TC}_1(\mathcal{V}^1_d(X)) = \mathcal{R}^1_d(X).$$

As a consequence, the irreducible components of $\mathcal{R}^1_d(X)$ are all rationally defined subspaces, while the components of $\mathcal{V}^1_d(X)$ passing through the origin are all rational subtori of the form $\exp(L)$, with L running through the irreducible components of $\mathcal{R}^1_d(X)$. The next corollary is immediate.

Corollary 8.2. *Every 1-formal space is locally 1-straight.*

In general, though, 1-formal spaces need not be 1-straight, as we shall see in Example 11.8. Conversely, 1-straight spaces need not be 1-formal, as we shall see in Example 10.9.

8.3. Formality and the Dwyer–Fried invariants. The formality of a space has definite implications on the nature of its Ω-invariants, and their relationship to the resonance varieties.

Corollary 8.3. *If X is 1-formal, then $\Omega^1_r(X) \subseteq \sigma_r(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$, for all $r \geq 1$.*

Proof. Follows from Corollaries 7.5 and 8.2. □

Corollary 8.4. *Let X be a 1-formal space. Suppose all positive-dimensional components of $\mathcal{W}^1(X)$ pass through 1. Then:*

(1) *X is 1-straight.*
(2) *$\Omega^1_r(X) = \sigma_r(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$, for all $r \geq 1$.*

Proof. Part (1) follows from Corollary 8.2 and the additional hypothesis. Part (2) now follows from Theorem 7.6 (with $k = 1$). □

The hypothesis on the components of $\mathcal{W}^1(X)$ is really needed in this corollary. Indeed, if X is the presentation 2-complex for the group G from Examples 6.5 and 7.8, then $H^*(X,\mathbb{Q}) \cong H^*(T^2,\mathbb{Q})$, and so X is formal. Yet, as we know, X is not 1-straight, and $\Omega^1_2(X) \neq \sigma_2(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$. We shall see another instance of this phenomenon in Example 11.8.

8.4. The case of infinite cyclic covers. In the case when $r = 1$, Theorem 7.3 has the following consequence.

Corollary 8.5 ([29])**.** *Let $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$ be an epimorphism, and let $\bar{\nu} \in H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{C})$ be the corresponding cohomology class. If the exponential map restricts to an isomorphism of analytic germs, $(\mathcal{R}^i_1(X), 0) \cong (\mathcal{V}^i_1(X), 1)$, for all $i \leq k$, then*

$$\sum_{i\leq k} b_i(X^\nu) < \infty \Longleftrightarrow \bar{\nu} \notin \mathcal{R}^k(X).$$

Using Theorem 8.1 together with Corollary 8.5, we obtain the following immediate consequence.

Corollary 8.6. *Suppose G is a 1-formal group. Then*

$$\Omega^1_1(G) = \overline{\mathcal{R}}^1(G,\mathbb{Q})^{\complement}.$$

In other words, if $X^\nu \to X$ is a regular, infinite cyclic cover of a 1-formal space, then $b_1(X^\nu)$ is finite if and only if the corresponding cohomology class, $\bar{\nu} \in H^1(X,\mathbb{Z})$, is non-resonant.

9. Toric complexes

In this section, we illustrate our techniques on a class of CW-complexes that are carved out of an n-torus in a manner prescribed by a simplicial complex on n vertices. Such "toric" complexes are minimal, formal, and straight—thus, ideal from our point of view.

9.1. Toric complexes and right-angled Artin groups. Let T^n be the n-torus, endowed with the standard cell decomposition, and with basepoint $*$ the unique 0-cell. For a simplex $\sigma \in [n]$, let $T^\sigma \subset T^n$ be the cellular subcomplex $T^\sigma = \{x \in T^n \mid x_i = * \text{ if } i \notin \sigma\}$.

Definition 9.1. Let K be a simplicial complex on n vertices. The associated *toric complex*, T_K, is the union of all T^σ, with σ running through the simplices of K.

The k-cells of T_K are in one-to-one correspondence with the $(k-1)$-simplices of K. Since the toric complex is a subcomplex of T^n, all boundary maps in $C_\bullet(T_K,\mathbb{Z})$ vanish; thus, T_K is a minimal cell complex. Evidently, $H_k(T_K,\mathbb{Z})$ is isomorphic to $C^{\text{simp}}_{k-1}(K,\mathbb{Z})$, the free abelian group on the $(k-1)$-simplices of K.

Denote by V be the set of 0-cells of K, and by E the set of 1-cells of K. The fundamental group of the toric complex T_K is then the *right-angled Artin group* associated to the graph $\Gamma = K^{(1)}$,

$$G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v,w\} \in \mathsf{E}\rangle. \tag{33}$$

Groups of this sort interpolate between $G_\Gamma = \mathbb{Z}^n$ in case Γ is a complete graph, and $G_\Gamma = F_n$ in case Γ is a discrete graph.

The toric complex construction behaves well with respect to simplicial joins: $T_{K*K'} = T_K \times T_{K'}$. Consequently, $G_{\Gamma*\Gamma'} = G_\Gamma \times G_{\Gamma'}$.

