Abelian Galois covers and rank one local systems

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Galois covers

Sample questions:

1. Given a (finite) CW-complex X, how to parametrize the Galois covers of X with fixed deck-transformation group A?
2. Given an infinite Galois A-cover, $Y \to X$, are the Betti numbers of Y finite?
   - If so, how to compute the Betti numbers of Y?
   - Furthermore, do the Galois covers of Y have finite Betti numbers?
3. Do the Galois A-covers that have finite Betti numbers form an open subspace of the parameter space?
4. Given a finite Galois A-cover, $Y \to X$, how to compute the Betti numbers of Y?

- Let X be a connected CW-complex with finite 1-skeleton. We may assume X has a single 0-cell, call it x_0. Set $G = \pi_1(X, x_0)$.
- Any epimorphism $\nu\colon G \twoheadrightarrow A$ gives rise to a (connected) Galois cover, $X^{\nu} \to X$, with group of deck transformations A.
- Moreover, if $\alpha \in \mathrm{Aut}(A)$, then $X^{\alpha\circ\nu} \cong X^{\nu}$ (A-equivariant homeo).
- Conversely, if $p\colon (Y, y_0) \to (X, x_0)$ is a Galois A-cover, we get a short exact sequence

$$1 \longrightarrow \pi_1(Y, y_0) \xrightarrow{p_\sharp} \pi_1(X, x_0) \xrightarrow{\nu} A \longrightarrow 1\ ,$$

  and an A-equivariant homeomorphism $Y \cong X^{\nu}$.
- Thus, the set of Galois A-covers of X can be identified with

$$\mathrm{Epi}(G, A)/\,\mathrm{Aut}(A).$$

Now assume A is a (finitely generated) Abelian group. Then $\mathrm{Hom}(G, A) \longleftrightarrow \mathrm{Hom}(H, A)$, where $H = G_{\mathrm{ab}}$.

Proposition (A.S.–Yang–Zhao)

There is a bijection

$$\mathrm{Epi}(H, A)/\,\mathrm{Aut}(A) \longleftrightarrow \mathrm{GL}_n(\mathbb{Z}) \times_{\mathrm{P}} \Gamma$$

where $n = \operatorname{rank} H$, $r = \operatorname{rank} A$, *and*

- P *is a parabolic subgroup of* $\mathrm{GL}_n(\mathbb{Z})$;
- $\mathrm{GL}_n(\mathbb{Z})/\,\mathrm{P} = \mathrm{Gr}_{n-r}(\mathbb{Z}^n)$;
- $\Gamma = \mathrm{Epi}(\mathbb{Z}^{n-r} \oplus \mathrm{Tors}(H), \mathrm{Tors}(A))/\,\mathrm{Aut}(\mathrm{Tors}(A))$*—a finite set;*
- $\mathrm{GL}_n(\mathbb{Z}) \times_{\mathrm{P}} \Gamma$ *is the twisted product under the diagonal* P*-action.*

- Simplest situation is when $A = \mathbb{Z}^r$.
- All Galois $\mathbb{Z}^r$-covers of X arise as pull-backs of the universal cover of the r-torus:

$$\begin{array}{ccc} X^{\nu} & \longrightarrow & \mathbb{R}^r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & T^r, \end{array}$$

where $f_\sharp \colon \pi_1(X) \to \pi_1(T^r)$ realizes the epimorphism $\nu \colon G \twoheadrightarrow \mathbb{Z}^r$.

- Hence:

$$\begin{array}{ccc} \{\text{Galois } \mathbb{Z}^r\text{-covers of } X\} & \longleftrightarrow & \{r\text{-planes in } H^1(X,\mathbb{Q})\} \\ X^{\nu} \to X & \longleftrightarrow & P_{\nu} \end{array}$$

where $P_{\nu} := \operatorname{im}(\nu^* \colon H^1(\mathbb{Z}^r,\mathbb{Q}) \to H^1(X,\mathbb{Q}))$.

