

Betti numbers of abelian covers

Alex Suciu

Northeastern University





Geometry and Topology Seminar
University of Wisconsin
May 6, 2011

- 1 Abelian covers
 - A parameter set for abelian covers
 - Free abelian covers
 - The Dwyer–Fried sets

- 2 Characteristic arrangements and Schubert varieties
 - Characteristic varieties
 - Computing the Ω -invariants
 - Characteristic subspace arrangements
 - Special Schubert varieties

- 3 Resonance varieties and straight spaces
 - The Aomoto complex
 - Resonance varieties
 - Straight spaces
 - Ω -invariants of straight spaces

References

-  S. Papadima and A. Suciuc, *Bieri–Neumann–Strebel–Renz invariants and homology jumping loci*, Proc. London Math. Soc. **100** (2010), no. 3, 795–834.
-  A. Suciuc, *Resonance varieties and Dwyer–Fried invariants*, to appear in Advanced Studies Pure Math., Kinokuniya, Tokyo, 2011.
-  A. Suciuc, *Characteristic varieties and Betti numbers of free abelian covers*, preprint 2011.
-  A. Suciuc, Y. Yang, and G. Zhao, *Homological finiteness of abelian covers*, preprint 2011.

Abelian covers

- Let X be a connected, finite-type CW-complex. We may assume X has a single 0-cell, call it x_0 . Set
 - $G = \pi_1(X, x_0)$.
 - $H = G_{\text{ab}} = H_1(X, \mathbb{Z})$.
- Let A be a finitely generated abelian group.
- Any epimorphism $\nu: G \twoheadrightarrow A$ gives rise to a connected, regular cover, $X^\nu \rightarrow X$, with group of deck transformations A .
- Conversely, any conn., regular A -cover $p: (Y, y_0) \rightarrow (X, x_0)$ yields

$$1 \longrightarrow \pi_1(Y, y_0) \xrightarrow{p_\#} \pi_1(X, x_0) \xrightarrow{\nu} A \longrightarrow 1$$

so that there is an A -equivariant homeomorphism $Y \cong X^\nu$.

- Thus, the set of conn., regular A -covers of X can be identified with

$$\text{Epi}(G, A) / \text{Aut}(A) = \text{Epi}(H, A) / \text{Aut}(A).$$

Theorem (S.–Yang–Zhao 2011)

There is a bijection

$$\text{Epi}(H, A) / \text{Aut}(A) \longleftrightarrow \text{GL}_n(\mathbb{Z}) \times_{\mathbb{P}} \Gamma$$

where $n = \text{rank } H$, $r = \text{rank } A$, and

- \mathbb{P} is a parabolic subgroup of $\text{GL}_n(\mathbb{Z})$.
- $\text{GL}_n(\mathbb{Z}) / \mathbb{P} = \text{Gr}_{n-r}(\mathbb{Z}^n)$.
- $\Gamma = \text{Epi}(\mathbb{Z}^{n-r} \oplus \text{Tors}(H), \text{Tors}(A)) / \text{Aut}(\text{Tors}(A))$ is a finite set.
- $\text{GL}_n(\mathbb{Z}) \times_{\mathbb{P}} \Gamma$ is the twisted product under the diagonal \mathbb{P} -action.

This bijection depends on a choice of splitting $H \cong \bar{H} \oplus \text{Tors}(H)$, and a choice of basis for $\bar{H} = H / \text{Tors}(H)$.

- Simplest situation is when $A = \mathbb{Z}^r$.
- Basic example: the universal cover of the r -torus, $\mathbb{Z}^r \rightarrow \mathbb{R}^r \rightarrow T^r$.
- All conn., regular \mathbb{Z}^r -covers of X arise as pull-backs of this cover:

$$\begin{array}{ccc} X^\nu & \longrightarrow & \mathbb{R}^r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & T^r, \end{array}$$

where $f_\# : \pi_1(X) \rightarrow \pi_1(T^r)$ realizes the epimorphism $\nu : G \twoheadrightarrow \mathbb{Z}^r$.
(Note: X^ν is the homotopy fiber of f).