A classifying space for the group G_K is the toric complex T_Δ, where $\Delta = \Delta_K$ is the flag complex of K, i.e., the maximal simplicial complex with 1-skeleton equal to the graph $\Gamma = K^{(1)}$.

Finally, it is known from the work of Notbohm and Ray [24] that all toric complexes are formal spaces. In particular, all right-angled Artin groups are 1-formal, a fact also proved in [25].

9.2. Cohomology jump loci. As noted above, $H_1(T_K, \mathbb{Z}) = \mathbb{Z}^n$, with generators indexed by the vertex set $\mathsf{V} = [n]$. Thus, we may identify $H^1(T_K, \mathbb{C})$ with the vector space $\mathbb{C}^{\mathsf{V}} = \mathbb{C}^n$, and $H^1(T_K, \mathbb{C}^\times)$ with the algebraic torus $(\mathbb{C}^\times)^{\mathsf{V}} = (\mathbb{C}^\times)^n$. For each subset $\mathsf{W} \subseteq \mathsf{V}$, let $\mathbb{C}^{\mathsf{W}}$ be the respective coordinate subspace, and let $(\mathbb{C}^\times)^{\mathsf{W}} = \exp(\mathbb{C}^{\mathsf{W}})$ be the respective algebraic subtorus.

Theorem 9.2 ([26])**.** *With notation as above,*

$$\mathcal{R}^i_d(T_K) = \bigcup_{\mathsf{W}} \mathbb{C}^{\mathsf{W}} \quad \text{and} \quad \mathcal{V}^i_d(T_K) = \bigcup_{\mathsf{W}} (\mathbb{C}^\times)^{\mathsf{W}},$$

where, in both cases, the union is taken over all subsets $\mathsf{W} \subset \mathsf{V}$ *for which* $\sum_{\sigma \in K_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{C}} \widetilde{H}_{i-1-|\sigma|}(\operatorname{lk}_{K_{\mathsf{W}}}(\sigma), \mathbb{C}) \geq d$.

In the above, K_{W} denotes the simplicial subcomplex induced by K on W, and $\operatorname{lk}_L(\sigma)$ denotes the link of a simplex σ in a subcomplex $L \subseteq K$.

In homological degree 1, the resonance formula from Theorem 9.2 takes a simpler form, already noted in [25]. Namely, $\mathcal{R}^1(G_\Gamma) = \bigcup_{\mathsf{W}} \mathbb{C}^{\mathsf{W}}$, where the union is taken over all (maximal) subsets $\mathsf{W} \subset \mathsf{V}$ for which the induced graph Γ_{W} is disconnected. In particular, the codimension of the resonance variety $\mathcal{R}^1(G_\Gamma)$ equals the connectivity of the graph Γ.

Corollary 9.3. *All toric complexes T_K are straight spaces.*

Proof. By the above theorem, each resonance variety $\mathcal{R}^i(T_K)$ is the union of a coordinate subspace arrangement, and each characteristic variety $\mathcal{V}^i(T_K)$ is the union of the corresponding arrangement of coordinate subtori. Consequently, all components of $\mathcal{V}^i(T_K)$ are algebraic subtori, and $\operatorname{TC}_1(\mathcal{V}^i(T_K)) = \mathcal{R}^i(T_K)$. □

9.3. Ω-invariants. In their landmark paper [3], Bestvina and Brady studied the geometric finiteness properties of certain subgroups of right-angled Artin groups G_Γ, arising as kernels of "diagonal" homomorphisms

$G_\Gamma \twoheadrightarrow \mathbb{Z}$. In [26], Papadima and Suciu computed the homology of such subgroups, and, more generally, the homology of regular $\mathbb{Z}$-covers of toric complexes. In a related vein, Denham [8] investigated the homology of covers of toric complexes T_K corresponding to "coordinate" homomorphisms $\pi_1(T_K) \twoheadrightarrow \mathbb{Z}^r$.

The study of homological finiteness properties of regular, free abelian covers of toric complexes was pursued in [29], where a general formula for the Dwyer–Fried sets of such complexes was given. In our setting, this result may be restated as follows.

Theorem 9.4 ([29]). *Let T_K be a toric complex. Then*

$$\Omega^k_r(T_K) = \sigma_r(\mathcal{R}^k(T_K, \mathbb{Q}))^{\complement},$$

for all $k, r \geq 1$. In particular, $\Omega^k_1(T_K) = \overline{\mathcal{R}}^k(T_K, \mathbb{Q})^{\complement}$.

Proof. Follows from Theorem 7.6 and Corollary 9.3. □

As a consequence, all the Ω-invariants of a toric complex T_K are Zariski open subsets of the Grassmannian $\mathrm{Gr}_r(\mathbb{Q}^n)$.

When combined with Theorem 9.2 and the discussion following it, Theorem 9.4 allows us to compute very explicitly the Dwyer–Fried sets of toric complexes. Let us provide one such computation.

