

On the Johnson filtration of the automorphism group of a free group

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Reference



Stefan Papadima and Alexander I. Suci, *Homological finiteness in the Johnson filtration of the automorphism group of a free group*, [arxiv:1011.5292](https://arxiv.org/abs/1011.5292)

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Filtrations and graded Lie algebras

Let G be a group, with commutator $(x, y) = xyx^{-1}y^{-1}$.
Suppose given a descending filtration

$$G = \Phi^1 \supseteq \Phi^2 \supseteq \dots \supseteq \Phi^s \supseteq \dots$$

by subgroups of G , satisfying

$$(\Phi^s, \Phi^t) \subseteq \Phi^{s+t}, \quad \forall s, t \geq 1.$$

Then $\Phi^s \triangleleft G$, and Φ^s/Φ^{s+1} is abelian. Set

$$\mathrm{gr}_\Phi(G) = \bigoplus_{s \geq 1} \Phi^s/\Phi^{s+1}.$$

This is a graded Lie algebra, with bracket $[\cdot, \cdot]: \mathrm{gr}_\Phi^s \times \mathrm{gr}_\Phi^t \rightarrow \mathrm{gr}_\Phi^{s+t}$
induced by the group commutator.

Basic example: the *lower central series*, $\Gamma^s = \Gamma^s(G)$, defined as

$$\Gamma^1 = G, \Gamma^2 = G', \dots, \Gamma^{s+1} = (\Gamma^s, G), \dots$$

Then for any filtration ϕ as above, $\Gamma^s \subseteq \phi^s$; thus, we have a morphism of graded Lie algebras,

$$\iota_\phi: \text{gr}_\Gamma(G) \longrightarrow \text{gr}_\phi(G).$$

Example (P. Hall, E. Witt, W. Magnus)

Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n . Then:

- F_n is residually nilpotent, i.e., $\bigcap_{s \geq 1} \Gamma^s(F_n) = \{1\}$.
- $\text{gr}_\Gamma(F_n)$ is isomorphic to the free Lie algebra $\mathcal{L}_n = \text{Lie}(\mathbb{Z}^n)$.
- $\text{gr}_\Gamma^s(F_n)$ is free abelian, of rank $\frac{1}{s} \sum_{d|s} \mu(d) n^{\frac{s}{d}}$.
- If $n \geq 2$, the center of \mathcal{L}_n is trivial.

Automorphism groups

Let $\text{Aut}(G)$ be the group of all automorphisms $\alpha: G \rightarrow G$, with $\alpha \cdot \beta := \alpha \circ \beta$. The *Johnson filtration*,

$$\text{Aut}(G) = F^0 \supseteq F^1 \supseteq \dots \supseteq F^s \supseteq \dots$$

with terms $F^s = F^s(\text{Aut}(G))$ consisting of those automorphisms which act as the identity on the s -th nilpotent quotient of G :

$$\begin{aligned} F^s &= \ker (\text{Aut}(G) \rightarrow \text{Aut}(G/\Gamma^{s+1})) \\ &= \{ \alpha \in \text{Aut}(G) \mid \alpha(x) \cdot x^{-1} \in \Gamma^{s+1}, \forall x \in G \} \end{aligned}$$

Kaloujnine [1950]: $(F^s, F^t) \subseteq F^{s+t}$.

First term is the *Torelli group*,

$$\mathcal{T}_G = F^1 = \ker (\text{Aut}(G) \rightarrow \text{Aut}(G_{ab})),$$

By construction, $F^1 = \mathcal{T}_G$ is a normal subgroup of $F^0 = \text{Aut}(G)$. The quotient group,

$$\mathcal{A}(G) = F^0/F^1 = \text{im}(\text{Aut}(G) \rightarrow \text{Aut}(G_{\text{ab}}))$$

is the *symmetry group* of \mathcal{T}_G ; it fits into exact sequence

$$1 \longrightarrow \mathcal{T}_G \longrightarrow \text{Aut}(G) \longrightarrow \mathcal{A}(G) \longrightarrow 1 .$$

The Torelli group comes endowed with two filtrations:

- The Johnson filtration $\{F^s(\mathcal{T}_G)\}_{s \geq 1}$, inherited from $\text{Aut}(G)$.
- The lower central series filtration, $\{\Gamma^s(\mathcal{T}_G)\}$.