- Let $\nu^* : \mathbb{Q}^r \rightarrow H^1(X, \mathbb{Q}) = \mathbb{Q}^n$. We get:

$$\{\mathbb{Z}^r\text{-covers } X^\nu \rightarrow X\} \longleftrightarrow \{r\text{-planes } P_\nu := \text{im}(\nu^*) \text{ in } H^1(X, \mathbb{Q})\}.$$

and so

$$\text{Epi}(H, \mathbb{Z}^r) / \text{Aut}(\mathbb{Z}^r) \cong \text{Gr}_{n-r}(\mathbb{Z}^n) \cong \text{Gr}_r(\mathbb{Q}^n).$$

The Dwyer–Fried sets

Moving about the parameter space for A -covers, and recording how the Betti numbers of those covers vary leads to:

Definition

The *Dwyer–Fried invariants* of X are the subsets

$$\Omega_A^i(X) = \{[\nu] \in \text{Epi}(H, A) / \text{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \leq i\}.$$

where, $X^\nu \rightarrow X$ is the cover corresponding to $\nu: H \rightarrow A$.

In particular, when $A = \mathbb{Z}^r$,

$$\Omega_r^i(X) = \{P_\nu \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i\},$$

with the convention that $\Omega_r^i(X) = \emptyset$ if $r > n = b_1(X)$. For a fixed $r > 0$, get filtration

$$\text{Gr}_r(\mathbb{Q}^n) = \Omega_r^0(X) \supseteq \Omega_r^1(X) \supseteq \Omega_r^2(X) \supseteq \cdots$$

The Ω -sets are homotopy-type invariants of X :

Lemma

Suppose $X \simeq Y$. For each $r > 0$, there is an isomorphism $\text{Gr}_r(H^1(Y, \mathbb{Q})) \cong \text{Gr}_r(H^1(X, \mathbb{Q}))$ sending each subset $\Omega_r^i(Y)$ bijectively onto $\Omega_r^i(X)$.

Thus, we may extend the definition of the Ω -sets from spaces to groups, by setting

$$\Omega_r^i(G) = \Omega_r^i(K(G, 1))$$

Similarly for the sets $\Omega_A^i(X)$.

- Especially manageable situation: $r = n$, where $n = b_1(X) > 0$.
- In this case, $\text{Gr}_n(H^1(X, \mathbb{Q})) = \{\text{pt}\}$.
- This single point corresponds to the maximal free abelian cover, $X^\alpha \rightarrow X$, where $\alpha: G \twoheadrightarrow \overline{G_{\text{ab}}} = \mathbb{Z}^n$.
- The sets $\Omega_n^i(X)$ are then given by

$$\Omega_n^i(X) = \begin{cases} \{\text{pt}\} & \text{if } b_j(X^\alpha) < \infty \text{ for } j \leq i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example

Let $X = S^1 \vee S^k$, for some $k > 1$. Then $X^\alpha \simeq \bigvee_{j \in \mathbb{Z}} S_j^k$. Thus,

$$\Omega_n^i(X) = \begin{cases} \{\text{pt}\} & \text{for } i < k, \\ \emptyset & \text{for } i \geq k. \end{cases}$$

Comparison diagram

- There is an commutative diagram,

$$\begin{array}{ccc}
 \Omega_A^i(X) & \hookrightarrow & \text{Epi}(G, A) / \text{Aut } A \cong \text{GL}_n(\mathbb{Z}) \times_{\text{P}} \Gamma \\
 \downarrow & & \downarrow \\
 \Omega_r^i(X) & \hookrightarrow & \text{Gr}_r(\mathbb{Q}^n)
 \end{array}$$

- If $\Omega_r^i(X) = \emptyset$, then $\Omega_A^i(X) = \emptyset$.
- The above is a pull-back diagram if and only if:

If X^ν is a \mathbb{Z}^r -cover with finite Betti numbers up to degree i , then any regular $\text{Tors}(A)$ -cover of X^ν has the same finiteness property.