Corollary 9.5. *Let Γ be a finite simple graph, and let κ be the connectivity of Γ. Then $\Omega^1_r(G_\Gamma) = \emptyset$, for all $r \geq \kappa + 1$.*

Proof. Recall that $\operatorname{codim} \mathcal{R}^1(G_\Gamma) = \kappa$. Thus, $\operatorname{codim} \sigma_r(\mathcal{R}^1(G_\Gamma)) = \kappa - r + 1$. The desired conclusion follows from Theorem 9.4. □

In particular, if Γ is disconnected, then $\Omega^1_r(G_\Gamma) = \emptyset$, for all $r \geq 1$.

Example 9.6. Let Γ be a tree on $n \geq 3$ vertices. Label the non-terminal vertices as $v_1, \dots, v_s$, and the terminal vertices as $v_{s+1}, \dots, v_n$. The cut sets of Γ are all singletons, consisting of the non-terminal vertices. Thus, the resonance variety $\mathcal{R}^1(G_\Gamma, \mathbb{Q})$ is the union of the coordinate hyperplanes $L_j = \{z \in \mathbb{Q}^n \mid z_j = 0\}$, with $1 \leq j \leq s$. Hence,

$$\Omega^1_r(G_\Gamma) = \begin{cases} \mathbb{QP}^{n-1} \setminus \bigcup_{j=1}^s \overline{L}_j & \text{if } r = 1, \\ \emptyset & \text{if } r \geq 2. \end{cases} \tag{34}$$

10. Kähler and quasi-Kähler manifolds

We now discuss the cohomology jumping loci and the Dwyer–Fried invariants of Kähler and quasi-Kähler manifolds.

10.1. Cohomology ring and formality. Let M be a compact, connected, complex manifold of complex dimension m. Such a manifold is called a *Kähler manifold* if it admits a Hermitian metric h for which the imaginary part $\omega = \Im(h)$ is a closed 2-form. The class of Kähler manifolds, includes smooth, complex projective varieties, such as Riemann surfaces. This class is closed under finite direct products and finite covers.

Hodge theory provides two sets of data on the cohomology ring of M. The first data, known as the Hodge decomposition on $H^i(M,\mathbb{C})$, depend only on the complex structure on M. The second data, known as the Lefschetz isomorphism and the Lefschetz decomposition on $H^i(M,\mathbb{R})$, depend only on the choice of a Kähler class $[\omega] \in H^{1,1}(M,\mathbb{C})$.

These data impose strong conditions on the possible Betti numbers $b_i = b_i(M)$, beyond the symmetry property $b_i = b_{2m-i}$ imposed by Poincaré duality. For example, the odd Betti numbers b_{2i+1} must be even, and increasing in the range $2i+1 \leq m$, while the even Betti numbers b_{2i} must be increasing in the range $2i \leq m$.

Another constraint on the topology of compact Kähler manifolds was established by Deligne, Griffiths, Morgan, and Sullivan in [7]. For such a manifold M, let d be the exterior derivative, J the complex structure, and $d^c = J^{-1}dJ$. Then the following holds: If η is a form which is closed for both d and d^c, and exact for either d or d^c, then η is exact for dd^c. As a consequence, all compact Kähler manifolds are formal.

A manifold X is said to be a *quasi-Kähler manifold* if there is a compact Kähler manifold $\overline{X}$ and a normal-crossings divisor D such that $X = \overline{X} \setminus D$. The class of quasi-Kähler manifolds includes smooth, irreducible, quasi-projective complex varieties, such as complements of plane algebraic curves.

Each quasi-Kähler manifold X inherits a mixed Hodge structure from its compactification $\overline{X}$. If X is a smooth, quasi-projective variety with vanishing degree 1 weight filtration on $H^1(X,\mathbb{C})$, then X is 1-formal. This happens, for instance, when X admits a non-singular compactification $\overline{X}$ with $b_1(\overline{X}) = 0$, e.g., when X is the complement of a hypersurface in $\mathbb{CP}^m$. In general, though, smooth, quasi-projective varieties need not be 1-formal. For a detailed treatment of the subject, we refer to Morgan [22].

10.2. Characteristic varieties. Foundational results on the structure of the cohomology support loci for local systems on smooth projective varieties, and more generally, on compact Kähler manifolds were obtained by Beauville, Green–Lazarsfeld, Simpson, and Campana. A further, wide-ranging generalization was obtained by Arapura in [1].

Theorem 10.1 ([1])**.** *Let $X = \overline{X} \setminus D$, where $\overline{X}$ is a compact Kähler manifold and D is a normal-crossings divisor. If either $D = \emptyset$ or $b_1(\overline{X}) = 0$, then each characteristic variety $\mathcal{V}^i_d(X)$ is a finite union of unitary translates of algebraic subtori of $H^1(X, \mathbb{C}^\times)$.*

In other words, for each $i \geq 0$ and $d > 0$, the characteristic variety $\mathcal{V}^i_d(X)$ admits a decomposition into irreducible components of the form $W = \rho \cdot T$, with

- direction subtorus $T = \operatorname{dir}(W)$, a connected algebraic subgroup of the character torus $\widehat{G}^0$, where $G = \pi_1(X)$;
- translation factor $\rho\colon G \to S^1$, a unitary character of G.