The respective associated graded Lie algebras, $\text{gr}_F(\mathcal{T}_G)$ and $\text{gr}_\Gamma(\mathcal{T}_G)$, come with natural actions of $\mathcal{A}(G)$, and the morphism

$$\iota_F: \text{gr}_\Gamma(\mathcal{T}_G) \rightarrow \text{gr}_F(\mathcal{T}_G)$$

is $\mathcal{A}(G)$ -equivariant.

Automorphism groups of free groups

- Identify $(F_n)_{ab} = \mathbb{Z}^n$, and $\text{Aut}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Z})$. The homomorphism $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$ is onto. Thus, $\mathcal{A}(F_n) = \text{GL}_n(\mathbb{Z})$.
- Denote the Torelli group by $\text{IA}_n = \mathcal{T}_{F_n}$, and the Johnson-Andreadakis filtration by $J_n^s = F^s(\text{Aut}(F_n))$.
- Magnus [1934]: IA_n is generated by the automorphisms

$$\alpha_{ij} : \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_\ell \mapsto x_\ell \end{cases} \quad \alpha_{ijk} : \begin{cases} x_i \mapsto x_i \cdot (x_j, x_k) \\ x_\ell \mapsto x_\ell \end{cases}$$

with $1 \leq i \neq j \neq k \leq n$.

- Thus, $\text{IA}_1 = \{1\}$ and $\text{IA}_2 = \text{Inn}(F_2) \cong F_2$ are finitely presented.
- Krstić and McCool [1997]: IA_3 is not finitely presentable.
- It is not known whether IA_n admits a finite presentation for $n \geq 4$.

The Johnson homomorphism

Given a graded Lie algebra \mathfrak{g} , let

$$\text{Der}^s(\mathfrak{g}) = \{ \delta: \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^{\bullet+s} \text{ linear} \mid \delta[x, y] = [\delta x, y] + [x, \delta y], \forall x, y \in \mathfrak{g} \}.$$

Then $\text{Der}(\mathfrak{g}) = \bigoplus_{s \geq 1} \text{Der}^s(\mathfrak{g})$ is a graded Lie algebra, with bracket $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$.

Theorem

Given a group G , there is a monomorphism of graded Lie algebras,

$$J: \text{gr}_F(\mathcal{T}_G) \longrightarrow \text{Der}(\text{gr}_\Gamma(G)),$$

given on homogeneous elements $\alpha \in F^s(\mathcal{T}_G)$ and $x \in \Gamma^t(G)$ by

$$J(\bar{\alpha})(\bar{x}) = \overline{\alpha(x) \cdot x^{-1}}.$$

Moreover, J is equivariant with respect to the natural actions of $\mathcal{A}(G)$.

The Johnson homomorphism informs on the Johnson filtration.

Theorem

Let G be a group. For each $q \geq 1$, the following are equivalent:

- 1 $J \circ \iota_F: \text{gr}_\Gamma^s(\mathcal{T}_G) \rightarrow \text{Der}^s(\text{gr}_\Gamma(G))$ is injective, for all $s \leq q$.
- 2 $\Gamma^s(\mathcal{T}_G) = F^s(\mathcal{T}_G)$, for all $s \leq q + 1$.

Proposition

Suppose G is residually nilpotent, $\text{gr}_\Gamma(G)$ is centerless, and $J \circ \iota_F: \text{gr}_\Gamma^1(\mathcal{T}_G) \rightarrow \text{Der}^1(\text{gr}_\Gamma(G))$ is injective. Then $F^2(\mathcal{T}_G) = \mathcal{T}'_G$.

Let $\text{Inn}(G) = \text{im}(\text{Ad}: G \rightarrow \text{Aut}(G))$, where $\text{Ad}_x: G \rightarrow G, y \mapsto xyx^{-1}$. Define the *outer* automorphism group of a group G by

$$1 \longrightarrow \text{Inn}(G) \longrightarrow \text{Aut}(G) \xrightarrow{\pi} \text{Out}(G) \longrightarrow 1 .$$

Obtain:

- Filtration $\{\tilde{F}^s\}_{s \geq 0}$ on $\text{Out}(G)$: $\tilde{F}^s := \pi(F^s)$.
- The *outer Torelli group* of G : subgroup $\tilde{\mathcal{T}}_G = \tilde{F}^1$ of $\text{Out}(G)$
- Exact sequence: $1 \longrightarrow \tilde{\mathcal{T}}_G \longrightarrow \text{Out}(G) \longrightarrow \mathcal{A}(G) \longrightarrow 1 .$