Example

Let $X = S^1 \vee \mathbb{R}P^2$. Then $G = \mathbb{Z} * \mathbb{Z}_2$, $G_{\text{ab}} = \mathbb{Z} \oplus \mathbb{Z}_2$, and

$$X^\alpha \simeq \bigvee_{j \in \mathbb{Z}} \mathbb{R}P_j^2,$$

$$X^{\text{ab}} \simeq \bigvee_{j \in \mathbb{Z}} S_j^1 \vee \bigvee_{j \in \mathbb{Z}} S_j^2.$$

Thus, $b_1(X^\alpha) = 0$, yet $b_1(X^{\text{ab}}) = \infty$.

Hence, $\Omega_1^1(X) \neq \emptyset$, but $\Omega_{\mathbb{Z} \oplus \mathbb{Z}_2}^1(X) = \emptyset$.

Remark

Finiteness of the Betti numbers of a free abelian cover X^ν does not imply finite-generation of the integral homology groups of X^ν .

E.g., let K be a knot in S^3 , with complement $X = S^3 \setminus K$, infinite cyclic cover X^{ab} , and Alexander polynomial $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$. Then

$$H_1(X^{\text{ab}}, \mathbb{Z}) = \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

Hence, $H_1(X^{\text{ab}}, \mathbb{Q}) = \mathbb{Q}^d$, where $d = \deg \Delta_K$. Thus,

$$\Omega_1^1(X) = \{\text{pt}\}.$$

But, if Δ_K is not monic, $H_1(X^{\text{ab}}, \mathbb{Z})$ need not be finitely generated.

Example (Milnor 1968)

Let K be the 5_2 knot, with Alex polynomial $\Delta_K = 2t^2 - 3t + 2$. Then $H_1(X^{\text{ab}}, \mathbb{Z}) = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ is not f.g., though $H_1(X^{\text{ab}}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$.

Characteristic varieties

- Consider the group of complex-valued characters of G ,

$$\widehat{G} = \text{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$$

- Let $G_{\text{ab}} = G/G' \cong H_1(X, \mathbb{Z})$ be the abelianization of G . The projection $\text{ab}: G \rightarrow G_{\text{ab}}$ induces an isomorphism $\widehat{G}_{\text{ab}} \xrightarrow{\cong} \widehat{G}$.
- The identity component, \widehat{G}^0 , is isomorphic to a complex algebraic torus of dimension $n = \text{rank } G_{\text{ab}}$.
- The other connected components are all isomorphic to $\widehat{G}^0 = (\mathbb{C}^\times)^n$, and are indexed by the finite abelian group $\text{Tors}(G_{\text{ab}})$.
- \widehat{G} parametrizes rank 1 local systems on X :

$$\rho: G \rightarrow \mathbb{C}^\times \rightsquigarrow \mathcal{L}_\rho$$

the complex vector space \mathbb{C} , viewed as a right module over the group ring $\mathbb{Z}G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$.

The homology groups of X with coefficients in \mathcal{L}_ρ are defined as

$$H_*(X, \mathcal{L}_\rho) = H_*(\mathcal{L}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\tilde{X}, \mathbb{Z})),$$

where $C_\bullet(\tilde{X}, \mathbb{Z})$ is the equivariant chain complex of the universal cover of X .

Definition

The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \hat{G} \mid H_j(X, \mathcal{L}_\rho) \neq 0, \text{ for some } j \leq i\}$$

- Get filtration $\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \dots \subseteq \hat{G}$.
- Each $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group \hat{G} .
- The characteristic varieties are homotopy-type invariants:
Suppose $X \simeq X'$. There is then an isomorphism $\hat{G}' \cong \hat{G}$, which restricts to isomorphisms $\mathcal{V}^i(X') \cong \mathcal{V}^i(X)$.

