

Rational homotopy groups and Koszul algebras

Stefan Papadima ^a, Alexander I. Suciu ^b

^a Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-70700 Bucharest, Romania
E-mail: spapadim@stoilow.imar.ro

^b Department of Mathematics, Northeastern University, Boston, MA 02115, USA
E-mail: alexsuciu@neu.edu

(Reçu le jour mois année, accepté après révision le jour mois année)

Abstract. Let X and Y be finite-type CW-spaces (X connected, Y simply connected), such that the ring $H^*(Y, \mathbb{Q})$ is a k -rescaling of $H^*(X, \mathbb{Q})$. If $H^*(X, \mathbb{Q})$ is a Koszul algebra, then the graded Lie algebra $\pi_*(\Omega Y) \otimes \mathbb{Q}$ is the k -rescaling of $\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}$. If Y is a formal space, then the converse holds, and Y is coformal. Furthermore, if X is formal, with Koszul cohomology algebra, there exist filtered group isomorphisms between the Malcev completion of $\pi_1 X$, the completion of $[\Omega S^{2k+1}, \Omega Y]$, and the Milnor-Moore group of coalgebra maps from $H_*(\Omega S^{2k+1}, \mathbb{Q})$ to $H_*(\Omega Y, \mathbb{Q})$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Groupes d'homotopie rationnels et algèbres de Koszul

Résumé. Soient X et Y deux CW-espaces de type fini (X connexe, Y simplement connexe), tels que l'anneau de cohomologie $H^*(Y, \mathbb{Q})$ soit un k -recalibrage de $H^*(X, \mathbb{Q})$. Si $H^*(X, \mathbb{Q})$ est une algèbre de Koszul, alors l'algèbre de Lie graduée $\pi_*(\Omega Y) \otimes \mathbb{Q}$ est le k -recalibrage de $\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}$. Si Y est un espace formel, alors l'implication réciproque est vraie aussi, et l'espace Y est coformel. De plus, si X est formel, avec algèbre de cohomologie de Koszul, on trouve des isomorphismes de groupes filtrés entre le complété de Malcev de $\pi_1 X$, le complété de $[\Omega S^{2k+1}, \Omega Y]$, et le groupe de Milnor-Moore d'applications de cogèbres entre $H_*(\Omega S^{2k+1}, \mathbb{Q})$ et $H_*(\Omega Y, \mathbb{Q})$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Cette note est un résumé des résultats de [9]. Commençons par définir la notion de recalibrage d'un espace topologique (ayant le type d'homotopie d'un CW-complexe connexe de type fini.)

Soient k un entier positif, et A^* une algèbre graduée. On définit l'algèbre graduée $A[k]$ par $A[k]^{q(2k+1)} = A^q$ et $A[k]^p = 0$ si $2k + 1 \nmid p$, avec la multiplication héritée de A . On dit qu'un espace Y est un k -recalibrage de X si $\pi_1 Y = 0$ et $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$, en tant qu'algèbres graduées. Un tel espace Y peut être construit à partir du modèle minimal de l'algèbre $H^*(X, \mathbb{Q})[k]$, munie de la différentielle nulle.

Note présentée par First name NAME

S0764-4442(00)0????-?/FLA

© 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés.

Cette construction donne un espace formel, mais X peut bien avoir des recalibrages non-formels. D'autre part, si $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) < \infty$, alors X a un k -recalibrage unique (à \mathbb{Q} -équivalence près), pour tout $k \gg 1$.

Soit L_* un espace vectoriel gradué, muni d'un crochet de Lie de degré 0. On définit l'algèbre de Lie graduée $L[k]$ par $L[k]_{2kq} = L_q$ et $L[k]_p = 0$ si $2k \nmid p$, avec le crochet hérité de L .

THÉORÈME 1. – *Soit Y un k -recalibrage d'un espace X . Soit $\mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}$ l'espace vectoriel gradué associé à la suite centrale descendante de $\pi_1 X$ (muni du crochet induit par le commutateur du groupe), et soit $\pi_*(\Omega Y) \otimes \mathbb{Q}$ l'algèbre de Lie d'homotopie de Y (munie du crochet de Samelson).*

- (a) *Si $A^* = H^*(X, \mathbb{Q})$ est une algèbre de Koszul (c'est à dire, si $\mathrm{Tor}_{p,q}^A(\mathbb{Q}, \mathbb{Q}) = 0$, pour tous $p \neq q$), alors il existe un isomorphisme d'algèbres de Lie graduées*

$$\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]. \quad (1)$$

De plus, $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\mathrm{rank} \pi_{2ki}(\Omega Y)} = P_X(-t^{2k+1})$.

