QUALIFYING EXAM IN TOPOLOGY

WINTER 1996

1. Let K be the Klein bottle — a square with opposite vertical edges identified in the same direction, and opposite horizontal edges identified in the opposite direction.

a) Compute the fundamental group of K.

b) Compute $H_*(K;\mathbb{Z})$ and $H_*(K;\mathbb{Z}_2)$.

2. Let $f : X \to Y$ be a covering map.

a) Is it true that $f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism? Give either a proof or a counterexample.

b) Is it true that $f_* : H_1(X) \to H_1(Y)$ is a monomorphism? Give either a proof or a counterexample.

3. Let X denote the set of all real numbers with the finite-complement topology, and define $f : E^1 \to X$ by $f(x) = x$. Show that f is continuous, but not a homeomorphism.

4. Given a continuous map $f : S^n \to S^n$, define the degree of f. Furthermore, show that:

a) If f has nonzero degree, then f is onto.

b) S^n admits maps of arbitrary degree. [Hint: Do this first for $n = 1$, and then suspend.]

5. Let X be a topological space, and G a group acting on it. Show that X may be Hausdorff yet X/G non-Hausdorff. If X is a compact topological group and G a closed subgroup acting on X by left translation, show that X/G is Hausdorff.

6. Let p, q, r, s integers such that $pr - qs = -1$. Take two solid tori $S^1 \times D^2$ and glue them along their boundary via the map

$$\begin{pmatrix} p & s \\ q & r \end{pmatrix}$$

Call the resulting space X. Find a cell decomposition for X and compute $H_*(X;\mathbb{Z})$.