

Qualifying Exam in Geometry

Spring 1998

Do all five problems. Give complete proofs or justifications for each statement you make. Show all your work.

1. Let $f : M^n \rightarrow N^n$ be a proper smooth map between oriented, smooth n -manifolds such that $f_* : T_p M \rightarrow T_{f(p)} N$ is orientation-preserving whenever p is a regular point. Show that if N is connected, then either f is surjective, or else all points are critical points of f .
2. Consider the Lie group $G = \mathrm{GL}_n(\mathbb{R})$, and its Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$. For any $A \in \mathfrak{g}$, define a vector field X_A on $C^\infty(G)$ by the formula

$$X_A f(B) = \left. \frac{\partial}{\partial t} \right|_{t=0} f(B + tAB).$$

- (a) Show that X_A is a right-invariant vector field, and that any right-invariant vector field on G arises in this way.
 - (b) What is the Lie bracket $[X_A, X_B]$?
3. Let $M = S^1$, viewed as the submanifold $\{(x, y) \mid x^2 + y^2 = 1\}$ of \mathbb{R}^2 . Let $\theta = \arctan(y/x)$ be the usual polar coordinate on $S^1 \setminus \{(1, 0)\}$.
 - (a) Show that the 1-form $d\theta$ extends uniquely to a smooth 1-form (also denoted by $d\theta$) on M .
 - (b) Show that $d\theta$ is a closed 1-form which is not exact.
 - (c) Let ω be a form in $E^1(M)$. Show there is a unique constant $c \in \mathbb{R}$ and a function $f \in E^0(M)$ such that $\omega - cd\theta = df$.
 - (d) Show that $E^1(M)/d(E^0(M)) \cong \mathbb{R}$.
 4. Consider the following vector fields on \mathbb{R} :
 - (a) $\xi(x) = x$
 - (b) $\xi(x) = x^2$

In each case, decide whether ξ is the tangent field of a global flow on \mathbb{R} . If not, give a reason why. If yes, find the flow.

5. Let $M = \mathbb{R}^3 \setminus \{0\}$. Consider the following 2-dimensional distributions on M :
 - (a) $E = \cup_{x \in M} E_x$, with $E_x = \{y \in T_x M \mid y \perp x\}$;
 - (b) $F = \cup_{x \in M} F_x$, with $F_x = \{(y_1, y_2, y_3) \in T_x M \mid (y_2, y_3, y_1) \perp x\}$.

In each case, decide whether the distribution is integrable or not, and, if it is, describe its integral manifolds.