- (b) *Si Y est formel et $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$, en tant qu'espaces vectoriels gradués, alors $H^*(X, \mathbb{Q})$ est une algèbre de Koszul. De plus, Y est coformal.*

La première partie du Théorème nous permet de décrire le type d'homotopie rationnelle de l'espace de lacets de Y , uniquement à partir du polynôme de Poincaré de X . En particulier, $P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$.

THÉORÈME 2. – *Soit Y un k -recalibrage d'un espace X , tel que $H^*(X, \mathbb{Q})$ soit une algèbre de Koszul. L'espace X est formel si et seulement s'il existe des isomorphismes de groupes filtrés*

$$\mathrm{Hom}^{\mathrm{cog\`ebre}}(H_*(\Omega S^{2k+1}, \mathbb{Q}), H_*(\Omega Y, \mathbb{Q})) \cong [\Omega S^{2k+1}, \Omega Y]^{\wedge} \cong \pi_1 X \otimes \mathbb{Q}. \quad (2)$$

Les groupes ci-dessus—le groupe de Milnor-Moore de morphismes de cogèbres entre $H_*(\Omega S^{2k+1}, \mathbb{Q})$ et $H_*(\Omega Y, \mathbb{Q})$, le complété du groupe de classes d'homotopie pointées entre ΩS^{2k+1} et ΩY , et le complété de Malcev du groupe fondamental de X —sont tous pourvus de filtrations canoniques de limite inverse. En passant aux gradués associés, l'isomorphisme de groupes filtrés $[\Omega S^{2k+1}, \Omega Y]^{\wedge} \cong \pi_1 X \otimes \mathbb{Q}$ donne l'isomorphisme d'algèbres de Lie graduées (1).

Parmi les espaces admettant des recalibrages intégraux, on trouve les compléments d'arrangements d'hyperplans dans \mathbb{C}^{ℓ} , et les compléments d'entrelacs de cercles dans S^3 . Le k -recalibrage (formel) d'un tel espace X est fourni par le complément Y d'un certain arrangement de sous-espaces de codimension $k + 1$ dans $\mathbb{C}^{(k+1)\ell}$, et par le complément d'un certain entrelacs de $(2k + 1)$ -sphères dans S^{4k+3} , respectivement.

La formule (1) est vraie pour les arrangements supersolvables (un résultat de [2]), ainsi que pour les entrelacs ayant un graphe d'enlacement connexe. Pour les arrangements génériques, la formule (1) n'est plus vraie, en général (à cause de la non-coformalité de Y , détectée par les produits de Whitehead d'ordre supérieur). Dans le cas des entrelacs, la formule (2) n'est pas toujours vraie (à cause de la non-formalité de X , détectée par les invariants de Campbell-Hausdorff), même si la formule (1) est valable.

1. Rescaling operations

This note is an announcement of [9]. We refer to that paper for full details, and complete proofs.

Let A^* be a graded algebra over a ring R . For each integer $k \geq 1$, the k -rescaling of A is the graded algebra $A[k]$ with $A[k]^{q(2k+1)} = A^q$, and $A[k]^p = 0$ otherwise, and with multiplication rescaled accordingly.

Let X be a connected space. A simply-connected space Y is called a k -rescaling of X (over R) if the cohomology algebra $H^*(Y, R)$ is the k -rescaling of $H^*(X, R)$. For example, the sphere S^{2k+1} is a k -rescaling of S^1 , the wedge $\bigvee^n S^{2k+1}$ is a k -rescaling of $\bigvee^n S^1$, and the connected sum $\#^g S^{2k+1} \times S^{2k+1}$ is a k -rescaling of a genus g orientable surface. Though here, and most throughout, the rescaling holds over $R = \mathbb{Z}$, the theory works best over $R = \mathbb{Q}$, and so this will be our default coefficients ring.

Using Sullivan's minimal models [14], it is easy to see that any connected CW-space of finite type, X , admits a rational k -rescaling, for each $k \geq 1$. Indeed, $(H^*(X, \mathbb{Q})[k], d = 0)$ is a 1-connected, finite-type differential graded algebra, with minimal model \mathcal{M} . Hence, there exists a finite-type, 1-connected CW-space Y such that $\mathcal{M}(Y) = \mathcal{M}$. In particular, $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$.