- Thus:

$$\operatorname{Epi}(H,\mathbb{Z}^r)/\operatorname{Aut}(\mathbb{Z}^r) \cong \operatorname{Gr}_{n-r}(\mathbb{Z}^n) \cong \operatorname{Gr}_r(\mathbb{Q}^n).$$

The Dwyer–Fried sets

Moving about the parameter space for A-covers, and recording how the Betti numbers of those covers vary leads to:

Definition

The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_A(X) = \{[\nu] \in \operatorname{Epi}(G, A)/\operatorname{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \leq i\}.$$

where $X^\nu \to X$ is the cover corresponding to $\nu\colon G \twoheadrightarrow A$.

In particular, when $A = \mathbb{Z}^r$,

$$\Omega^i_r(X) = \{P_\nu \in \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i\},$$

with the convention that $\Omega^i_r(X) = \emptyset$ if $r > n = b_1(X)$. For a fixed $r > 0$, get filtration

$$\operatorname{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots .$$

The Ω-sets are homotopy-type invariants: If $X \simeq Y$, then, for each $r > 0$, there is an isomorphism $\mathrm{Gr}_r(H^1(Y,\mathbb{Q})) \cong \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$ sending each subset $\Omega^i_r(Y)$ bijectively onto $\Omega^i_r(X)$.

Thus, we may extend the definition of the Ω-sets from spaces to groups: $\Omega^i_r(G) = \Omega^i_r(K(G,1))$, and similarly for $\Omega^i_A(X)$.

Example

Let $X = S^1 \vee S^k$, for some $k > 1$. Then $X^{\mathrm{ab}} \simeq \bigvee_{j\in\mathbb{Z}} S^k_j$. Thus,

$$\Omega^i_1(X) = \begin{cases} \{\mathrm{pt}\} & \text{for } i < k, \\ \emptyset & \text{for } i \geq k. \end{cases}$$

Comparison diagram

- There is an commutative diagram,

$$\begin{array}{ccc} \Omega^i_A(X) & \hookrightarrow & \operatorname{Epi}(G,A)/\operatorname{Aut} A \cong \operatorname{GL}_n(\mathbb{Z}) \times_{\mathrm{P}} \Gamma \\ \downarrow & & \downarrow \\ \Omega^i_r(X) & \hookrightarrow & \operatorname{Gr}_r(\mathbb{Q}^n) \end{array}$$

- If $\Omega^i_r(X) = \emptyset$, then $\Omega^i_A(X) = \emptyset$.

- The above is a pull-back diagram if and only if:

  If X^ν is a $\mathbb{Z}^r$-cover with finite Betti numbers up to degree i, then any regular $\operatorname{Tors}(A)$-cover of X^ν has the same finiteness property.

Example

Let $X = S^1 \vee \mathbb{RP}^2$. Then $G = \mathbb{Z} * \mathbb{Z}_2$, $G_{\text{ab}} = \mathbb{Z} \oplus \mathbb{Z}_2$, $G_{\text{fab}} = \mathbb{Z}$, and

$$X^{\text{fab}} \simeq \bigvee_{j\in\mathbb{Z}} \mathbb{RP}^2_j, \qquad X^{\text{ab}} \simeq \bigvee_{j\in\mathbb{Z}} S^1_j \vee \bigvee_{j\in\mathbb{Z}} S^2_j.$$

Thus, $b_1(X^{\text{fab}}) = 0$, yet $b_1(X^{\text{ab}}) = \infty$.

Hence, $\Omega^1_1(X) \neq \emptyset$, but $\Omega^1_{\mathbb{Z}\oplus\mathbb{Z}_2}(X) = \emptyset$.

Characteristic varieties

- Group of complex-valued characters of G:

$$\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$$

- Let $G_{\text{ab}} = G/G' \cong H_1(X, \mathbb{Z})$ be the abelianization of G. The map $\text{ab}\colon G \twoheadrightarrow G_{\text{ab}}$ induces an isomorphism $\widehat{G}_{\text{ab}} \xrightarrow{\simeq} \widehat{G}$.
- $\widehat{G}^0 = (\mathbb{C}^\times)^n$, an algebraic torus of dimension $n = \operatorname{rank} G_{\text{ab}}$.
- $\widehat{G} = \coprod_{\operatorname{Tors}(G_{\text{ab}})} (\mathbb{C}^\times)^n$.
- $\widehat{G}$ parametrizes rank 1 local systems on X:

$$\rho\colon G \to \mathbb{C}^\times \quad \rightsquigarrow \quad \mathbb{C}_\rho$$

the complex vector space $\mathbb{C}$, viewed as a right module over the group ring $\mathbb{Z}G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$.

