

## Qualifying Exam in Geometry

(1.5 hours)

Spring 1996

1. If  $\omega = f dx$  is a 1-form on  $[0, 1]$ , with  $f(0) = f(1)$ , show that there is a unique number  $\lambda$  such that

$$\omega = \lambda dx + dg$$

for some function  $g$  with  $g(0) = g(1)$ .

2. i) Prove that  $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$  is not of the form  $e^A$ , for any  $A \in \mathfrak{gl}(2, \mathbb{R})$ .

ii) Find matrices  $A, B \in \mathfrak{gl}(2, \mathbb{C})$  such that  $e^{A+B} \neq e^A e^B$ .

3. For each of the following spheres:  $S^1$ ,  $S^2$ , and  $S^3$ , show that either

a) There is no nowhere-vanishing vector field, or

b) Exhibit a nowhere-vanishing vector field.

4. i) Let  $\xi : E \xrightarrow{\pi} B$  be a  $C^\infty$  vector bundle. What does it mean that  $\xi$  is orientable?

ii) Let  $M$  be a  $C^\infty$  manifold. What does it mean that  $M$  is orientable?

iii) Show that there is an exact sequence of vector bundles

$$0 \rightarrow \pi^*(\xi) \rightarrow TE \rightarrow \pi^*(TB) \rightarrow 0.$$

iv) If  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  is exact, then each bundle  $E_i$  is orientable if the other two are.

v) Show that  $T(TM)$  is always orientable.

vi) Assume the bundle  $\xi : E \xrightarrow{\pi} B$  is not orientable. Show that the manifold  $E$  is not orientable.