The characteristic varieties may be reinterpreted as the support varieties for the Alexander invariants of X .

- Let $X^{\text{ab}} \rightarrow X$ be the maximal abelian cover. View $H_*(X^{\text{ab}}, \mathbb{C})$ as a module over $\mathbb{C}[G_{\text{ab}}]$. Then (Papadima–S. 2010),

$$\mathcal{V}^i(X) = V\left(\text{ann}\left(\bigoplus_{j \leq i} H_j(X^{\text{ab}}, \mathbb{C})\right)\right).$$

- Set $\mathcal{W}^i(X) = \mathcal{V}^i(X) \cap \widehat{G}^0$. View $H_*(X^\alpha, \mathbb{C})$ as a module over $\mathbb{C}[G_\alpha] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $n = b_1(G)$. Then

$$\mathcal{W}^i(X) = V\left(\text{ann}\left(\bigoplus_{j \leq i} H_j(X^\alpha, \mathbb{C})\right)\right).$$

Example

Let $L = (L_1, \dots, L_n)$ be a link in S^3 , with complement $X = S^3 \setminus \bigcup_{i=1}^n L_i$ and Alexander polynomial $\Delta_L = \Delta_L(t_1, \dots, t_n)$. Then

$$\mathcal{V}^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\}.$$

Computing the Ω -invariants

- Given an epimorphism $\nu: G \twoheadrightarrow \mathbb{Z}^r$, let

$$\hat{\nu}: \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{G}, \quad \hat{\nu}(\rho)(g) = \nu(\rho(g))$$

be the induced monomorphism between character groups.

- Its image, $\mathbb{T}_\nu = \hat{\nu}(\widehat{\mathbb{Z}^r})$, is a complex algebraic subtorus of \widehat{G} , isomorphic to $(\mathbb{C}^\times)^r$.

Theorem (Dwyer–Fried 1987, Papadima–S. 2010)

Let X be a connected CW-complex with finite k -skeleton, $G = \pi_1(X)$. For an epimorphism $\nu: G \twoheadrightarrow \mathbb{Z}^r$, the following are equivalent:

- The vector space $\bigoplus_{i=0}^k H_i(X^\nu, \mathbb{C})$ is finite-dimensional.
- The algebraic torus \mathbb{T}_ν intersects the variety $\mathcal{W}^k(X)$ in only finitely many points.

Let $\exp: H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}^\times)$ be the coefficient homomorphism induced by the homomorphism $\mathbb{C} \rightarrow \mathbb{C}^\times, z \mapsto e^z$.

Lemma

Let $\nu: G \rightarrow \mathbb{Z}^r$ be an epimorphism. Under the universal coefficient isomorphism $H^1(X, \mathbb{C}^\times) \cong \text{Hom}(G, \mathbb{C}^\times)$, the complex r -torus $\exp(P_\nu \otimes \mathbb{C})$ corresponds to $\mathbb{T}_\nu = \hat{\nu}(\widehat{\mathbb{Z}^r})$.

Thus, we may reinterpret the Ω -invariants, as follows:

Theorem

$$\Omega_r^i(X) = \{P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid \dim(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)) = 0\}.$$

More generally,

Theorem (S.–Yang–Zhao 2011)

$$\Omega_A^i(X) = \{ [\nu] \in \text{Epi}(H, A) / \text{Aut}(A) \mid \text{im}(\hat{\nu}) \cap \mathcal{V}^i(X) \text{ is finite} \}.$$

Corollary

Suppose $\mathcal{W}^i(X)$ is finite. Then $\Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q}))$, $\forall r \leq b_1(X)$.