The homology groups of X with coefficients in $\mathbb{C}_\rho$ are defined as

$$H_*(X, \mathbb{C}_\rho) = H_*(\mathbb{C}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X}, \mathbb{Z})),$$

where $C_\bullet(\widetilde{X}, \mathbb{Z})$ is the $\mathbb{Z}G$-equivariant cellular chain complex of the universal cover of X.

Definition

The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_j(X, \mathbb{C}_\rho) \neq 0, \text{ for some } j \leq i\}.$$

- Get filtration $\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \widehat{G}$.
- If X has finite k-skeleton, then $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$, for each $i \leq k$.
- The varieties $\mathcal{V}^i(X)$ are homotopy-type invariants of X.

The characteristic varieties may be reinterpreted as the support varieties for the Alexander invariants of X.

- Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}}, \mathbb{C})$ as a module over $\mathbb{C}[G_{\mathrm{ab}}]$. Then
$$\mathcal{V}^i(X) = V\Big(\operatorname{ann}\Big(\bigoplus_{j\le i} H_j(X^{\mathrm{ab}}, \mathbb{C})\Big)\Big).$$
- Let $X^{\mathrm{fab}} \to X$ be the max free abelian cover. View $H_*(X^{\mathrm{fab}}, \mathbb{C})$ as a module over $\mathbb{C}[G_{\mathrm{fab}}] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $n = b_1(G)$. Then
$$\mathcal{W}^i(X) := \mathcal{V}^i(X) \cap \widehat{G}^0 = V\Big(\operatorname{ann}\Big(\bigoplus_{j\le i} H_j(X^{\mathrm{fab}}, \mathbb{C})\Big)\Big).$$

Example

Let $L = (L_1, \dots, L_n)$ be a link in S^3, with complement $X = S^3 \setminus \bigcup_{i=1}^n L_i$ and Alexander polynomial $\Delta_L = \Delta_L(t_1, \dots, t_n)$. Then
$$\mathcal{V}^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\}.$$

The characteristic varieties

$$\mathcal{V}^i_j(X,\mathbb{k}) = \{\rho \in \mathrm{Hom}(\pi_1(X),\mathbb{k}^\times) \mid \dim_{\mathbb{k}} H_i(X,\mathbb{k}_\rho) \geq j\}$$

can be used to compute the homology of finite abelian Galois covers (work of A. Libgober, E. Hironaka, P. Sarnak–S. Adams, M. Sakuma, D. Matei–A. S. from the 1990s). E.g.:

Theorem (Matei–A.S. 2002)

Let $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}_n$*. Suppose* $\bar{\mathbb{k}} = \mathbb{k}$ *and* $\mathrm{char}\,\mathbb{k} \nmid n$*, so that* $\mathbb{Z}_n \subset \mathbb{k}^\times$*. Then:*

$$\dim_{\mathbb{k}} H_1(X^\nu,\mathbb{k}) = \dim_{\mathbb{k}} H_1(X,\mathbb{k}) + \sum_{1\neq k\mid n} \varphi(k)\cdot \mathrm{depth}_{\mathbb{k}}(\nu^{n/k}),$$

where $\mathrm{depth}_{\mathbb{k}}(\rho) = \max\{j \mid \rho \in \mathcal{V}^1_j(X,\mathbb{k})\}$*.*

Computing the Ω-invariants

Theorem (Dwyer–Fried 1987, Papadima–S. 2010)

Let X be a connected CW-complex with finite k-skeleton. For an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$, the following are equivalent:

1. *The vector space $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ is finite-dimensional.*
2. *The algebraic torus $\mathbb{T}_\nu = \operatorname{im}\left(\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{\pi_1(X)}\right)$ intersects the variety $\mathcal{W}^k(X)$ in only finitely many points.*

Let $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^\times)$ be the coefficient homomorphism induced by the homomorphism $\mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^z$.

