Geometric and homological finiteness in free abelian covers

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Alex Suciu

Northeastern University

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THE BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

Let G be a finitely generated group. Set $S(G) = (\operatorname{Hom}(G, \mathbb{R}) \setminus \{0\})/\mathbb{R}^+$.

DEFINITION (BIERI, NEUMANN, STREBEL 1987)

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\}.$$

Here, $\mathcal{C}(G)$ is the Cayley graph, and $\mathcal{C}_\chi(G)$ the induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

$\Sigma^1(G)$ is an open set, independent of choice of generating set for G.

DEFINITION (BIERI, RENZ 1988)

$$\Sigma^k(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}.$$

Here, G is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq k$.

Similar definition for $\Sigma^k(G, \Bbbk)$, with $\Bbbk$ a field.

- The Σ-invariants form a descending chain of open subsets,
$$S(G) \supseteq \Sigma^1(G,\mathbb{Z}) \supseteq \Sigma^1(G,\mathbb{Z}) \supseteq \cdots$$
- $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- $\Sigma^k(G,\mathbb{Z}) \neq \emptyset \implies G$ is of type FP_k.
- Note that a non-zero $\chi\colon G \to \mathbb{R}$ has image $\mathbb{Z}^r$, for some $r \geq 1$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which G/N is free abelian:
$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$
where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$.
- In particular: $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

Let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$. Let $G = \pi_1(X, x_0)$. For each $\chi \in S(X) = S(G)$, set

$$\widehat{\mathbb{Z}G}_\chi = \left\{ \lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp} \lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R} \right\}$$

This is a ring, which contains $\mathbb{Z}G$ as a subring; hence, a $\mathbb{Z}G$-module.

DEFINITION (FARBER, GEOGHEGAN, SCHÜTZ 2010)

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leq q\}$$

Bieri: If G is of type FP_k, then $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \leq k$.

The sphere $S(X)$ parametrizes all regular, free abelian covers of X. The Σ-invariants of X keep track of the geometric finiteness properties of these covers.

THE DWYER–FRIED INVARIANTS

- Another tack was taken by Dwyer and Fried, also in 1987. Instead of looking at all regular, free abelian covers of X at once, they fix the rank, say r, of the deck-transformation group.
- Any epimorphism $\nu\colon \pi_1(X, x_0) \twoheadrightarrow \mathbb{Z}^r$ gives rise to such a cover, by pulling back the universal cover of the r-torus,

$$\begin{array}{ccc} X^{\nu} & \longrightarrow & \mathbb{R}^r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & T^r, \end{array}$$

where $f_\sharp\colon \pi_1(X, x_0) \to \pi_1(T^r, *)$ realizes ν. Moreover, all regular $\mathbb{Z}^r$-covers of X arise in this manner.

- Thus, such covers are parameterized by the Grassmannian $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$, via the correspondence

$$\{\text{regular } \mathbb{Z}^r\text{-covers of } X\} \longleftrightarrow \{r\text{-planes in } H^1(X, \mathbb{Q})\}$$

$$X^{\nu} \to X \longleftrightarrow P_{\nu} := \mathrm{im}(\nu^*\colon \mathbb{Q}^r \to H^1(X, \mathbb{Q}))$$

Moving about the rational Grassmannian, and recording how the Betti numbers of the corresponding covers vary leads to:

DEFINITION

The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_r(X) = \{P_\nu \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \le i\},$$

defined for all $i \ge 0$ and all $r > 0$, with the convention that $\Omega^i_r(X) = \emptyset$ if $r > b_1(X)$.

For a fixed $r > 0$, we get a descending filtration of the Grassmanian of r-planes in $\mathbb{Q}^n$, where $n = b_1(X)$:

$$\mathrm{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots .$$

- The Ω-sets are homotopy-type invariants:
  Suppose $X \simeq Y$. For each $r > 0$, there is an isomorphism $\mathrm{Gr}_r(H^1(Y, \mathbb{Q})) \cong \mathrm{Gr}_r(H^1(X, \mathbb{Q}))$ sending each subset $\Omega^i_r(Y)$ bijectively onto $\Omega^i_r(X)$.
- Hence, if G is f.g, we may define $\Omega^i_r(G) = \Omega^i_r(K(G, 1))$.
- $\Omega^i_1(X)$ is always open, but $\Omega^i_r(X)$ may be non-open for $r > 1$.
- If $r = n$, where $n = b_1(X) > 0$, then $\mathrm{Gr}_n(H^1(X, \mathbb{Q})) = \{\mathrm{pt}\}$.
  This single point corresponds to the maximal free abelian cover, $X^\alpha \to X$, where $\alpha\colon G \twoheadrightarrow G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}}) = \mathbb{Z}^n$.