Example

Let M be a nilmanifold. By (Macinic–Papadima 2009): $\mathcal{W}^i(M) = \{1\}$, for all $i \geq 0$. Hence,

$$\Omega_r^i(M) = \text{Gr}_r(\mathbb{Q}^n), \quad \forall i \geq 0, r \leq n = b_1(M).$$

Example

Let X be the complement of a knot in S^m , $m \geq 3$. Then

$$\Omega_1^i(X) = \{\text{pt}\}, \quad \forall i \geq 0.$$

Corollary

Let $n = b_1(X)$. Suppose $\mathcal{W}^i(X)$ is infinite, for some $i > 0$. Then $\Omega_n^q(X) = \emptyset$, for all $q \geq i$.

Example

Let S_g be a Riemann surface of genus $g > 1$. Then

$$\Omega_r^i(S_g) = \emptyset, \quad \text{for all } i, r \geq 1$$

$$\Omega_r^n(S_{g_1} \times \cdots \times S_{g_n}) = \emptyset, \quad \text{for all } r \geq 1$$

Example

Let $Y_m = \bigvee^m S^1$ be a wedge of m circles, $m > 1$. Then

$$\Omega_r^i(Y_m) = \emptyset, \quad \text{for all } i, r \geq 1$$

$$\Omega_r^n(Y_{m_1} \times \cdots \times Y_{m_n}) = \emptyset, \quad \text{for all } r \geq 1$$

Tangent cones

Let $W = V(I)$ be a Zariski closed subset in $(\mathbb{C}^\times)^n$.

Definition

- The *tangent cone* at $\mathbf{1}$ to W :

$$TC_1(W) = V(\text{in}(I))$$

- The *exponential tangent cone* at $\mathbf{1}$ to W :

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \forall \lambda \in \mathbb{C}\}$$

Both types of tangent cones

- are homogeneous subvarieties of \mathbb{C}^n ;
- are non-empty iff $1 \in W$;
- depend only on the analytic germ of W at 1 ;
- commute with finite unions.

Moreover,

- τ_1 commutes with (arbitrary) intersections;
- $\tau_1(W) \subseteq TC_1(W)$
 - ▶ = if all irred components of W are subtori
 - ▶ \neq in general
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces of \mathbb{C}^n .

Characteristic subspace arrangements

Set $n = b_1(X)$, and identify $H^1(X, \mathbb{C}) = \mathbb{C}^n$ and $H^1(X, \mathbb{C}^\times)^0 = (\mathbb{C}^\times)^n$.

Definition

The i -th characteristic arrangement of X , denoted $\mathcal{C}_i(X)$, is the subspace arrangement in $H^1(X, \mathbb{Q})$ whose complexified union is the exponential tangent cone to $\mathcal{W}^i(X)$:

$$\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}.$$

- We get a sequence $\mathcal{C}_0(X), \mathcal{C}_1(X), \dots$ of rational subspace arrangements, all lying in $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$.
- The arrangements $\mathcal{C}_i(X)$ depend only on the homotopy type of X .

Theorem

$$\Omega_r^i(X) \subseteq \left(\bigcup_{L \in \mathcal{C}_i(X)} \{P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid P \cap L \neq \{0\}\} \right)^c.$$

Proof.

Fix an r -plane $P \in \text{Gr}_r(H^1(X, \mathbb{Q}))$, and let $T = \exp(P \otimes \mathbb{C})$. Then:

$$\begin{aligned} P \in \Omega_r^i(X) &\iff T \cap \mathcal{W}^i(X) \text{ is finite} \\ &\implies \tau_1(T \cap \mathcal{W}^i(X)) = \{0\} \\ &\iff (P \otimes \mathbb{C}) \cap \tau_1(\mathcal{W}^i(X)) = \{0\} \\ &\iff P \cap L = \{0\}, \text{ for each } L \in \mathcal{C}_i(X), \end{aligned}$$

□

- For “straight” spaces, the inclusion holds as an equality.
- If $r = 1$, the inclusion always holds as an equality.
- In general, though, the inclusion is strict. E.g., there exist finitely presented groups G for which $\Omega_2^1(G)$ is *not* open.

