Northeastern University
Department of Mathematics

Qualifying Exam in Geometry

Winter 1999

Do all six problems. Give complete proofs or justifications for each statement you make. Show all your work.

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1. Let M be an n-dimensional manifold. Let $\omega \in \Omega^n(M)$ be an n-form on M, and let X be a complete vector field on M, with flow ϕ_t. Prove that $\phi_t^*\omega = \omega$ if and only if $i_X\omega$ is closed.

2. Let M be a compact orientable manifold of dimension n. Let $\alpha \in \Omega^n(M)$ be an n-form on M and X a vector field on M. Show that $L_X\alpha$ vanishes at some point.

3. Let M and N be smooth manifolds, and $f : M \to N$ a C^∞ map. Suppose that M is compact, N is connected, f is injective, and df_x is an isomorphism for each $x \in M$. Show that f is a diffeomorphism.

4. Let $u = x^2 - y^3,\ v = 3xy + y^2 - x^2$.
   (a) For which (a, b) in $\mathbb{R}^2$ is there a neighborhood U of (a, b) such that $(U, (u, v))$ is a coordinate system?
   (b) For which real numbers c is the locus $y^2 - x(x-1)(x-c) = 0$ a submanifold of $\mathbb{R}^2$?

5. Let $T^2 = S^1 \times S^1$ be the torus, with coordinates (x, y). Let $H(x, y) \in C^\infty(T^2)$ be a smooth function. Consider the flow ϕ_t generated by the following linear system:
$$\begin{cases} \frac{dx}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dx}{dt} &= -\frac{\partial H}{\partial x} \end{cases}$$
   Prove that:
   (a) ϕ_t exists for all $t \in \mathbb{R}$.
   (b) $\phi_t^*(dx \wedge dy) = dx \wedge dy$, for all $t \in \mathbb{R}$.

6. Let H be the Heisenberg group
$$H = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid x_{12}, x_{13}, x_{23} \in \mathbb{R} \right\}$$
   of upper-diagonal 3×3 real matrices with 1's on the diagonal. This group has natural coordinates (x_{12}, x_{13}, x_{23}), and it acts on itself by left translations. Let $v_{12}, v_{13}, v_{,23}$ be the left-invariant vector-fields on H, with values at the identity $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. Consider the 2-dimensional distributions E and F on H generated by v_{12}, v_{13} and v_{12}, v_{23}, respectively. Show that E is integrable and F is not.