EXAMPLE

Let $X = S^1 \vee S^k$, for some $k > 1$. Then $X^\alpha \simeq \bigvee_{j\in\mathbb{Z}} S^k_j$. Thus,

$$\Omega^i_n(X) = \{\mathrm{pt}\} \text{ if } i < k \text{ and is empty if } i \geq k$$

CHARACTERISTIC VARIETIES

Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$ be the character group of $G = \pi_1(X)$.

DEFINITION

The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_j(X, \mathbb{C}_\rho) \neq 0, \text{ for some } j \leq i\}$$

- Get a filtration $\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \widehat{G}$.
- If X has finite k-skeleton, then $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$, for each $i \leq k$.
- The varieties $\mathcal{V}^i(X)$ are homotopy-type invariants of X.
- Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}}, \mathbb{C})$ as a module over $\mathbb{C}[G_{\mathrm{ab}}]$. Then $\mathcal{V}^i(X) = V(\mathrm{ann}\,(\bigoplus_{j\leq i} H_j(X^{\mathrm{ab}}, \mathbb{C})))$.

EXAMPLE

Let $L = (L_1, \dots, L_n)$ be a link in S^3, with complement X_L, and Alexander poly Δ_L. Then $\mathcal{V}^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\}$.

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^\times)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^z$.
- Given a Zariski closed subset $W \subset H^1(X,\mathbb{C}^\times)$, set

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$$

- $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- $\tau_1(W)$ is non-empty iff $1 \in W$.
- $\tau_1(W) = T_1(W) \cong \mathbb{C}^r$ if $W \cong (\mathbb{C}^\times)^r$ is an algebraic subtorus of $H^1(X,\mathbb{C}^\times)$.
- Set $\tau_1^{\mathbb{Q}}(W) = \tau_1(W) \cap H^1(X,\mathbb{Q})$ and $\tau_1^{\mathbb{R}}(W) = \tau_1(W) \cap H^1(X,\mathbb{R})$.

AN UPPER BOUND FOR THE Σ-INVARIANTS

- Let X be a CW-complex with finite k-skeleton, and let $\chi \in S(X)$.
- Let $\Gamma = \operatorname{im}(\chi) \cong \mathbb{Z}^r$. A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is χ-monic if the greatest element in $\chi(\operatorname{supp}(p))$ is 0, and $n_0 = 1$.
- Let $\mathcal{R}\Gamma_\chi$ be the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all χ-monic polynomials; it's both a $\mathbb{Z}G$-module and a PID.
- For each $i \le k$, set $b_i(X,\chi) = \operatorname{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$.

THEOREM (PAPADIMA–S. 2010)

1. $-\chi \in \Sigma^k(X,\mathbb{Z}) \implies b_i(X,\chi) = 0, \forall i \le k.$
2. $\chi \notin \tau_1^{\mathbb{R}}(\mathcal{V}^k(X)) \iff b_i(X,\chi) = 0, \forall i \le k.$

Hence:

$$\Sigma^i(X,\mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))$$

Thus, $\Sigma^i(X,\mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.

A formula and a bound for the Ω-invariants

Theorem (Dwyer–Fried 1987, Papadima–S. 2010)

For an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$, the following are equivalent:

1. *The vector space $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ is finite-dimensional.*
2. *The algebraic torus $\mathbb{T}_\nu = \operatorname{im}\left(\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{\pi_1(X)}\right)$ intersects the variety $\mathcal{V}^k(X)$ in only finitely many points.*

Note that $\exp(P_\nu \otimes \mathbb{C}) = \mathbb{T}_\nu$. Thus:

Corollary

$$\Omega_r^i(X) = \left\{P \in \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \mid \dim\left(\exp(P \otimes \mathbb{C}) \cap \mathcal{V}^i(X)\right) = 0\right\}$$

Corollary

Set $n = b_1(X)$.

1. *If $\mathcal{V}^i(X)$ is finite, then $\Omega_r^i(X) = \operatorname{Gr}_r(\mathbb{Q}^n)$.*
2. *If $\mathcal{V}^i(X)$ is infinite, then $\Omega_n^q(X) = \emptyset$, for all $q \geq i$.*

- Let V be a homogeneous variety in $\mathbb{k}^n$. The set $\sigma_r(V) = \{P \in \operatorname{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\}\}$ is Zariski closed.
- If $L \subset \mathbb{k}^n$ is a linear subspace, $\sigma_r(L)$ is the *special Schubert variety* defined by L. If $\operatorname{codim} L = d$, then $\operatorname{codim} \sigma_r(L) = d - r + 1$.

THEOREM

$$\Omega^i_r(X) \subseteq \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r\big(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))\big)$$

- Thus, each set $\Omega^i_r(X)$ is contained in the complement of a finite union of special Schubert varieties.
- If $r = 1$, the inclusion always holds as an equality. In general, though, the inclusion is strict.