Example

Let $G = \langle x_1, x_2, x_3 \mid [x_1^2, x_2], [x_1, x_3], x_1[x_2, x_3]x_1^{-1}[x_2, x_3] \rangle$. Then $G_{\text{ab}} = \mathbb{Z}^3$, and

$$\mathcal{V}^1(G) = \{1\} \cup \{t \in (\mathbb{C}^\times)^3 \mid t_1 = -1\}.$$

Let $T = (\mathbb{C}^\times)^2$ be an algebraic 2-torus in $(\mathbb{C}^\times)^3$. Then

$$T \cap \mathcal{V}^1(G) = \begin{cases} \{1\} & \text{if } T = \{t_1 = 1\} \\ \mathbb{C}^\times & \text{otherwise} \end{cases}$$

Thus, $\Omega_2^1(G)$ consists of a single point in $\text{Gr}_2(H^1(G, \mathbb{Q})) = \mathbb{Q}\mathbb{P}^2$, and so it's not open.

Example

- Let C_1 be a curve of genus 2 with an elliptic involution σ_1 .
- Let C_2 be a curve of genus 3 with a free involution σ_2 .

Then

- $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1.
- $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2.

We let \mathbb{Z}_2 act freely on the product $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$. The quotient space,

$$M = (C_1 \times C_2)/\mathbb{Z}_2,$$

is a smooth, minimal, complex projective surface of general type with $p_g(M) = q(M) = 3$, and $K_M^2 = 8$. The projection $\text{pr}_2: C_1 \times C_2 \rightarrow C_2$ induces a smooth fibration,

$$C_1 \rightarrow M \rightarrow \Sigma_2.$$

Example (Continued)

Let $\pi = \pi_1(M)$. Then $\pi_{\text{ab}} = \mathbb{Z}^6$, $\hat{\pi} = (\mathbb{C}^\times)^6$, and

$$\mathcal{V}^1(\pi) = \{t \mid t_1 = t_2 = 1\} \cup \{t_4 = t_5 = t_6 = 1, t_3 = -1\}.$$

It follows that $\Omega_2^1(\pi)$ consists of a single point in $\text{Gr}_2(\mathbb{Q}^6)$, corresponding to the plane spanned by the vectors e_1 and e_2 .

In particular, $\Omega_2^1(\pi)$ is not open.

Special Schubert varieties

- Let V be a homogeneous variety in \mathbb{k}^n . The set

$$\sigma_r(V) = \{P \in \text{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\}\}$$

is a Zariski closed subset of $\text{Gr}_r(\mathbb{k}^n)$, called the *variety of incident r -planes* to V .

- When V is a linear subspace $L \subset \mathbb{k}^n$, the variety $\sigma_r(L)$ is called the *special Schubert variety* defined by L .
- If L has codimension d in \mathbb{k}^n , then $\sigma_r(L)$ has codimension $d - r + 1$ in $\text{Gr}_r(\mathbb{k}^n)$.

Example

The Grassmannian $\text{Gr}_2(\mathbb{k}^4)$ is the hypersurface in $\mathbb{P}(\mathbb{k}^6)$ with equation $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$. Let L be a plane in \mathbb{k}^4 , represented as the row space of a 2×4 matrix. Then $\sigma_2(L)$ is the 3-fold in $\text{Gr}_2(\mathbb{k}^4)$ cut out by the hyperplane

$$p_{12}L_{34} - p_{13}L_{24} - p_{23}L_{14} + p_{14}L_{23} - p_{24}L_{13} + p_{34}L_{12} = 0.$$

Theorem

$$\Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \left(\bigcup_{L \in \mathcal{C}_i(X)} \sigma_r(L) \right).$$

Thus, each set $\Omega_r^i(X)$ is contained in the complement of a Zariski closed subset of $\text{Gr}_r(H^1(X, \mathbb{Q}))$: the union of the special Schubert varieties corresponding to the subspaces comprising $\mathcal{C}_i(X)$.

Corollary

Suppose $\mathcal{C}_i(X)$ contains a subspace of codimension d . Then $\Omega_r^i(X) = \emptyset$, for all $r \geq d + 1$.

