QUALIFYING EXAM IN TOPOLOGY

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1. Let (X, d) be a metric space, and let $f : X \to X$ be an isometry (that is, $d(f(x), f(y)) = d(x, y)$, for all $x, y \in X$).

(i) Show that, if X is compact, then f is surjective.
(ii) Show that, in general, f is not surjective.

2. Let X and Y be topological spaces, and let $f : X \to Y$ be a homotopy equivalence. Show the following to be either true (provide a proof), or false (provide a counter-example):

(i) If X is countable, so is Y.
(ii) If X is connected, so is Y.
(iii) If X is path-connected, so is Y.
(iv) If X is compact, so is Y.
(v) If X is contractible, so is Y.

3. Consider two labeled octogons with edges identified in pairs as follows:

(i) $abcda^{-1}b^{-1}c^{-1}d^{-1}$;
(ii) $abcda^{-1}b^{-1}c^{-1}d$.

For each of the resulting surfaces, determine its orientability status (i.e., orientable or not), its genus (i.e., number of handles or crosscaps), and its Euler characteristic.

4. Let $X = S^1 \bigvee S^1$ be the one-point union of two circles.

(i) Compute $\pi_1(X)$.
(ii) Let Y be a connected n-fold cover of X. Compute $\pi_1(Y)$.
(iii) Draw at least 3 distinct connected 3-fold covers of X Determine which ones are regular, and which ones are not.

5. Consider the following subset Σ of $\mathbb{R}^3$:

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid ((x^2 + y^2)^{\frac{1}{2}} - 1)^2 + z^2 = 1\}.$$

Construct a cellular decomposition of Σ and calculate its homology groups.

6. Let $f : S^n \to S^n$ be a continuous map.

(i) Define the *degree* of f, $\deg(f)$.
(ii) Show that, if f has no fixed points, then $\deg(f) = (-1)^{n+1}$.
(iii) Show that, if $\deg(f) \neq 0$, then f is surjective.