Northeastern University—Mathematics Department
Graduate Qualifying Exam in Analysis
April 1995

Part I: Basics of Analysis

State clearly the hypothesis and conclusion of any theorem you use. You should completely explain your answers; a simple yes or no answer will not suffice. If you are asked to prove a certain statement, merely citing a reference for that statement is not sufficient; you may however cite basic results as part of your proof.

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1) Let d be the usual Euclidean metric on $\mathbb{R}^n$. Consider the following functions:

$$b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad b(x, y) = \min\{1, d(x, y)\}$$
$$m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad m(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}.$$

Show:

a) The functions b and m are metrics on $\mathbb{R}^n$.

b) The metrics b and m induce the same topology on $\mathbb{R}^n$ as d.

c) The metrics d and b are *numerically equivalent*, i.e., there exist positive constants c and k such that, for all x and y in $\mathbb{R}^n$,

$$c \cdot d(x, y) < b(x, y), \quad \text{and} \quad k \cdot b(x, y) < d(x, y).$$

d) The metrics d and m are *not* numerically equivalent.

2) Let f be a function from a metric space (X, d) to a metric space (X', d'). Suppose that F_1 and F_2 are two closed subsets of X such that $X = F_1 \cup F_2$, and that the restrictions $f \mid F_1$ and $f \mid F_2$ are continuous. Show that f is continuous.

3) Consider the real functions defined by

$$f_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \text{and} \quad f(x) = \sin x.$$

a) Does $f_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$?
b) Does $f_n \longrightarrow f$ uniformly on $[-r, r]$, for $r > 0$?
c) Does $f_n \longrightarrow f$ uniformly on $\mathbb{R}$?

4) Let $f_n : [-1, 1] \to \mathbb{R}$ be the functions defined by:

$$f_n(x) = \frac{x}{1 + nx^2}.$$

a) Show that f_n are differentiable functions, and converge uniformly on $[-1, 1]$ to a differentiable function.
b) Compute

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) \qquad \text{and} \qquad \frac{d}{dx} \lim_{n\to\infty} f_n(x).$$

c) Explain why the answers in part b) agree, or disagree.

5) Let $f_n : [0, 1] \to \mathbb{R}$ be the function defined by:

$$f_n(x) = \begin{cases} n^2 x & \text{for} \quad 0 \le x \le 1/n \\ -n^2 x + 2n & \text{for} \quad 1/n < x \le 2/n \\ 0 & \text{for} \quad 2/n < x \le 1. \end{cases}$$

a) Compute

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \qquad \text{and} \qquad \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

b) Explain why the answers in part a) agree, or disagree.

6) Let $F(x, y, z) = (x^2 + z^2 - 4)^2 + y - 16$.
a) Identify and draw the set of points (x_0, y_0, z_0) in $\mathbb{R}^3$ at which it is *not* possible to use the Implicit Function Theorem in order to find a local implicit function $z = f(x, y)$ so that $F(x, y, f(x, y)) = 0$.
b) Find a point (x_1, y_1, z_1) in $\mathbb{R}^3$ and a neighborhood U of (x_1, y_1) in $\mathbb{R}^2$ so that the following condition holds: there is a function $f : U \to \mathbb{R}$ such that $F(x, y, f(x, y)) = 0$ for all $(x, y) \in U$.