Corollary

Let X^α be the maximal free abelian cover of X . If $\tau_1(\mathcal{W}^1(X)) \neq \{0\}$, then $b_1(X^\alpha) = \infty$.

The Aomoto complex

Consider the cohomology algebra $A = H^*(X, \mathbb{C})$, with product operation given by the cup product of cohomology classes.

For each $a \in A^1$, we have $a^2 = 0$, by graded-commutativity of the cup product.

Definition

The *Aomoto complex* of A (with respect to $a \in A^1$) is the cochain complex of finite-dimensional, complex vector spaces,

$$(A, a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \dots \xrightarrow{a} A^k,$$

with differentials given by left-multiplication by a .

Alternative interpretation: Pick a basis $\{e_1, \dots, e_n\}$ for $A^1 = H^1(X, \mathbb{C})$, and let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for $A_1 = H_1(X, \mathbb{C})$. Identify $\text{Sym}(A_1)$ with $S = \mathbb{C}[x_1, \dots, x_n]$.

Definition

The *universal Aomoto complex* of A is the cochain complex of free S -modules,

$$\cdots \longrightarrow A^i \otimes_{\mathbb{C}} S \xrightarrow{d^i} A^{i+1} \otimes_{\mathbb{C}} S \xrightarrow{d^{i+1}} A^{i+2} \otimes_{\mathbb{C}} S \longrightarrow \cdots,$$

where the differentials are defined by $d^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^i$, and then extended by S -linearity.

Lemma

The evaluation of the universal Aomoto complex at an element $a \in A^1$ coincides with the Aomoto complex (A, a) .

Let X be a connected, finite-type CW-complex.

The CW-structure on X is *minimal* if the number of i -cells of X equals the Betti number $b_i(X)$, for every $i \geq 0$.

Equivalently, all boundary maps in $C_\bullet(X, \mathbb{Z})$ are zero.

Theorem (Papadima–S. 2010)

If X is a minimal CW-complex, the linearization of the cochain complex $C^\bullet(X^{\text{ab}}, \mathbb{C})$ coincides with the universal Aomoto complex of $H^(X, \mathbb{C})$.*

Resonance varieties

Definition

The *resonance varieties* of X are the sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^j(A, \cdot a) \neq 0, \text{ for some } j \leq i\},$$

defined for all integers $0 \leq i \leq k$.

- Get filtration
 $\{0\} = \mathcal{R}^0(X) \subseteq \mathcal{R}^1(X) \subseteq \cdots \subseteq \mathcal{R}^k(X) \subseteq H^1(X, \mathbb{C}) = \mathbb{C}^n$.
- Each $\mathcal{R}^i(X)$ is a homogeneous algebraic subvariety of \mathbb{C}^n .
- These varieties are homotopy-type invariants of X :
 If $X \simeq Y$, there is an isomorphism $H^1(Y, \mathbb{C}) \cong H^1(X, \mathbb{C})$ which restricts to isomorphisms $\mathcal{R}^i(Y) \cong \mathcal{R}^i(X)$, for all $i \geq 0$.
- (Libgober 2002) $\text{TC}_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X)$.

Straight spaces

Let X be a connected CW-complex with finite k -skeleton.

Definition

We say X is k -straight if the following conditions hold, for each $i \leq k$:

- 1 All positive-dimensional components of $\mathcal{W}^i(X)$ are algebraic subtori.
- 2 $\mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.

If X is k -straight for all $k \geq 1$, we say X is a *straight space*.

- The k -straightness property depends only on the homotopy type of a space.
- Hence, we may declare a group G to be k -straight if there is a $K(G, 1)$ which is k -straight; in particular, G must be of type F_k .
- X is 1-straight if and only if $\pi_1(X)$ is 1-straight.