Let \mathfrak{g} be a graded Lie algebra, and $\text{ad}: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$, where $\text{ad}_x: \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]$. Define the Lie algebra of outer derivations of \mathfrak{g} by

$$0 \longrightarrow \text{im}(\text{ad}) \longrightarrow \text{Der}(\mathfrak{g}) \xrightarrow{q} \widetilde{\text{Der}}(\mathfrak{g}) \longrightarrow 0 .$$

Theorem

Suppose $Z(\text{gr}_\Gamma(G)) = 0$. Then the Johnson homomorphism induces an $\mathcal{A}(G)$ -equivariant monomorphism of graded Lie algebras,

$$\tilde{J}: \text{gr}_{\tilde{F}}(\tilde{\mathcal{T}}_G) \longrightarrow \widetilde{\text{Der}}(\text{gr}_\Gamma(G)).$$

To summarize:

$$\begin{array}{ccccc}
 \text{gr}_\Gamma(G) & \xrightarrow{=} & \text{gr}_\Gamma(G) & \xrightarrow{=} & \text{gr}_\Gamma(G) \\
 \downarrow \text{gr}_\Gamma(\text{Ad}) & & \downarrow \overline{\text{Ad}} & & \downarrow \text{ad} \\
 \text{gr}_\Gamma(\mathcal{T}_G) & \xrightarrow{\iota_F} & \text{gr}_F(\mathcal{T}_G) & \xrightarrow{J} & \text{Der}(\text{gr}_\Gamma(G)) \\
 \downarrow \text{gr}_\Gamma(\pi) & & \downarrow \bar{\pi} & & \downarrow q \\
 \text{gr}_\Gamma(\tilde{\mathcal{T}}_G) & \xrightarrow{\iota_{\tilde{F}}} & \text{gr}_{\tilde{F}}(\tilde{\mathcal{T}}_G) & \xrightarrow{\tilde{J}} & \widetilde{\text{Der}}(\text{gr}_\Gamma(G)),
 \end{array}$$

The Torelli group of F_n

Let $\mathcal{T}_{F_n} = J_n^1 = IA_n$ be the Torelli group of F_n . Recall we have an equivariant $GL_n(\mathbb{Z})$ -homomorphism,

$$J: \text{gr}_F(IA_n) \rightarrow \text{Der}(\mathcal{L}_n),$$

In degree 1, this can be written as

$$J: \text{gr}_F^1(IA_n) \rightarrow H^* \otimes (H \wedge H),$$

where $H = (F_n)_{\text{ab}} = \mathbb{Z}^n$, viewed as a $GL_n(\mathbb{Z})$ -module via the defining representation. Composing with ι_F , we get a homomorphism

$$J \circ \iota_F: (IA_n)_{\text{ab}} \longrightarrow H^* \otimes (H \wedge H).$$

Theorem (Andreadakis, Cohen–Pakianathan, Farb, Kawazumi)

For each $n \geq 3$, the map $J \circ \iota_F$ is a $GL_n(\mathbb{Z})$ -equivariant isomorphism.

Thus, $H_1(IA_n, \mathbb{Z})$ is free abelian, of rank $b_1(IA_n) = n^2(n-1)/2$.

How about the homology groups of the terms deeper in the Johnson filtration of $\text{Aut}(F_n)$?

Conjecture (F. Cohen, A. Heap, A. Pettet 2010)

If $n \geq 3$, $s \geq 2$, and $1 \leq i \leq n - 2$, the cohomology group $H^i(J_n^s, \mathbb{Z})$ is not finitely generated.

We disprove this conjecture, at least rationally, in the case when $n \geq 5$, $s = 2$, and $i = 1$.

Theorem

If $n \geq 5$, then $\dim_{\mathbb{Q}} H^1(J_n^2, \mathbb{Q}) < \infty$.

To start with, note that $J_n^2 = IA'_n$. Thus, it remains to prove that $b_1(IA'_n) < \infty$, i.e., $(IA'_n/IA''_n) \otimes \mathbb{Q}$ is finite dimensional.

We first work with the “outer” groups,

$$\begin{array}{ccccccc}
 \text{Inn}(F_n) & \xrightarrow{=} & \text{Inn}(F_n) & & & & \\
 \downarrow & & \downarrow & & & & \\
 1 \longrightarrow & \text{IA}_n & \longrightarrow & \text{Aut}(F_n) & \longrightarrow & \text{GL}_n(\mathbb{Z}) & \longrightarrow 1 \\
 \downarrow \pi & & \downarrow \pi & & & \downarrow = & \\
 1 \longrightarrow & \text{OA}_n & \longrightarrow & \text{Out}(F_n) & \longrightarrow & \text{GL}_n(\mathbb{Z}) & \longrightarrow 1
 \end{array}$$