By construction, the space Y is *formal*, i.e., its rational homotopy type is a formal consequence of its rational cohomology algebra. Hence, Y is uniquely determined (up to rational homotopy equivalence) among spaces with the same cohomology ring. But there may be other, non-formal rescalings of X . For example, take $X = S^1 \vee S^1 \vee S^{2k+2}$. Clearly, the formal k -rescaling is $Y = S^{2k+1} \vee S^{2k+1} \vee S^{(2k+1)(2k+2)}$. A non-formal rescaling is $Z = (S_x^{2k+1} \vee S_y^{2k+1}) \cup_\alpha e^{(2k+1)(2k+2)}$, where α is the iterated Whitehead product $\text{ad}_x^{2k+2}(y) = [x, [\cdot \cdot [x, y]]]$. Even so, if $H^{>d}(X, \mathbb{Q}) = 0$, then X has a unique k -rescaling (up to \mathbb{Q} -equivalence), for all $k > (d-1)/2$; see [13].

A graded \mathbb{Q} -vector space L_* , endowed with a bilinear operation $[\cdot, \cdot]: L_p \otimes L_q \rightarrow L_{p+q}$ is called a *Lie algebra with grading* if the bracket satisfies the anti-commutativity and Jacobi identities. If the Lie identities are satisfied only up to sign (following the Koszul convention), then L_* is called a *graded Lie algebra*.

We are interested in two main examples. The *associated graded Lie algebra* of a finitely generated group G is the Lie algebra with grading $\text{gr}_*(G) \otimes \mathbb{Q} := \bigoplus_{r \geq 1} (\Gamma_r G / \Gamma_{r+1} G) \otimes \mathbb{Q}$, where $\{\Gamma_r G\}_{r \geq 1}$ is the lower central series of G , and the bracket is induced by the group commutator. The *homotopy Lie algebra* of a based, simply-connected space Y is the graded Lie algebra $\pi_*(\Omega Y) \otimes \mathbb{Q} := \bigoplus_{r \geq 1} \pi_r(\Omega Y) \otimes \mathbb{Q}$, where ΩY is the loop space of Y , and the bracket is the Samelson product, obtained from the Whitehead product on $\pi_* Y$ via the boundary map in the path fibration over Y .

Given a Lie algebra with grading L_* , and a positive integer k , the k -rescaling of L is the graded Lie algebra $L[k]$, with $L[k]_{2kq} = L_q$ and $L[k]_p = 0$ otherwise, and with Lie bracket rescaled accordingly.

2. The Rescaling Formula

Let A^* be a connected, graded algebra over \mathbb{Q} . By definition, A is a *Koszul algebra* if $\text{Tor}_{p,q}^A(\mathbb{Q}, \mathbb{Q}) = 0$, for all $p \neq q$. A necessary condition is that A be the quotient of a free algebra on generators in degree 1 by an ideal I generated in degree 2. A sufficient condition is that I admit a quadratic Gröbner basis. A topological interpretation of Koszulness is as follows. Let X be a formal space. Then, $H^*(X, \mathbb{Q})$ is a Koszul algebra if and only if the (Bousfield-Kan) rationalization $X_{\mathbb{Q}}$ is aspherical; see [10].

From now on, all spaces will be assumed to be connected, well-pointed, and homotopy equivalent to some finite-type CW-complex. Recall that a k -rescaling of a space X is a simply-connected space Y with $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$, as graded algebras. Our first result shows that, under a Koszulness assumption, this homological rescaling passes to a homotopical rescaling.

THEOREM 2.1. – *Let Y be a k -rescaling of a space X . If $H^*(X, \mathbb{Q})$ is a Koszul algebra, then the following Rescaling Formula holds:*

$$\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \text{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k], \quad \text{as graded Lie algebras.} \quad (3)$$

We sketch the proof in the particular case when both X and Y are formal.

Let $A^* = H^*(X, \mathbb{Q})$, and let $\mathcal{H}_*(A) = \mathbb{L}^*(A_1)/(\text{im } \nabla)$ be its holonomy Lie algebra, defined as the quotient of the free Lie algebra on the dual of A^1 by the Lie ideal generated by the image of the comultiplication map, $\nabla: A_2 \rightarrow A_1 \wedge A_1 = \mathbb{L}^2(A_1)$, and with grading given by bracket length. Since X is formal, $\text{gr}_*(\pi_1 X) \otimes \mathbb{Q} \cong \mathcal{H}_*(A)$, as Lie algebras with grading (see for instance [6]).