In degree 1 and depth 1, the condition that $b_1(\overline{X}) = 0$ if $D \neq \emptyset$ may be lifted. Furthermore, each positive-dimensional component of $\mathcal{V}^1_1(X)$ is of the form $\rho \cdot T$, with ρ a *torsion* character.

In the quasi-projective setting, more can be said. The next theorem summarizes several recent results in this direction: the first two parts are from [10], the third part is from [4] and [2], while the last part is from [9] and [2].

Theorem 10.2 ([9, 10, 4, 2])**.** *Let X be a smooth, quasi-projective variety. Then:*

1. *If W and W' are two distinct components of $\mathcal{V}^1(X)$, then either $\operatorname{dir}(W) = \operatorname{dir}(W')$, or $T_1 \operatorname{dir}(W) \cap T_1 \operatorname{dir}(W') = \{0\}$.*
2. *For each pair of distinct components, W and W', the intersection $W \cap W'$ is a finite set of torsion characters.*
3. *The isolated points in $\mathcal{V}^1(X)$ are also torsion characters.*
4. *If $W = \rho T$, with $\dim T = 1$ and $\rho \neq 1$, then T is not a component of $\mathcal{V}^1(X)$.*

10.3. Resonance varieties. In the presence of 1-formality, the quasi-Kähler condition also imposes stringent conditions on the degree 1 resonance varieties.

Theorem 10.3 ([11])**.** *Let X be a 1-formal, quasi-Kähler manifold, and let $\{L_\alpha\}$ be the collection of positive-dimensional, irreducible components of $\mathcal{R}^1_1(X)$. Then:*

1. *Each L_α is a linear subspace of $H^1(X, \mathbb{C})$ of dimension at least $2\varepsilon(\alpha) + 2$, for some $\varepsilon(\alpha) \in \{0, 1\}$.*
2. *The restriction of the cup-product map $H^1(X, \mathbb{C}) \wedge H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ to $L_\alpha \wedge L_\alpha$ has rank $\varepsilon(\alpha)$.*
3. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
4. *$\mathcal{R}^1_d(X) = \{0\} \cup \bigcup_{\alpha : d \leq \dim L_\alpha - \varepsilon(\alpha)} L_\alpha$.*

Remark 10.4. Suppose X is a smooth quasi-projective variety with $W_1(H^1(X,\mathbb{C}))=0$. Then, as mentioned in §10.1, X is 1-formal. In this situation, each subspace L_α is isotropic, i.e., $\epsilon(\alpha)=0$.

Remark 10.5. Suppose M is a compact Kähler manifold. Then of course M is formal, and Theorem 10.3 again applies. In this situation, each subspace L_α has dimension $2g(\alpha)$, for some $g(\alpha)\geq 2$, and the restriction of the cup-product map to $L_\alpha\wedge L_\alpha$ has rank $\epsilon(\alpha)=1$.

It is now a straightforward exercise to enumerate the possibilities for the first resonance variety of a Kähler manifold M, at least for low values of $n=b_1(M)$:

- If $n=0$ or 2, then $\mathcal{R}^1(M)=\{0\}$.
- If $n=4$, then $\mathcal{R}^1(M)=\{0\}$ or $\mathbb{C}^4$.
- If $n=6$, then $\mathcal{R}^1(M)=\{0\}$, $\mathbb{C}^4$, or $\mathbb{C}^6$.
- If $n=8$, then $\mathcal{R}^1(M)=\{0\}$, $\mathbb{C}^4$, $\mathbb{C}^6$, $\mathbb{C}^8$, or $\mathcal{R}^1(M)$ consists of two transverse, 4-dimensional subspaces.

Using the computations from Examples 3.4 and 3.10, it is readily seen that all these possibilities can be realized by manifolds of the form $M=S_g$ or $M=S_g\times S_h$, with S_g and S_h Riemann surfaces of suitable genera $g,h\geq 0$.

10.4. Straightness. We now discuss in detail the straightness properties of Kähler and quasi-Kähler manifolds, and how these properties relate to 1-formality.

Proposition 10.6. *Let X be a 1-formal, quasi-Kähler manifold (for instance, a compact Kähler manifold). Then:*

(1) *X is locally 1-straight.*
(2) *X is 1-straight if and only if $\mathcal{W}^1(X)$ contains no positive-dimensional translated subtori.*

PROOF. Part (1) follows from Corollary 8.2, using only the 1-formality assumption, while part (2) follows from Theorem 10.1. □

In general, quasi-Kähler manifolds may fail to be locally 1-straight, as illustrated in Example 10.7 below. Perhaps more surprisingly, compact Kähler manifolds may fail to be 1-straight; for a concrete example, we refer to [34].

Example 10.7. Let X be the complex Heisenberg manifold, i.e., the total space of the $\mathbb{C}^\times$-bundle over $\mathbb{C}^\times\times\mathbb{C}^\times$ with Euler number 1. Then X is a smooth, quasi-projective variety with $\mathcal{V}^1(X)=\{1\}$, yet $\mathcal{R}^1(X)=\mathbb{C}^2$. Thus, the tangent cone formula fails in this instance, and so X is neither 1-formal, nor locally 1-straight.