Example

- Let $f \in \mathbb{Z}[t]$ with $f(1) = 0$. Then $X_f = (S^1 \vee S^2) \cup_f e^3$ is minimal.
- $\mathcal{W}^1(X_f) = \{1\}$, $\mathcal{W}^2(X_f) = V(f)$: finite subsets of $H^1(X, \mathbb{C}^\times) = \mathbb{C}^\times$.
- $\mathcal{R}^1(X_f) = \{0\}$, and

$$\mathcal{R}^2(X_f) = \begin{cases} \{0\}, & \text{if } f'(1) \neq 0, \\ \mathbb{C}, & \text{otherwise.} \end{cases}$$

- Therefore, X_f is always 1-straight, but

$$X_f \text{ is 2-straight} \iff f'(1) \neq 0.$$

Proposition

For each $k \geq 2$, there is a minimal CW-complex which has the integral homology of $S^1 \times S^k$ and which is $(k-1)$ -straight, but not k -straight.

Alternate description of straightness:

Proposition

The space X is k -straight if and only if the following equalities hold, for all $i \leq k$:

$$\mathcal{W}^i(X) = \left(\bigcup_{L \in \mathcal{C}_i(X)} \exp(L \otimes \mathbb{C}) \right) \cup Z_i$$

$$\mathcal{R}^i(X) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}$$

for some finite (algebraic) subsets $Z_i \subset H^1(X, \mathbb{C}^\times)^0$.

Corollary

Let X be a k -straight space. Then, for all $i \leq k$,

$$\textcircled{1} \quad \tau_1(\mathcal{W}^i(X)) = \text{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X).$$

$$\textcircled{2} \quad \mathcal{R}^i(X, \mathbb{Q}) = \bigcup_{L \in \mathcal{C}_i(X)} L.$$

In particular, the resonance varieties $\mathcal{R}^i(X)$ are unions of rationally defined subspaces.

Example

Let G be the group with generators x_1, x_2, x_3, x_4 and relators $r_1 = [x_1, x_2]$, $r_2 = [x_1, x_4][x_2^{-2}, x_3]$, $r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Then

$$\mathcal{R}^1(G) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\},$$

which splits into two linear subspaces defined over \mathbb{R} , but not over \mathbb{Q} . Thus, G is not 1-straight.

Ω -invariants of straight spaces

Theorem

Suppose X is k -straight. Then, for all $i \leq k$ and $r \geq 1$,

$$\Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X, \mathbb{Q})).$$

In particular, if all components of $\mathcal{R}^i(X)$ have the same codimension r , then $\Omega_r^i(X)$ is the complement of the Chow divisor of $\mathcal{R}^i(X, \mathbb{Q})$.

Corollary

Let X be k -straight space, with $b_1(X) = n$. Then each set $\Omega_r^i(X)$ is the complement of a finite union of special Schubert varieties in $\text{Gr}_r(\mathbb{Q}^n)$. In particular, $\Omega_r^i(X)$ is a Zariski open set in $\text{Gr}_r(\mathbb{Q}^n)$.

Example

- Let $L = (L_1, L_2)$ be a 2-component link in S^3 , with $\text{lk}(L_1, L_2) = 1$, and Alexander polynomial $\Delta_L(t_1, t_2) = t_1 + t_1^{-1} - 1$.
- Let X be the complement of L . Then $\mathcal{W}^1(X) \subset (\mathbb{C}^\times)^2$ is given by

$$\mathcal{W}^1(X) = \{1\} \cup \{t \mid t_1 = e^{\pi i/3}\} \cup \{t \mid t_1 = e^{-\pi i/3}\}$$

Hence, X is not 1-straight.

- Since $\mathcal{W}^1(X)$ is infinite, we have

$$\Omega_2^1(X) = \emptyset.$$

- On the other hand, \cup_X is non-trivial, and so $\mathcal{R}^1(X, \mathbb{Q}) = \{0\}$. Hence,

$$\sigma_2(\mathcal{R}^1(X, \mathbb{Q}))^{\text{G}} = \{\text{pt}\}.$$