Under the isomorphism $H^1(X, \mathbb{C}^\times) \cong \widehat{\pi_1(X)}$, we have

$$\exp(P_\nu \otimes \mathbb{C}) = \mathbb{T}_\nu.$$

Thus, we may reinterpret the Ω-invariants, as follows:

Corollary

$$\Omega^i_r(X) = \{P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid \dim\left(\exp(P\otimes\mathbb{C})\cap \mathcal{W}^i(X)\right) = 0\}.$$

More generally, for any abelian group A:

Theorem ([?])

$$\Omega^i_A(X) = \{\, [\nu] \in \mathrm{Epi}(H,A)/\,\mathrm{Aut}(A) \mid \mathrm{im}(\hat{\nu}) \cap \mathcal{V}^i(X) \textit{ is finite}\,\}.$$

Characteristic subspace arrangements

Set $n = b_1(X)$, and identify $H^1(X, \mathbb{C}) = \mathbb{C}^n$ and $H^1(X, \mathbb{C}^\times)^0 = (\mathbb{C}^\times)^n$.

Given a Zariski closed subset $W \subset (\mathbb{C}^\times)^n$, define the *exponential tangent cone* at 1 to W as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C}\}.$$

Lemma (Dimca–Papadima–A.S. 2009)

$\tau_1(W)$ *is a finite union of rationally defined linear subspaces of* $\mathbb{C}^n$.

The *i-th characteristic arrangement* of X, is the subspace arrangement $\mathcal{C}_i(X)$ in $H^1(X, \mathbb{Q})$ defined as:

$$\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}.$$

Theorem

$$\Omega^i_r(X) \subseteq \Big(\bigcup_{L \in \mathcal{C}_i(X)} \{ P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid P \cap L \neq \{0\} \} \Big)^{\complement}.$$

Proof.

Fix an r-plane $P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$, and let $T = \exp(P \otimes \mathbb{C})$. Then:

$$\begin{aligned} P \in \Omega^i_r(X) &\iff T \cap \mathcal{W}^i(X) \text{ is finite} \\ &\implies \tau_1(T \cap \mathcal{W}^i(X)) = \{0\} \\ &\iff (P \otimes \mathbb{C}) \cap \tau_1(\mathcal{W}^i(X)) = \{0\} \\ &\iff P \cap L = \{0\}, \text{ for each } L \in \mathcal{C}_i(X), \end{aligned}$$

□

- For "straight" spaces, the inclusion holds as an equality.
- If $r = 1$, the inclusion always holds as an equality.
- In general, though, the inclusion is strict. E.g., there exist finitely presented groups G for which $\Omega^1_2(G)$ is *not* open.

Example

Let $G = \langle x_1, x_2, x_3 \mid [x_1^2, x_2],\ [x_1, x_3],\ x_1[x_2, x_3]x_1^{-1}[x_2, x_3]\rangle$. Then $G_{\text{ab}} = \mathbb{Z}^3$, and

$$\mathcal{V}^1(G) = \{1\} \cup \big\{t \in (\mathbb{C}^\times)^3 \mid t_1 = -1\big\}.$$

Let $T = (\mathbb{C}^\times)^2$ be an algebraic 2-torus in $(\mathbb{C}^\times)^3$. Then

$$T \cap \mathcal{V}^1(G) = \begin{cases} \{1\} & \text{if } T = \{t_1 = 1\} \\ \mathbb{C}^\times & \text{otherwise} \end{cases}$$

Thus, $\Omega^1_2(G)$ consists of a single point in $\operatorname{Gr}_2(H^1(G, \mathbb{Q})) = \mathbb{QP}^2$, and so it's not open.

Special Schubert varieties

- Let V be a homogeneous variety in $\Bbbk^n$. The set $\sigma_r(V) = \{P \in \mathrm{Gr}_r(\Bbbk^n) \mid P \cap V \neq \{0\}\}$ is Zariski closed.
- If $L \subset \Bbbk^n$ is a linear subspace, $\sigma_r(L)$ is the *special Schubert variety* defined by L. If $\operatorname{codim} L = d$, then $\operatorname{codim} \sigma_r(L) = d - r + 1$.

Theorem

$$\Omega^i_r(X) \subseteq \mathrm{Gr}_r\left(H^1(X,\mathbb{Q})\right) \setminus \left(\bigcup_{L \in \mathcal{C}_i(X)} \sigma_r(L)\right).$$

Thus, each set $\Omega^i_r(X)$ is contained in the complement of a Zariski closed subset of $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$: the union of the special Schubert varieties corresponding to the subspaces comprising $\mathcal{C}_i(X)$.