Let $(Y, 0)$ be a quasi-homogeneous isolated surface singularity. Then $X = Y \setminus \{0\}$ is a smooth, quasi-projective variety which supports a "good" $\mathbb{C}^{\times}$-action, with orbit space a smooth projective curve. Moreover, X deform-retracts onto the singularity link, which is a closed, smooth, orientable 3-manifold.

Proposition 10.8. *Suppose the curve $X/\mathbb{C}^{\times}$ has genus $g > 1$. Then X is straight, yet X is not 1-formal.*

PROOF. According to formula (14) from [12], we have $\mathcal{V}^1(X) = H^1(X, \mathbb{C}^{\times})$. Hence, by Lemma 6.6, X is straight. The fact that X is not 1-formal follows from [12, Proposition 7.1]. □

We illustrate this result with a concrete family of examples.

Example 10.9. Let X be the total space of the $\mathbb{C}^{\times}$-bundle over the Riemann surface of genus g, with Euler number -1. Then X is homotopy equivalent to the Brieskorn manifold $\Sigma(2, 2g+1, 2(2g+1))$. Assume now $g > 1$; then clearly X fits into the setup of Proposition 10.8. Hence, X is straight, but not 1-formal.

10.5. Dwyer–Fried invariants. We conclude this section with a discussion of the Ω-invariants of (quasi-) Kähler manifolds, and the extent to which these invariants are determined by the resonance varieties.

Theorem 10.10. *Let X be a 1-formal, quasi-Kähler manifold (for instance, a compact Kähler manifold). Then:*

1. $\Omega^1_1(X) = \overline{\mathcal{R}}^1(X, \mathbb{Q})^{\complement}$ *and* $\Omega^1_r(X) \subseteq \sigma_r(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$, *for $r \geq 2$.*
2. *If $\mathcal{W}^1(X)$ contains no positive-dimensional translated subtori, then $\Omega^1_r(X) = \sigma_r(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$, for all $r \geq 1$.*

PROOF. Part (1). Using only the 1-formality assumption, the two statements follow from Corollaries 8.6 and 8.3, respectively.

Part (2). In view of Theorem 10.1, the hypothesis is equivalent to $\mathcal{W}^1(X)$ containing no positive-dimensional component not passing through 1. The conclusion follows from Corollary 8.4. □

In general, though, the inclusion $\Omega^1_r(X) \subseteq \sigma_r(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$ can be strict, provided $2 \leq r \leq b_1(X)$. The next theorem identifies a fairly broad class of smooth, quasi-projective varieties for which this is the case. An explicit example will be given at the end of Section 11.

Theorem 10.11. *Let X be a 1-formal, smooth, quasi-projective variety. Suppose*

1. *$\mathcal{W}^1(X)$ has a 1-dimensional component not passing through 1;*
2. *$\mathcal{R}^1(X)$ has no codimension-1 components.*

Then $\Omega^1_2(X)$ *is strictly contained in* $\sigma_2(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$.

PROOF. Set $n = b_1(X)$, and identify $H = H^1(X,\mathbb{Q})$ with $\mathbb{Q}^n$. By assumption (1), the characteristic variety $\mathcal{W}^1(X) \subset (\mathbb{C}^\times)^n$ has a component of the form $W = \rho \cdot T$, with

- $T = \exp(\ell \otimes \mathbb{C})$, where ℓ is a 1-dimensional subspace in H;
- $\rho = \exp(2\pi i q)$, where q is a vector in $H \setminus \ell$.

We claim that the line ℓ is not contained in the resonance variety $\mathcal{R}^1(X,\mathbb{Q})$. For, if it were, the 1-dimensional algebraic torus T would be contained in $\mathcal{W}^1(X)$. But we know from Theorem 10.2(4) that T is not a component of $\mathcal{W}^1(X)$. Hence, there would exist a component W' with $T \subsetneqq W'$. Therefore,

$$\begin{aligned} \operatorname{dir}(W) = T \neq W' = \operatorname{dir}(W'), \ \text{ and}\\ T_1(\operatorname{dir}(W)) \cap T_1(\operatorname{dir}(W')) = \ell \otimes \mathbb{C} \neq \{0\}. \end{aligned}$$

This contradicts Theorem 10.2(1), thereby establishing the claim.

In view of the above, and of assumption (2), the resonance variety $\mathcal{R}^1(X,\mathbb{R})$ has codimension at least 2 in $H_{\mathbb{R}} = H^1(X,\mathbb{R})$. Therefore, the set

$$Z = \{P \in \operatorname{Gr}_2(H_{\mathbb{R}}) \mid \ell \subset P \text{ and } P \cap \mathcal{R}^1(X,\mathbb{R}) \neq \{0\}\}$$

is a proper subvariety of $\operatorname{Gr}_2(H_{\mathbb{R}})$. Hence, there is a non-zero vector $r_0 \in \mathbb{R}^n$, and an open cone U containing r_0, such that, for all $r \in U$, the plane P spanned by r and ℓ intersects $\mathcal{R}^1(X,\mathbb{R})$ only at 0.

