Northeastern University
Department of Mathematics

Qualifying Exam in Topology

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Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $f\colon X \to Y$ be a continuous surjection, and let $g\colon Y \to Z$ be a function so that $g \circ f$ is continuous. Show that:
   (a) If f is a closed map, then g is continuous.
   (b) If f is not a closed map, then g may fail to be continuous.

2. Let $X = [0,1]/(\frac{1}{4}, \frac{3}{4})$ be the quotient space of the unit interval, where the open interval $(\frac{1}{4}, \frac{3}{4})$ is identified to a single point. Show that:
   (a) X is connected.
   (b) X is compact.
   (c) X is not Hausdorff.

3. (a) Define the two notions: "homotopy between two maps" and "homotopy equivalence between two topological spaces."
   (b) Give an example of topological spaces X and Y that have the same homotopy type but are not homeomorphic.
   (c) Give an example of topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.
   (d) Give an example of topological spaces X and Y that have isomorphic homology groups (in all degrees) but are not homotopy equivalent.

4. Let M be a compact, connected, orientable surface of genus 3 with 2 boundary circles.
   (a) Compute the fundamental group and the homology groups of M.
   (b) Suppose N is a compact, connected, orientable surface that is a covering of M. Show that N has an even number of boundary circles.

5. Let K be the 2-skeleton of the 4-simplex.
   (a) Write down the simplicial chain complex of K.
   (b) Compute all the homology groups of K.

6. Let $X = S^1 \bigvee \mathbb{RP}^2$ be the one-point union of a circle and a projective plane. Draw all the 3-fold covering spaces of X. In each case, identify the corresponding subgroup of $\pi_1(X)$, and specify whether the cover is regular, or not.