Corollary

1. *If* $\operatorname{codim} \mathcal{C}_i(X) \geq d$*, then* $\Omega^i_r(X) = \emptyset$*, for all* $r \geq d+1$.
2. *If* $\tau_1(\mathcal{W}^1(X)) \neq \{0\}$*, then* $b_1(X^{\mathrm{fab}}) = \infty$.

Resonance varieties

Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have $a^2 = 0$. Thus, we get a cochain complex of finite-dimensional, complex vector spaces,

$$(A, a)\colon \ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots .$$

Definition

The *resonance varieties* of X are the sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^j(A, \cdot a) \neq 0, \text{ for some } j \leq i\}.$$

- Get filtration $\mathcal{R}^0(X) \subseteq \mathcal{R}^1(X) \subseteq \cdots \subseteq \mathcal{R}^k(X) \subseteq H^1(X, \mathbb{C}) = \mathbb{C}^n$.
- If X has finite k-skeleton, then $\mathcal{R}^i(X)$ is a homogeneous algebraic subvariety of $\mathbb{C}^n$, for each $i \leq k$
- These varieties are homotopy-type invariants of X.
- $\tau_1(\mathcal{W}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X)$.

Straight spaces

Let X be a connected CW-complex with finite k-skeleton.

Definition

We say X is *k-straight* if the following conditions hold, for each $i \leq k$:

1. All positive-dimensional components of $\mathcal{W}^i(X)$ are algebraic subtori.
2. $\mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.

If X is k-straight for all $k \geq 1$, we say X is a *straight space*.

- The k-straightness property depends only on the homotopy type of a space.
- Hence, we may declare a group G to be k-straight if there is a $K(G,1)$ which is k-straight; in particular, G must be of type F_k.
- X is 1-straight if and only if $\pi_1(X)$ is 1-straight.

Theorem

Let X be a k-straight space. Then, for all $i \leq k$,

1. $\tau_1(\mathcal{W}^i(X)) = \mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.
2. $\mathcal{R}^i(X, \mathbb{Q}) = \bigcup_{L \in \mathcal{C}_i(X)} L$.

In particular, the resonance varieties $\mathcal{R}^i(X)$ are unions of rationally defined subspaces.

Example

Let G be the group with generators x_1, x_2, x_3, x_4 and relators $r_1 = [x_1, x_2]$, $r_2 = [x_1, x_4][x_2^{-2}, x_3]$, $r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Then

$$\mathcal{R}^1(G) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\},$$

which splits into two linear subspaces defined over $\mathbb{R}$, but not over $\mathbb{Q}$. Thus, G is not 1-straight.

Theorem

Suppose X is k-straight. Then, for all $i \leq k$ and $r \geq 1$,

$$\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X, \mathbb{Q})).$$

In other words, each set $\Omega^i_r(X)$ is the complement of a finite union of special Schubert varieties in the rational Grassmannian; in particular, $\Omega^i_r(X)$ is a Zariski open set.

Characteristic varieties

The structure of the characteristic varieties of smooth, complex projective and quasi-projective varieties (and, more generally, Kähler and quasi-Kähler manifolds) was determined by Beauville, Green–Lazarsfeld, Simpson, Campana, and Arapura in the 1990s.

Theorem (Arapura 1997)

Let $X = \overline{X} \setminus D$, where $\overline{X}$ is a compact Kähler manifold and D is a normal-crossings divisor. If either $D = \emptyset$ or $b_1(\overline{X}) = 0$, then each characteristic variety $\mathcal{V}^i(X)$ is a finite union of unitary translates of algebraic subtori of $H^1(X, \mathbb{C}^\times)$.

In degree 1, the condition that $b_1(\overline{X}) = 0$ if $D \neq \emptyset$ may be lifted. Furthermore, each positive-dimensional component of $\mathcal{V}^1(X)$ is of the form $\rho \cdot T$, with T an algebraic subtorus, and ρ a *torsion* character.