Let $\pi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$ be the projection map. Clearly, $\pi(\mathbb{Z}^n \setminus \{0\})$ is a dense subset of $\mathbb{RP}^{n-1}$. Thus, $\pi(q + \mathbb{Z}^n)$ is also dense, and so intersects $\pi(U)$. Hence, there is a lattice point $\lambda \in \mathbb{Z}^n$ such that $\pi(q+\lambda)$ belongs to $\pi(U)$. The rational vector $q_0 := q + \lambda$ then belongs to U.

Let P_0 be the 2-dimensional subspace of H spanned by ℓ and q_0. By construction, $P_0 \cap \mathcal{R}^1(X,\mathbb{Q}) = \{0\}$, and so $P_0 \in \sigma_2(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$. On the other hand, the algebraic torus $T_0 = \exp(P_0 \otimes \mathbb{C})$ contains both $\exp(\ell \otimes \mathbb{C}) = T$ and $\exp(2\pi i q_0) = \rho$; therefore, $T_0 \supset \rho T$. Consequently, $\dim(T_0 \cap \mathcal{W}^1(X)) > 0$, showing that $P_0 \notin \Omega^1_2(X)$. ☐

11. Hyperplane arrangements

We conclude with a class of spaces exhibiting a strong interplay between the resonance varieties and the finiteness properties of free abelian covers. These spaces, obtained by deleting a finite number of hyperplanes from a complex affine space, are minimal, formal, and locally straight, but not always straight.

11.1. Cohomology jump loci. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. To start with, we will assume that all hyperplanes in $\mathcal{A}$ pass through the origin; non-central arrangements can be handled much the same way, using standard coning and deconing constructions.

Let $X = X(\mathcal{A})$ be the complement of the union of the hyperplanes comprising $\mathcal{A}$. Then X can be viewed as the complement of a normal-crossing divisor in a suitably blown-up $\mathbb{CP}^\ell$. In particular, X has the homotopy type of an ℓ-dimensional CW-complex. Moreover, this CW-complex can be chosen to be minimal (Dimca–Papadima, Randell).

Let $G = G(\mathcal{A})$ be the fundamental group of the complement. Its abelianization, G_{ab}, is the free abelian group of rank $n = |\mathcal{A}|$. Thus, we may identify the character group $\widehat{G}$ with the complex algebraic torus $(\mathbb{C}^\times)^n$. Let $\mathcal{V}^i_d(\mathcal{A}) = \mathcal{V}^i_d(X(\mathcal{A}))$ be the characteristic varieties of the arrangement. By Arapura's work [1], these varieties consist of subtori of $(\mathbb{C}^\times)^n$, possibly translated by unitary characters, together with a finite number of isolated unitary characters.

The cohomology ring $A = H^*(X(\mathcal{A}), \mathbb{Z})$ was computed by Brieskorn in the early 1970s, building on pioneering work of Arnol'd on the cohomology ring of the braid arrangement. It follows from Brieskorn's work that the space X is formal; in particular, the fundamental group of X is 1-formal. In 1980, Orlik and Solomon gave a simple description of the ring A, solely in terms of the intersection lattice $L(\mathcal{A})$, i.e., the poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion.

The resonance varieties $\mathcal{R}^i_d(\mathcal{A}) = \mathcal{R}^i_d(X(\mathcal{A}))$ were first defined and studied by Falk in [17]. Identifying $H^1(X, \mathbb{C})$ with $\mathbb{C}^n$, we may view the resonance varieties of $\mathcal{A}$ as homogeneous subvarieties of $\mathbb{C}^n$. It is known from work of Yuzvinsky that these varieties actually lie in the hyperplane $\{x \in \mathbb{C}^n \mid \sum_{i=1}^n x_i = 0\}$. Moreover, it follows from [15] that resonance "propagates" for the Orlik-Solomon algebra. More precisely, if $i \leq k \leq \ell$, then $\mathcal{R}^i_1(\mathcal{A}) \subseteq \mathcal{R}^k_1(\mathcal{A})$; in particular, $\mathcal{R}^k(\mathcal{A}) = \mathcal{R}^k_1(\mathcal{A})$.

11.2. Straightness. As above, let $X = X(\mathcal{A})$ be an arrangement complement. Using work of Esnault, Schechtman, and Viehweg [16], one can show that the exponential map, $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^\times)$, induces an isomorphism of analytic germs from $(\mathcal{R}^i_d(X), 0)$ to $(\mathcal{V}^i_d(X), 1)$, for all $i \geq 0$ and $d > 0$. We then have

$$\mathrm{TC}_1(\mathcal{V}^i_d(\mathcal{A})) = \mathcal{R}^i_d(\mathcal{A}), \quad \text{for all } d > 0. \tag{35}$$

In particular, all the resonance varieties $\mathcal{R}^i_d(\mathcal{A})$ are finite unions of rationally defined linear subspaces.

The tangent cone formula (35) was first proved in degree $i = 1$ (using different methods) by Cohen and Suciu [6] and Libgober [21], and was generalized to the higher-degree jump loci by Cohen and Orlik [5].