Let $B^* = H^*(Y, \mathbb{Q}) = A^*[k]$, and let $\mathcal{L}(B, 0) = (\mathbb{L}(s^{-1}(\sharp B^{>0})), \partial)$ be the corresponding Quillen differential graded Lie algebra, defined as the free Lie algebra on the desuspension of the dual of the augmentation ideal of B , with differential ∂ arising from the dual of the multiplication map. Since Y is formal, $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong H_*(\mathcal{L}(B, 0))$, as graded Lie algebras (see [11]).

Now define a morphism of graded Lie algebras, $\lambda: \mathcal{L}(B, 0) \rightarrow (\mathcal{H}_*(A)[k], 0)$, by sending A_1 identically to A_1 (in degree $2k$) and $A_{>1}$ to zero. It is readily checked that λ commutes with the differentials and

induces a surjection in homology. Since the algebra A is Koszul, results from [11] and [10] insure that the induced map, $\lambda_*: \pi_*(\Omega Y) \otimes \mathbb{Q} \longrightarrow \text{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$, is in fact an isomorphism (of graded Lie algebras).

Our next result shows that the Rescaling Formula (even at the level of graded vector spaces) is strong enough to imply—under a formality assumption—the Koszulness of $H^*(X, \mathbb{Q})$.

THEOREM 2.2. – *Let Y be a formal k -rescaling of a space X . If $\text{Hilb}(\pi_*(\Omega Y) \otimes \mathbb{Q}, t)$ equals $\text{Hilb}(\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}, t^{2k})$, then $H^*(X, \mathbb{Q})$ is a Koszul algebra. Moreover, Y is a coformal space (i.e., its rational homotopy type is determined by its homotopy Lie algebra).*

Let $P_X(t) = \text{Hilb}(H^*(X; \mathbb{Q}), t)$ be the Poincaré series of X , and set $\phi_r = \text{rank gr}_r(\pi_1 X)$. The following *Lower Central Series formula* has received considerable attention: $\prod_{r \geq 1} (1 - t^r)^{\phi_r} = P_X(-t)$. This formula was established for classifying spaces of pure braids by Kohno, and then for complements of arbitrary fiber-type arrangements by Falk and Randell. The LCS formula was related to Koszul duality in [12], and extended to formal spaces X with Koszul cohomology algebra in [10]. Our next result gives an LCS-type formula for the rational homotopy groups of a rescaling of X (under no formality assumptions).

THEOREM 2.3. – *Let Y be a simply-connected CW-space of finite type. Assume $H^*(Y, \mathbb{Q})$ is the k -rescaling of a Koszul algebra. Set $\Phi_r := \text{rank } \pi_r(\Omega Y)$. Then $\Phi_r = 0$, if r is not a multiple of $2k$, and the following homotopy LCS formula holds:*

$$\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = P_Y(-t). \quad (4)$$

Consequently, $\Omega Y \simeq_{\mathbb{Q}} \prod_{i \geq 1}^w K(\mathbb{Q}, 2ki)^{\Phi_{2ki}}$. If $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$, and $H^*(X, \mathbb{Q})$ is a Koszul algebra, it follows that the rational homotopy type of ΩY is determined by the Poincaré polynomial of X . In particular, the Poincaré series of ΩY is given by $P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$. In fact, by Milnor-Moore [7], $H_*(\Omega Y, \mathbb{Q}) \cong U(\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k])$, as Hopf algebras.

We illustrate these results with some simple examples. In each case, X is a formal space, with $H^*(X, \mathbb{Q})$ a Koszul algebra, and Y is the (unique up to \mathbb{Q} -equivalence) formal k -rescaling of X .