$\text{GL}_n(\mathbb{Z})$ acts on $(\text{OA}_n)_{\text{ab}}$, and the outer Johnson homomorphism defines a $\text{GL}_n(\mathbb{Z})$ -equivariant isomorphism

$$\tilde{J} \circ \iota_{\tilde{F}} : (\text{OA}_n)_{\text{ab}} \xrightarrow{\cong} H^* \otimes (H \wedge H) / H.$$

Moreover, $\tilde{J}_n^2 = \text{OA}'_n$, and we have an exact sequence

$$1 \longrightarrow F'_n \xrightarrow{\text{Ad}} \text{IA}'_n \longrightarrow \text{OA}'_n \longrightarrow 1.$$

The Alexander invariant

Let G be a group. Recall $G' = (G, G)$ and $G_{\text{ab}} = G/G'$ is the maximal abelian quotient of G .

Similarly, $G'' = (G', G')$ and G/G'' is the maximal metabelian quotient.

Get exact sequence $0 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G_{\text{ab}} \longrightarrow 0$.

Conjugation in G/G'' turns the abelian group

$$B(G) := G'/G'' = H_1(G', \mathbb{Z})$$

into a module over $R = \mathbb{Z}G_{\text{ab}}$, called the *Alexander invariant* of G .

Since both G' and G'' are characteristic subgroups of G , the action of $\text{Aut}(G)$ on G induces an action on $B(G)$. Although this action need not respect the R -module structure, we have:

Proposition

The Torelli group \mathcal{T}_G acts R -linearly on the Alexander invariant $B(G)$.

Characteristic varieties

Let G be a finitely generated group.

- The *character group* $\widehat{G} = \text{Hom}(G, \mathbb{C}^\times)$ is an algebraic group.
- The projection $\text{ab}: G \rightarrow G_{\text{ab}}$ induces an isomorphism $\widehat{G}_{\text{ab}} \xrightarrow{\cong} \widehat{G}$.
- The identity component, \widehat{G}^0 , is isomorphic to a complex algebraic torus of dimension $n = \text{rank } G_{\text{ab}}$.
- The coordinate ring of $\widehat{G} = H^1(G, \mathbb{C}^\times)$ is $R_{\mathbb{C}} = \mathbb{C}[G_{\text{ab}}]$.
- The (first) *characteristic variety* of G is the support of the Alexander invariant: $\mathcal{V}(G) = V(\text{ann } B) \cup \{1\} \subset \widehat{G}$.
- $\mathcal{V}(G)$ finite $\iff \dim_{\mathbb{Q}} B(G) \otimes \mathbb{Q} < \infty$.

Example

If $G = \mathbb{Z}^n$, then $B(G) = 0$ and $\mathcal{V}(G) = \{1\} \subset (\mathbb{C}^\times)^n$.

If $G = F_n$, $n \geq 2$, then $\mathcal{V}(G) = (\mathbb{C}^\times)^n$.

Resonance varieties

- Let $\cup: H^1(G, \mathbb{C}) \wedge H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C})$ be the cup-product map.
- The (first) *resonance variety* of G is defined as

$$\mathcal{R}(G) = \{z \in H^1(G, \mathbb{C}) \mid \exists u \in H^1(G, \mathbb{C}), u \neq \lambda z \text{ and } z \cup u = 0\}.$$

- This is a homogeneous algebraic subvariety of $H^1(G, \mathbb{C}) = \mathbb{C}^n$, where $n = b_1(G)$.
- Let $\text{TC}_1(\mathcal{V}(G))$ be the tangent cone to $\mathcal{V}(G)$ at $\mathbf{1}$, viewed as a subset of $T_1(\mathbb{T}(G)) = H^1(G, \mathbb{C})$. Then:

$$\text{TC}_1(\mathcal{V}(G)) \subseteq \mathcal{R}(G).$$

Example

If $G = \mathbb{Z}^n$, then $\mathcal{R}(G) = \{0\}$.

If $G = F_n$, $n \geq 2$, then $\mathcal{R}(G) = \mathbb{C}^n$.

Representations of $\mathfrak{sl}_n(\mathbb{C})$

- \mathfrak{h} : the Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, with coordinates t_1, \dots, t_n .
- $\{t_i - t_j \mid 1 \leq i < j \leq n\}$: the positive roots of $\mathfrak{sl}_n(\mathbb{C})$.
- $\lambda_i = t_1 + \dots + t_i$.
- $V(\lambda)$: the irreducible, finite dimensional representation of $\mathfrak{sl}_n(\mathbb{C})$ with highest weight $\lambda = \sum_{i < n} a_i \lambda_i$, with $a_i \in \mathbb{Z}_{\geq 0}$.