Proposition 11.1. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with complement X.*

(1) *X is locally straight.*
(2) *X is k-straight if and only if $\mathcal{V}^k(X)$ contains no positive-dimensional translated tori.*

Proof. For part (1), we must verify the two properties from Definition 6.1. Property (a) follows at once from Arapura's theorem 10.1, while property (b) follows from the tangent cone formula (35).

For part (2), we must verify the additional property (c) from Definition 6.2. The conclusion follows again from Theorem 10.1. □

As first noted in [31], there do exist arrangements $\mathcal{A}$ for which $\mathcal{V}^1(\mathcal{A})$ contains positive-dimensional translated components. By Proposition 11.1, such arrangements are not 1-straight. We will come back to this point in Example 11.8.

11.3. Dwyer–Fried invariants. Let us define the Dwyer–Fried invariants of an arrangement $\mathcal{A}$ as $\Omega^i_r(\mathcal{A}) = \Omega^i_r(X(\mathcal{A}))$. Suppose $\mathcal{A}$ consists of n hyperplanes in $\mathbb{C}^\ell$. For each $1 \leq r \leq n$, there is then a filtration

$$\operatorname{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(\mathcal{A}) \supseteq \Omega^1_r(\mathcal{A}) \supseteq \Omega^2_r(\mathcal{A}) \supseteq \cdots \supseteq \Omega^\ell_r(\mathcal{A}). \tag{36}$$

The next result establishes a comparison between the terms of this filtration and the incidence varieties to the resonance varieties of $\mathcal{A}$.

Theorem 11.2. *Let $\mathcal{A}$ be an arrangement of n hyperplanes, and fix an integer $k \geq 1$. Then,*

(1) *$\Omega^k_r(\mathcal{A}) \subseteq \operatorname{Gr}_r(\mathbb{Q}^n) \setminus \sigma_r(\mathcal{R}^k(\mathcal{A},\mathbb{Q}))$, for all $r \geq 1$.*
(2) *$\Omega^1_1(\mathcal{A}) = \mathbb{QP}^{n-1} \setminus \overline{\mathcal{R}}^1(\mathcal{A},\mathbb{Q})$.*
(3) *If $\mathcal{V}^k(\mathcal{A})$ contains no positive-dimensional translated tori, then $\Omega^k_r(\mathcal{A}) = \operatorname{Gr}_r(\mathbb{Q}^n) \setminus \sigma_r(\mathcal{R}^k(\mathcal{A},\mathbb{Q}))$, for all $r \geq 1$.*

Proof. Part (1) follows from Proposition 11.1(1) and Corollary 7.5.

Part (2) follows from the 1-formality of the complement of $\mathcal{A}$ and Corollary 8.6.

Part (3) follows from Proposition 11.1(2) and Theorem 7.6. □

In other words, each Dwyer–Fried invariant $\Omega^k_r(\mathcal{A})$ is included in the complement of a union of special Schubert varieties of the form $\sigma_r(L)$, where L runs through the components of $\mathcal{R}^k(\mathcal{A},\mathbb{Q})$, with the inclusion being an equality when $\mathcal{V}^k(\mathcal{A}) = \bigcup_L \exp(L \otimes \mathbb{C})$.

Remark 11.3. In [18], Falk gives a decomposition of the resonance variety $\mathcal{R}^1(\mathcal{A})$ into combinatorial pieces and shows that, projectively, each of these pieces is the ruled variety corresponding to an intersection of special Schubert varieties in special position in the Grassmannian of lines in projective space. It would be interesting to see if Falk's description sheds additional light on the Dwyer–Fried invariants $\Omega^1_r(\mathcal{A})$.

11.4. Resonance varieties of line arrangements. For the rest of this section, we will concentrate on the resonance varieties $\mathcal{R}^1(\mathcal{A}) = \mathcal{R}^1(G(\mathcal{A}))$ and their relation to the Dwyer–Fried invariants $\Omega^1_r(\mathcal{A}) = \Omega^1_r(G(\mathcal{A}))$. We start with a brief review of the former.

By the Lefschetz-type theorem of Hamm and Lê, taking a generic two-dimensional section does not change the group of the arrangement. Thus, we may assume $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ is an affine line arrangement in $\mathbb{C}^2$, for which no two lines are parallel.

The variety $\mathcal{R}^1(\mathcal{A})$ is a union of linear subspaces in $\mathbb{C}^n$. Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0. The simplest components of $\mathcal{R}^1(\mathcal{A})$ are the *local* components: to an intersection point $v_J = \bigcap_{j\in J} \ell_j$ of multiplicity $|J| \geq 3$, there corresponds a subspace L_J of dimension $|J| - 1$, given by equations of the form $\sum_{j\in J} x_j = 0$, and $x_i = 0$ if $i \notin J$. The remaining components correspond to certain "neighborly partitions" of sub-arrangements of $\mathcal{A}$.

If $|\mathcal{A}| \leq 5$, then all components of $\mathcal{R}^1(\mathcal{A})$ are local. For $|\mathcal{A}| \geq 6$, though, the resonance variety $\mathcal{R}^1(\mathcal{A})$ may have non-local components.