- $X = S^1$, $Y = S^{2k+1}$. We have $\pi_1 X = \mathbb{Z}$, and so $\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^*(x)$, the free Lie algebra on a generator x in degree $2k$. Thus, $\Omega S^{2k+1} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2k)$, a result that goes back to Serre’s thesis.
- $X = \bigvee^n S^1$, $Y = \bigvee^n S^{2k+1}$. The associated graded of $\pi_1 X$ was computed by Magnus. We obtain: $\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^*(x_1, \dots, x_n)$. Hence, $\Phi_r = 0$ if $2k \nmid r$, and $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = 1 - nt^{2k+1}$. Thus, $P_{\Omega Y}(t) = (1 - nt^{2k})^{-1}$, a result that goes back to Bott and Samelson.
- $X = \#^g S^1 \times S^1$, $Y = \#^g S^{2k+1} \times S^{2k+1}$. The associated graded of $\pi_1 X$ was computed by Labute. We obtain: $\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^*(x_1, \dots, x_{2g}) / ([x_1, x_2] + \dots + [x_{2g-1}, x_{2g}] = 0)$. Hence, $\Phi_r = 0$ if $2k \nmid r$, and $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = 1 - 2gt^{2k+1} + t^{4k+2}$. Thus, $P_{\Omega Y}(t) = (1 - 2gt^{2k} + t^{4k})^{-1}$.

3. Malcev completions and Milnor-Moore groups

Let K be a connected, finite-type CW-complex K , with base-point \star . Fix an increasing, exhaustive filtration of K by connected, finite subcomplexes, $\{K_r\}_{r \geq 0}$, starting with $K_0 = \star$. Let Y be a based, simply-connected CW-space of finite type, and denote by $[K, \Omega Y]$ the group (under composition of loops) of based homotopy classes of based maps. Since K_r is a finite complex, $[K_r, \Omega Y]$ is a finitely-generated nilpotent group. Define the completion $[K, \Omega Y]^\wedge := \varprojlim_r ([K_{r-1}, \Omega Y] \otimes \mathbb{Q})$, and endow it with the inverse limit filtration, $F_r [K, \Omega Y]^\wedge := \ker([K, \Omega Y]^\wedge \longrightarrow [K_{r-1}, \Omega Y] \otimes \mathbb{Q})$.

For example, $K = \Omega S^m$ ($m \geq 2$) has a cell decomposition with one cell of dimension $r(m-1)$, for each $r \geq 0$. Setting K_r equal to the $r(m-1)$ -th skeleton, we obtain the filtered group $[\Omega S^m, \Omega Y]^\wedge$.

Now let G be an arbitrary finitely-generated group. Then G has a *Malcev completion*, defined as $G \otimes \mathbb{Q} := \varprojlim_r ((G/\Gamma_r G) \otimes \mathbb{Q})$. This group comes equipped with the inverse limit filtration; see [11].

The next theorem lifts the Rescaling Formula (3) from the level of associated graded Lie algebras to the level of filtered groups.

Rational homotopy groups and Koszul algebras

THEOREM 3.1. – *Let Y be a k -rescaling of a space X . Assume that $H^*(X, \mathbb{Q})$ is a Koszul algebra. Then, X is formal if and only if the following Malcev Formula holds:*

$$[\Omega S^{2k+1}, \Omega Y]^\wedge \cong \pi_1 X \otimes \mathbb{Q}, \quad \text{as filtered groups.} \quad (5)$$

Let us sketch the proof of the forward implication. To each complete Lie algebra L , one associates, in a functorial way, a filtered group, called the *exponential group* of L . The underlying set of $\exp(L)$ is just L , while the group law is given by the Campbell-Hausdorff formula; see [11]. We then have:

$$\begin{aligned} [\Omega S^{2k+1}, \Omega Y]^\wedge &\cong \exp(\operatorname{Hom}(H_{>0}(\Omega S^{2k+1}, \mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q})) \\ &\cong \exp(\pi_*(\Omega Y) \otimes \mathbb{Q}\{2k+1\}^\wedge) \cong \exp(\operatorname{gr}_*(\pi_1 X) \otimes \mathbb{Q}^\wedge) \cong \pi_1 X \otimes \mathbb{Q}. \end{aligned}$$

The key is the first isomorphism, which follows from a theorem of H. Baues [1]. The second isomorphism requires a “rebracketing” of the homotopy Lie algebra. The third one is provided by Theorem 2.1, while the last one uses the formality of X (see [14], [6]).

Consider now the Milnor-Moore group of degree 0 coalgebra maps from $H_*(K, \mathbb{Q})$ to $H_*(\Omega Y, \mathbb{Q})$, as defined in [7]. There is a natural filtration on $\operatorname{Hom}^{\operatorname{coalg}}(H_*(K, \mathbb{Q}), H_*(\Omega Y, \mathbb{Q}))$, with r -th term equal to the kernel of the map induced by the inclusion $K_{r-1} \rightarrow K$; see [3]. Using results of Hilton-Mislin-Roitberg and Scheerer, we show that $\operatorname{Hom}^{\operatorname{coalg}}(H_*(K, \mathbb{Q}), H_*(\Omega Y, \mathbb{Q})) \cong [K, \Omega Y]^\wedge$. Combined with Theorem 3.1, this proves the following theorem.