Theorem (Dimca–Papadima–A.S. 2009)

Let X be a 1-formal, quasi-Kähler manifold, and let $\{L_\alpha\}$ be the positive-dimensional, irreducible components of $\mathcal{R}^1(X)$. Then:

1. *Each L_α is a linear subspace of $H^1(X,\mathbb{C})$ of dimension at least $2\varepsilon(\alpha)+2$, for some $\varepsilon(\alpha)\in\{0,1\}$.*
2. *The restriction of $\cup\colon H^1(X,\mathbb{C})\wedge H^1(X,\mathbb{C})\to H^2(X,\mathbb{C})$ to $L_\alpha\wedge L_\alpha$ has rank $\varepsilon(\alpha)$.*
3. *If $\alpha\neq\beta$, then $L_\alpha\cap L_\beta=\{0\}$.*

If M is a compact Kähler manifold, then M is formal, and so the theorem applies: each L_α has dimension $2g(\alpha)\geq 4$, and the restriction of the cup-product map to $L_\alpha\wedge L_\alpha$ has rank $\epsilon(\alpha)=1$.

Theorem

Let X be a 1-formal, quasi-Kähler manifold (for instance, a compact Kähler manifold). Then:

1. $\Omega^1_1(X) = \overline{\mathcal{R}}^1(X,\mathbb{Q})^{\complement}$ *and* $\Omega^1_r(X) \subseteq \sigma_r(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$*, for* $r \geq 2$.
2. *If* $\mathcal{W}^1(X)$ *contains no positive-dimensional translated subtori, then* $\Omega^1_r(X) = \sigma_r(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$*, for all* $r \geq 1$.

In general, though, this last inclusion can be strict.

Theorem

Let X be a 1-formal, smooth, quasi-projective variety. Suppose

1. $\mathcal{W}^1(X)$ *has a 1-dimensional component not passing through 1;*
2. $\mathcal{R}^1(X)$ *has no codimension-1 components.*

Then $\Omega^1_2(X)$ *is strictly contained in* $\sigma_2(\mathcal{R}^1(X,\mathbb{Q}))^{\complement}$.

Concrete example: the complement of the "deleted B_3" arrangement.

The Dwyer–Fried sets of a compact Kähler manifold need not be open.

Example

- Let C_1 be a curve of genus 2 with an elliptic involution σ_1. Then $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1.
- Let C_2 be a curve of genus 3 with a free involution σ_2. Then $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2.
- We let $\mathbb{Z}_2$ act freely on the product $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$. The quotient space, M, is a smooth, minimal, complex projective surface of general type with $p_g(M) = q(M) = 3$, $K_M^2 = 8$.
- The group $\pi = \pi_1(M)$ can be computed by method due to I. Bauer, F. Catanese, F. Grunewald. Identifying $\pi_{\text{ab}} = \mathbb{Z}^6$, $\widehat{\pi} = (\mathbb{C}^\times)^6$, get

$$\mathcal{V}^1(\pi) = \{t \mid t_1 = t_2 = 1\} \cup \{t_4 = t_5 = t_6 = 1,\ t_3 = -1\}.$$

- It follows that $\Omega_2^1(\pi)$ is not open.

Proposition ([?])

Suppose $\mathcal{V}^i(X)$ is a union of algebraic subgroups. If X^ν is a free abelian cover with finite Betti numbers up to degree i, then any finite regular abelian cover of X^ν has the same finiteness property.

For general quasi-projective varieties, the conclusion does not hold.

Example

- The Brieskorn 3-manifold $M = \Sigma(3,3,6)$ is the singularity link of a weighted homogeneous polynomial; thus, it has the homotopy type of a smooth (non-formal) quasi-projective variety.
- A shown in [Dimca–Papadima-A.S. 2011], the variety $\mathcal{V}^1(M)$ has 3 positive-dimensional irreducible components, all of dimension 2, none of which passes through the identity.
- It follows that $b_1(\Sigma(3,3,6)^{\mathrm{fab}}) < \infty$, while $b_1(\Sigma(3,3,6)^{\mathrm{ab}}) = \infty$.

Hyperplane arrangements

- Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^d$, defined by a polynomial $f = \prod_{H \in \mathcal{A}} \alpha_H$, with α_H linear forms.
- The complement, $X = X(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H$, is a smooth, quasi-projective variety. It is also a formal space.
- The homology groups $H_*(X, \mathbb{Z})$ are torsion-free.
- The cohomology ring $A = H^*(X, \mathbb{C})$ is the quotient $A = E/I$ of the exterior algebra on n generators, modulo an ideal determined by the intersection lattice $L(\mathcal{A})$.
- The fundamental group $G = \pi_1(X(\mathcal{A}))$ has a presentation associated to a generic plane section, with generators corresponding to the lines, and commutator relators corresponding to the multiple points. In particular, $G_{\text{ab}} = \mathbb{Z}^n$.