EXAMPLE

Let $G = \langle x_1, x_2, x_3 \mid [x_1^2, x_2],\ [x_1, x_3],\ x_1[x_2, x_3]x_1^{-1}[x_2, x_3]\rangle$. Then

$$\mathcal{V}^1(G) = \{1\} \cup \{t \in (\mathbb{C}^\times)^3 \mid t_1 = -1\}.$$

Thus, $\Omega^1_2(G)$ is a single point in $\operatorname{Gr}_2(H^1(G, \mathbb{Q})) = \mathbb{QP}^2$, hence *not* open.

COMPARING THE Σ- AND Ω-BOUNDS

THEOREM

Suppose that $\Sigma^i(X,\mathbb{Z}) = S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))$.
Then $\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(X)))$, *for all* $r \geq 1$.

COROLLARY

Suppose there is an integer $r \geq 2$ *such that* $\Omega^i_r(X)$ *is* not *Zariski open. Then* $\Sigma^i(X,\mathbb{Z}) \neq S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X)))^{\complement}$.

In general, the implication from the theorem cannot be reversed.

EXAMPLE

Let $G = \langle x_1, x_2 \mid x_1 x_2 x_1^{-1} = x_2^2 \rangle$. Then $\mathcal{V}^1(G) = \{1,2\} \subset \mathbb{C}^{\times}$.
Thus, $\Omega^1_1(G) = \{\mathrm{pt}\}$, and so $\Omega^1_1(G) = \sigma_1(\tau_1^{\mathbb{Q}}(\mathcal{V}^1(X)))^{\complement}$.
On the other hand, $\Sigma^1(G) = \{-1\}$, whereas $S(\tau_1^{\mathbb{Q}}(\mathcal{V}^1(X)))^{\complement} = \{\pm 1\}$.

TORIC COMPLEXES AND RAAGS

- L simplicial complex on n vertices $\rightsquigarrow$ *toric complex* T_L (subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L).
- $\pi_1(T_L)$ is the *right-angled Artin group* associated to graph $\Gamma = L^{(1)}$:

$$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \rangle.$$

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
- $H^*(T_L, \Bbbk)$ is the *exterior Stanley-Reisner ring* of L, with generators the duals v^*, and relations the monomials corresponding to the missing simplices of L.

EXAMPLE

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
- $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
- $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

- Identify $\widehat{G_\Gamma} = H^1(T_L, \mathbb{C}^\times)$ with the algebraic torus $(\mathbb{C}^\times)^{\mathsf{V}} = (\mathbb{C}^\times)^n$.
- Each subset $\mathsf{W} \subseteq \mathsf{V}$ yields an algebraic subtorus $(\mathbb{C}^\times)^{\mathsf{W}} \subset (\mathbb{C}^\times)^{\mathsf{V}}$.
- Denote by L_{W} the subcomplex induced by L on W, and by $\mathrm{lk}_K(\sigma)$ the link of a simplex $\sigma \in L$ in a subcomplex $K \subseteq L$.

THEOREM (PAPADIMA–S. 2009)

For each $i \geq 1$,

$$\mathcal{V}^i(T_L) = \bigcup_{\mathsf{W}} (\mathbb{C}^\times)^{\mathsf{W}} \quad \textit{and} \quad \tau_1(\mathcal{V}^i(T_L)) = \bigcup_{\mathsf{W}} \mathbb{C}^{\mathsf{W}}$$

where the union is taken over all $\mathsf{W} \subseteq \mathsf{V}$ *for which there is a simplex* $\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}$ *and an index* $j \leq i$ *such that* $\widetilde{H}_{j-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \mathbb{C}) \neq 0$.

In particular:

$$\mathcal{V}^1(G_\Gamma) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} (\mathbb{C}^\times)^{\mathsf{W}}.$$

COROLLARY

For each $i \geq 0$ *and* $r \geq 1$,

$$\Omega^i_r(T_L) = \mathrm{Gr}_r(\mathbb{Q}^{\mathsf{V}}) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(T_L))).$$

Using now the computation of the Σ-invariants of RAAGs by Meier, Meinert, VanWyk (1998), we get:

COROLLARY

For each $i \geq 0$,

1. $\Sigma^i(G_L, \mathbb{R}) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(G_L))^{\complement}$.
2. $\Sigma^i(G_L, \mathbb{Z}) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(G_L))^{\complement}$, *provided that, for every* $\sigma \in \Delta$, *and every* $\mathsf{W} \subseteq \mathsf{V}$ *with* $\sigma \cap W = \emptyset$, *the groups* $\widetilde{H}_j(\mathrm{lk}_{\Delta_{\mathsf{W}}}(\sigma), \mathbb{Z})$ *are torsion-free, for all* $j \leq i - \dim(\sigma) - 2$.

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