THEOREM 3.2. – *Let Y be a k -rescaling of a formal space X . If $H^*(X, \mathbb{Q})$ is a Koszul algebra, then*

$$\operatorname{Hom}^{\operatorname{coalg}}(H_*(\Omega S^{2k+1}, \mathbb{Q}), H_*(\Omega Y, \mathbb{Q})) \cong \pi_1 X \otimes \mathbb{Q}, \quad \text{as filtered groups.} \quad (6)$$

As noted by Cohen and Gitler in [3], the filtered group $[\Omega S^2, \Omega Y]$ is a particularly interesting object. As a set, it equals $\prod_{r \geq 1} \pi_r(\Omega Y)$, thus reassembling all the homotopy groups of Y into a single group, called the “group of homotopy groups” of Y . In this context, we obtain a result similar to Theorem 3.2, with ΩS^{2k+1} replaced by ΩS^2 . In the case when X is the configuration space of ℓ distinct points in \mathbb{C} , and Y is the configuration space of ℓ distinct points in \mathbb{C}^{k+1} , this result answers a question posed by Cohen and Gitler. In the case when $X = \bigvee^n S^1$ and $Y = \bigvee^n S^{2k+1}$, we recover a result of Sato (see [4]).

4. Rescaling hyperplane arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in \mathbb{C}^ℓ , with complement $X = M(\mathcal{A})$. For each $k \geq 1$, let $\mathcal{A}^k := \{H_1^{\times k}, \dots, H_n^{\times k}\}$ be the corresponding *redundant* arrangement of codimension k subspaces in $\mathbb{C}^{k\ell}$. Then, as shown by Cohen, Cohen and Xicoténcatl [2], the complement $Y = M(\mathcal{A}^{k+1})$ is an integral k -rescaling of X . By work of Brieskorn and Yuzvinsky, respectively, it is known that both X and Y are formal spaces.

By Theorems 2.1 and 2.2, the Rescaling Formula (3) holds precisely for the class of arrangements for which $H^*(X, \mathbb{Q})$ is a Koszul algebra. In this case, Y is coformal, and the Malcev Formula (5) also applies.

Presently, the only arrangements which are known to be Koszul are the fiber-type (or, supersolvable) arrangements; see [12]. For such arrangements, the Rescaling Formula was first established in [2], as a generalization of previous results of Fadell-Husseini and Cohen-Gitler on configuration spaces.

The Poincaré polynomial of the complement of a fiber-type arrangement \mathcal{A} , with exponents d_1, \dots, d_ℓ , factors as $P_X(t) = \prod_{j=1}^\ell (1 + d_j t)$. From Theorem 2.3, we see that $\Phi_r = \operatorname{rank} \pi_r(\Omega Y)$ vanishes if $2k \nmid r$, and $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = \prod_{j=1}^\ell (1 - d_j t^{2k+1})$. As a consequence, the rational homotopy type of ΩY is determined solely by the exponents of \mathcal{A} . In particular, $P_{\Omega Y}(t) = \prod_{j=1}^\ell (1 - d_j t^{2k})^{-1}$.

If \mathcal{A} is an affine, generic arrangement of n hyperplanes in \mathbb{C}^{n-1} ($n > 2$), then the Rescaling Formula fails for $X = M(\mathcal{A})$, due to the non-coformality of $Y = M(\mathcal{A}^{k+1})$. The absence of coformality is detected by higher-order Whitehead products, which also account for the deviation from equality in (3) and (4). For example, $\Phi_{(2k+1)n-2} = 1$, whereas $\operatorname{gr}_{>1}(\pi_1 X) = 0$.

5. Rescaling links in S^3

Let $K = (K_1, \dots, K_n)$ be a link of oriented circles in S^3 . For each $k \geq 1$, we define the k -rescaling $K^{\otimes k}$ to be the link of $(2k + 1)$ -spheres in S^{4k+3} obtained by taking the iterated join (in the sense of Koschorke and Rolfsen [5]) of the link K with k copies of the n -component Hopf link.