- Identify $\widehat{G} = H^1(X, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$ and $H^1(X, \mathbb{C}) = \mathbb{C}^n$.
- Set $\mathcal{V}^i(\mathcal{A}) = \mathcal{V}^i(X)$, etc.
- Tangent cone formula holds:

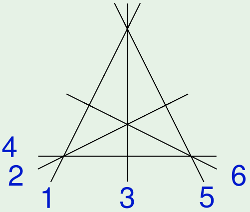
$$\tau_1(\mathcal{V}^i(\mathcal{A})) = \mathrm{TC}_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A}).$$

- Components of $\mathcal{R}^i(\mathcal{A})$ are rationally defined linear subspaces of $\mathbb{C}^n$, depending only on $L(\mathcal{A})$.
- Components of $\mathcal{V}^i(\mathcal{A})$ are subtori of $(\mathbb{C}^\times)^n$, possibly translated by roots of 1.
- Components passing through 1 are combinatorially determined:

$$L \subset \mathcal{R}^i(\mathcal{A}) \rightsquigarrow T = \exp(L) \subset \mathcal{V}^i(\mathcal{A}).$$

- $\mathcal{V}^1(\mathcal{A})$ may contain translated subtori.

Example (Braid arrangement A_3)



$\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from triple points), and one non-local component, from neighborly partition $\Pi = (16|25|34)$:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L_{\Pi} = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

There are no translated components.

Theorem

Suppose $\mathcal{V}^k(\mathcal{A})$ contains no translated components. Then:

1. $X(\mathcal{A})$ *is* k*-straight.*
2. $\Omega_r^k(\mathcal{A}) = \mathrm{Gr}_r(\mathbb{Q}^n) \setminus \sigma_r(\mathcal{R}^k(\mathcal{A}, \mathbb{Q}))$, *for all* $1 \leq r \leq n$.

Proposition

Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, and let m be the maximum multiplicity of its intersection points.

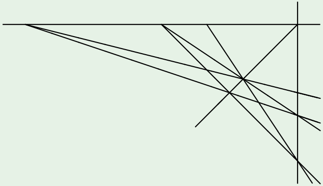
1. *If* $m = 2$, *then* $\Omega_r^1(\mathcal{A}) = \mathrm{Gr}_r(\mathbb{Q}^n)$, *for all* $r \geq 1$.
2. *If* $m \geq 3$, *then* $\Omega_r^1(\mathcal{A}) = \emptyset$, *for all* $r \geq n - m + 2$.

Proposition

Suppose $\mathcal{A}$ has 1 or 2 lines which contain all the intersection points of multiplicity 3 and higher. Then $X(\mathcal{A})$ is 1-straight, and

$$\Omega_r^1(\mathcal{A}) = \sigma_r(\mathcal{R}^1(\mathcal{A}, \mathbb{Q}))^{\complement}.$$

Example (Deleted B_3 arrangement)



Let $\mathcal{A}$ be defined by $f = z_0 z_1 (z_0^2 - z_1^2)(z_0^2 - z_2^2)(z_1^2 - z_2^2)$. Then:

- $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^8$ contains 7 local components (from 6 triple points and 1 quadruple point), and 5 non-local components (from braid sub-arrangements). In particular, $\operatorname{codim} \mathcal{R}^1(\mathcal{A}) = 5$.
- In addition to the corresponding 12 subtori, $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^\times)^8$ also contains $\rho \cdot T$, where $T \cong \mathbb{C}^\times$, and ρ is a root of unity of order 2.
- Thus, the complement X is not 1-straight.
- But X is formal, so $\Omega_2^1(\mathcal{A})$ is strictly contained in $\sigma_2(\mathcal{R}^1(\mathcal{A}))^\complement$.

