Northeastern University
Department of Mathematics

Qualifying Exam in Topology

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Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$, and let $f\colon X \to \mathbb{R}$ be the function defined by $f(x, y) = y$. Prove or disprove the following:
   (a) f is a continuous map.
   (b) f is an open map.
   (c) f is a closed map.

2. Prove or disprove the following:
   (a) If X and Y are path-connected, then $X \times Y$ is path-connected.
   (b) If X is path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.
   (c) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.

3. Find path-connected spaces X and Y, and a continuous map $f\colon X \to Y$, such that f induces an isomorphism between $H_1(X, \mathbb{Z})$ and $H_1(Y, \mathbb{Z})$, but the fundamental group of X is not isomorphic to the fundamental group of Y.

4. Let $Y = \{(z, w) \in \mathbb{C}^2 \mid z \neq w\}$. Let $X = Y/S_2$, where the symmetric group S_2 acts on Y by permutating the coordinates. Let $p\colon Y \to X$ be the projection map.
   (a) Find the fundamental group of Y.
   (b) Find the fundamental group of X.
   (c) Determine the induced homomorphim $p_*\colon \pi_1(Y) \to \pi_1(X)$.

5. If M and N are two connected, oriented manifolds of dimension n, their *connected sum*, $M \sharp N$, is obtained by removing an open n-disk from each manifold, and identifying the boundaries of the two disks by an orientation-preserving homeomorphism.
   (a) Express the Euler characteristic $\chi(M \sharp N)$ in terms of $\chi(M)$ and $\chi(N)$.
   (b) Suppose $n > 2$. Express the fundamental group $\pi_1(M \sharp N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.

6. Consider two labeled dodecagons with edges identified in pairs as follows:
   (i) $abcdefa^{-1}b^{-1}c^{-1}d^{-1}e^{-1}f^{-1}$.
   (ii) $aba^{-1}b^{-1}cd^{-1}efe^{-1}f^{-1}cd^{-1}$.

   For each of the resulting surfaces, do the following.
   (a) Determine its orientability status (i.e., orientable or not).
   (b) Compute its genus (i.e., number of handles or crosscaps).
   (c) Compute its homology groups.
   (d) Compute its Euler characteristic.