Let X and Y be the complements of K and $K^{\otimes k}$. Clearly, $\pi_1(Y) = 0$. Interpreting cup products in X and Y in terms of linking numbers, we show that Y is an integral k -rescaling of X . Since $H^{>2}(X, \mathbb{Z}) = 0$, this rescaling is unique (up to \mathbb{Q} -equivalence), and so Y is a formal space.

Associated to K there is a linking graph, G_K , with vertices corresponding to the components K_i , and edges connecting pairs of distinct vertices for which $\text{lk}(K_i, K_j) \neq 0$. It is known that G_K is connected if and only if $H^*(X, \mathbb{Q})$ is Koszul; see [6]. Examples of links with complete (hence, connected) linking graphs include algebraic links and singularity links of central arrangements of transverse planes in \mathbb{R}^4 .

The Rescaling Formula (3) holds for X and Y if and only if G_K is connected. In that case, Y is also coformal. Its homotopy Lie algebra is a semidirect product of free Lie algebras generated in degree $2k$, with non-zero ranks given by $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = (1 - t^{2k+1})(1 - (n-1)t^{2k+1})$.

On the other hand, the Malcev Formula (5) may fail (due to the non-formality of X), even when the Rescaling Formula does hold. To illustrate this phenomenon, we use the *Campbell-Hausdorff invariants* of links, introduced in [8]. If K_0 and K are two links with the same connected weighted linking graph, and if both link complements are formal, we show that $p^r(K_0) = p^r(K)$, for all $r \geq 1$.

Now take K_0 to be the n -component Hopf link ($n \geq 4$), and add the Borromean braid on three of its strands to get K . Then K_0 and K have the same weighted linking graph (the complete graph on n vertices, with all linking numbers equal to 1), but $p^3(K_0) \neq p^3(K)$. Since X_0 is obviously formal, X must be non-formal. Hence, if Y is the complement of the k -rescaling of K , then $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \text{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$, as graded Lie algebras, but $[\Omega S^{2k+1}, \Omega Y]^\wedge \not\cong \pi_1 X \otimes \mathbb{Q}$, as filtered groups.

Acknowledgements. The first author was partially supported by a CERES grant. The second author was partially supported by NSF grant DMS-0105342.

References

- [1] Baues H.J., Commutator calculus and groups of homotopy classes, London Math. Soc. Lecture Note Series, vol. 50, Cambridge Univ. Press, Cambridge-New York, 1981.
- [2] Cohen D.C., Cohen F.R., Xicoténcatl M., Lie algebras associated to fiber-type arrangements, preprint (2000), available at arXiv:math.AT/0005091.
- [3] Cohen F.R., Gitler S., Loop spaces of configuration spaces, braid-like groups, and knots, in: Cohomological methods in homotopy theory, Progress in Math., vol. 196, Birkhäuser, Basel-Berlin-Boston, 2001, pp. 59–78.
- [4] Cohen F.R., Sato T., On groups of homotopy groups, loop spaces, and braid-like groups, preprint (2001).
- [5] Koschorke U., Rolfsen D., Higher dimensional link operations and stable homotopy, Pacific J. Math. 139 (1989) 87–106.
- [6] Markl M., Papadima S., Homotopy Lie algebras and fundamental groups via deformation theory, Annales Inst. Fourier 42 (1992) 905–935.
- [7] Milnor J.W., Moore J.C., On the structure of Hopf algebras, Ann. of Math. 81 (1965) 211–264.
- [8] Papadima S., Campbell-Hausdorff invariants of links, Proc. London Math. Soc. 75 (1997) 641–670.
- [9] Papadima S., Suciuc A.I., Homotopy Lie algebras, lower central series, and the Koszul property, preprint (2001), available at arXiv:math.AT/0110303.
- [10] Papadima S., Yuzvinsky S., On rational $K[\pi, 1]$ spaces and Koszul algebras, J. Pure Appl. Alg. 144 (1999) 157–167.
- [11] Quillen D., Rational homotopy theory, Ann. of Math. 90 (1969) 205–295.
- [12] Shelton B., Yuzvinsky S., Koszul algebras from graphs and hyperplane arrangements, J. London Math. Soc. 56 (1997) 477–490.
- [13] Shiga H., Yagita N., Graded algebras having a unique rational homotopy type, Proc. Amer. Math. Soc. 85 (1982) 623–632.
- [14] Sullivan D., Infinitesimal computations in topology, Inst. Hautes Études Sci. Publ. Math. 47 (1977) 269–331.