Milnor fibration

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $\mathbb{C}^d$, defined by a polynomial $f = \alpha_1 \cdots \alpha_n$.
- Milnor fibration: $f \colon \mathbb{C}^d \setminus V(f) \to \mathbb{C} \setminus \{0\}$.
- Milnor fiber: $F = f^{-1}(1)$, a smooth, affine variety, with the homotopy type of a $(d-1)$-dimensional, finite CW-complex (not necessarily formal: H. Zuber 2010).
- F is a Galois, $\mathbb{Z}$-cover of $X = \mathbb{C}^d \setminus V(f)$; it is also a Galois, $\mathbb{Z}_n$-cover of $U = \mathbb{CP}^{d-1} \setminus V(f)$.
- Hence, we may compute $H_1(F, \Bbbk)$ by counting certain torsion points on the varieties $\mathcal{V}^1_j(U, \Bbbk)$, provided $\operatorname{char} \Bbbk \nmid n$.
- Let $s = (s_1, \dots, s_n)$ be positive integers with $\gcd(s) = 1$. The polynomial $f_s = \alpha_1^{s_1} \cdots \alpha_n^{s_n}$ defines a multi-arrangement $\mathcal{A}_s$, with $X(\mathcal{A}_s) = X(\mathcal{A})$, but $F(\mathcal{A}_s) \not\simeq F(\mathcal{A})$, in general.

Question (Dimca–Némethi 2002)

Let $f\colon \mathbb{C}^d \to \mathbb{C}$ be a homogeneous polynomial, $X = \mathbb{C}^d \setminus V(f)$, and $F = f^{-1}(1)$. If $H_*(X, \mathbb{Z})$ is torsion-free, is $H_*(F, \mathbb{Z})$ also torsion-free?

Answer (Cohen–Denham–A.S. 2003, Denham–A.S. 2011)

Not for $H_1(F(\mathcal{A}_s), \mathbb{Z})$, nor for $H_*(F(\mathcal{A}), \mathbb{Z})$.

Example

Take $\mathcal{A}$ to be the deleted B_3 arrangement, with weights $s = (2, 1, 3, 3, 2, 2, 1, 1)$, so that

$$f_s = z_0^2 z_1 (z_0^2 - z_1^2)^3 (z_0^2 - z_2^2)^2 (z_1^2 - z_2^2).$$

Then $\dim_{\Bbbk} H_1(F(\mathcal{A}_s), \Bbbk) = 7$ if $\operatorname{char} \Bbbk \neq 2, 3, 5$, yet $\dim_{\Bbbk} H_1(F(\mathcal{A}_s), \Bbbk) = 9$ if $\operatorname{char} \Bbbk = 2$. In fact:

$$H_1(F(\mathcal{A}_s), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Example

Let $\mathcal{A}$ be the arrangement of 24 hyperplanes in $\mathbb{C}^8$, defined by

$$f = z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - z_3^2)(z_2^2 - z_3^2) y_1 y_2 y_3 y_4 y_5 (z_1 - y_1)(z_1 - y_2) \cdot (z_1^2 - 4y_1^2)(z_1 - y_3)(z_1^2 - y_4^2)(z_1 - 2y_4)(z_1^2 - y_5^2)(z_1 - 2y_5).$$

The 2-torsion part of $H_6(F(\mathcal{A}), \mathbb{Z})$ is $(\mathbb{Z}_2)^{54}$.

Question

Are any of the following determined by the intersection lattice $L(\mathcal{A})$:

1. The translated components in $\mathcal{V}^k(\mathcal{A})$.
2. The Dwyer–Fried sets $\Omega_r^i(\mathcal{A})$.
3. The Betti numbers of $F(\mathcal{A})$.
4. The torsion in $H_*(F(\mathcal{A}), \mathbb{Z})$.

References

- G. Denham and A. Suciu, *Torsion in Milnor fiber homology* II, preprint 2011.
- S. Papadima and A. Suciu, *Bieri–Neumann–Strebel–Renz invariants and homology jumping loci*, Proc. London Math. Soc. **100** (2010), no. 3, 795–834.
- A. Suciu, *Resonance varieties and Dwyer–Fried invariants*, to appear in Advanced Studies Pure Math., Kinokuniya, Tokyo, 2011.
- A. Suciu, *Characteristic varieties and Betti numbers of free abelian covers*, preprint 2011.
- A. Suciu, Y. Yang, and G. Zhao, *Homological finiteness of abelian covers*, preprint 2011.