Example 11.4. Let $\mathcal{A}$ be the braid arrangement, with defining polynomial $Q(\mathcal{A}) = z_0z_1z_2(z_0 - z_1)(z_0 - z_2)(z_1 - z_2)$. Take a generic plane section, and label the corresponding lines as $6, 2, 4, 3, 5, 1$. The variety $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components, corresponding to the triple points $124, 135, 236, 456$, and one non-local component, $L_\Pi = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}$, corresponding to the neighborly partition $\Pi = (16|25|34)$.

For an arbitrary arrangement $\mathcal{A}$, work of Falk, Pereira, and Yuzvinsky [**19, 30, 36**] shows that any non-local component in $\mathcal{R}^1(\mathcal{A})$ has dimension either 2 or 3.

11.5. Ω-invariants of line arrangements. We are in a position now where we can compute explicitly the Dwyer–Fried invariants of some line arrangements. We start with a simple example.

Example 11.5. Let $\mathcal{A}$ be the arrangement with defining polynomial $Q(\mathcal{A}) = z_0z_1z_2(z_1 - z_2)$. The variety $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^4$ has a single component, namely, the 2-plane $L = \{x \mid x_1 + x_2 + x_3 = x_4 = 0\}$, while

$\mathcal{V}^1(\mathcal{A}) = \exp(L)$. Using Theorem 11.2, we obtain

$$\Omega^1_r(\mathcal{A}) = \begin{cases} \mathbb{QP}^3 \setminus \overline{L} & \text{if } r = 1, \\ \operatorname{Gr}_2(\mathbb{Q}^4) \setminus \sigma_2(L) & \text{if } r = 2, \\ \emptyset & \text{if } r \geq 3. \end{cases}$$

Here, the Grassmannian $\operatorname{Gr}_2(\mathbb{Q}^4)$ is the hypersurface in $\mathbb{P}(\bigwedge^2 \mathbb{Q}^4)$ with equation $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$, while the Schubert variety $\sigma_2(L)$ is the 3-fold in $\operatorname{Gr}_2(\mathbb{Q}^4)$ cut out by the hyperplane $p_{12} - p_{13} + p_{23} = 0$.

Proposition 11.6. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, and let m be the maximum multiplicity of its intersection points.*

(1) *If $m = 2$, then $\Omega^1_r(\mathcal{A}) = \operatorname{Gr}_r(\mathbb{Q}^n)$, for all $r \geq 1$.*
(2) *If $m \geq 3$, then $\Omega^1_r(\mathcal{A}) = \emptyset$, for all $r \geq n - m + 2$.*

Proof. If $\mathcal{A}$ has only double points, then $G(\mathcal{A}) = \mathbb{Z}^n$, by a well-known theorem of Zariski. Assertion (1) follows from Example 7.2.

Now suppose $\mathcal{A}$ has an intersection point of multiplicity $m \geq 3$. Then $\mathcal{R}^1(\mathcal{A})$ has a (local) component L of dimension $m - 1$. The corresponding Schubert variety, $\sigma_r(L)$, has codimension $n - m + 2 - r$. Assertion (2) follows from Theorem 11.2(1). □

Proposition 11.7. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$. Suppose $\mathcal{A}$ has 1 or 2 lines which contain all the intersection points of multiplicity 3 and higher. Then $\Omega^1_r(\mathcal{A}) = \sigma_r(\mathcal{R}^1(\mathcal{A}, \mathbb{Q}))^{\complement}$, for all $1 \leq r \leq n$.*

Proof. As shown by Nazir and Raza in [23], the characteristic variety $\mathcal{V}^1(\mathcal{A})$ of such an arrangement has no translated components. The conclusion follows from Theorem 11.2(3). □

In general, though, the "resonance upper bound" for the Dwyer–Fried invariants of arrangements is not attained. We illustrate this claim with the smallest possible example.

Example 11.8. Let $\mathcal{A}$ be the deleted B_3 arrangement, with defining polynomial $Q(\mathcal{A}) = z_0 z_1 (z_0^2 - z_1^2)(z_0^2 - z_2^2)(z_1^2 - z_2^2)$. The jump loci of $\mathcal{A}$ were computed in [32]. Briefly, the resonance variety $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^8$ contains 7 local components, corresponding to 6 triple points and one quadruple point, and 5 non-local components, corresponding to braid sub-arrangements. In particular, $\operatorname{codim} \mathcal{R}^1(\mathcal{A}) = 5$.

In addition to the 12 subtori arising from the subspaces in $\mathcal{R}^1(\mathcal{A})$, the characteristic variety $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^{\times})^8$ also contains a component of the form $\rho \cdot T$, where T is a 1-dimensional algebraic subtorus, and ρ is a root of unity of order 2.

Let $X = X(\mathcal{A})$ be complement of the arrangement. Then X is formal, yet not 1-straight. Moreover, the hypothesis of Theorem 10.11 are satisfied for X. We conclude that $\Omega^1_2(\mathcal{A})$ is strictly contained in $\sigma_2(\mathcal{R}^1(\mathcal{A}))^{\complement}$.

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