Set $H_{\mathbb{C}} = H_1(F_n, \mathbb{C}) = \mathbb{C}^n$, and

$$V := H^1(\mathrm{OA}_n, \mathbb{C}) = H_{\mathbb{C}} \otimes (H_{\mathbb{C}}^* \wedge H_{\mathbb{C}}^*) / H_{\mathbb{C}}^*.$$

$$K := \ker(\cup: V \wedge V \rightarrow H^2(\mathrm{OA}_n, \mathbb{C})).$$

Theorem (Pettet 2005)

Fix $n \geq 4$, and set $\lambda = \lambda_1 + \lambda_{n-2}$ and $\mu = \lambda_1 + \lambda_{n-2} + \lambda_{n-1}$. Then $V = V(\lambda)$ and $K = V(\mu)$, as $\mathfrak{sl}_n(\mathbb{C})$ -modules.

Theorem

$\mathcal{R}(\mathrm{OA}_n) = \{0\}$, for all $n \geq 4$.

Proof.

- Let $u_0 \in V(\mu)$ be a maximal vector.
- Suppose $\mathcal{R} \neq \{0\}$. Then, since \mathcal{R} is a Zariski closed, $\mathfrak{sl}_n(\mathbb{C})$ -invariant cone in $V(\lambda)$, it must contain a maximal vector $v_0 \in V(\lambda)$. (This follows from the Borel fixed point theorem.)
- Since $v_0 \in \mathcal{R}$, there is a $w \in V(\lambda)$ such that $u_0 = v_0 \wedge w$.
- Let $x \in \mathfrak{sl}_n(\mathbb{C})^+$. Since u_0, v_0 are max vectors, $xu_0 = xv_0 = 0$.
- Since $u_0 = v_0 \wedge w$, we have $xu_0 = xv_0 \wedge w + v_0 \wedge xw$.
- Hence, $v_0 \wedge xw = 0$, and thus $xw \in \mathbb{C} \cdot v_0$.
- This implies $w = 0$, and so $u_0 = v_0 \wedge w = 0$, contradicting the maximality of u_0 .



Let S be a complex, simple linear algebraic group defined over \mathbb{Q} , with \mathbb{Q} -rank(S) ≥ 1 , and let Γ be an arithmetic subgroup of S .

Theorem (Dimca, Papadima 2010)

Suppose Γ acts on a lattice L , such that the action of Γ on $L \otimes \mathbb{C}$ extends to a rational, irreducible S -representation. Then, the corresponding action of Γ on the complex algebraic torus $\widehat{L} = \mathrm{Hom}(L, \mathbb{C}^\times)$ is geometrically irreducible, i.e., the only Γ -invariant, Zariski closed subsets of \widehat{L} are either equal to \widehat{L} , or finite.

Theorem

If $n \geq 4$, then $\mathcal{V}(\mathrm{OA}_n)$ is finite, and so $b_1(\mathrm{OA}'_n) < \infty$.

Proof.

- Set $\mathcal{S} = \mathfrak{sl}_n(\mathbb{C})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $L = (\mathrm{OA}_n)_{\mathrm{ab}}$. By above result: $\widehat{L} = H^1(\mathrm{OA}_n, \mathbb{C}^\times)$ is geometrically Γ -irreducible.
- The variety $\mathcal{V} = \mathcal{V}(\mathrm{OA}_n)$ is a Γ -invariant, Zariski closed subset of \widehat{L} .
- Suppose \mathcal{V} is infinite. Then $\mathcal{V} = \widehat{L}$, and so $\mathcal{R}(\mathrm{OA}_n) = H^1(\mathrm{OA}_n, \mathbb{C})$, contradicting $\mathcal{R} = \{0\}$.



Theorem

If $n \geq 5$, then $b_1(\mathbf{IA}'_n) < \infty$.

Proof.

For each n , the Hochschild-Serre spectral sequence of the extension

$$1 \longrightarrow F'_n \longrightarrow \mathbf{IA}'_n \longrightarrow \mathbf{OA}'_n \longrightarrow 1$$

gives rise to exact sequence

$$H_1(F'_n, \mathbb{C})_{\mathbf{IA}'_n} \longrightarrow H_1(\mathbf{IA}'_n, \mathbb{C}) \longrightarrow H_1(\mathbf{OA}'_n, \mathbb{C}) \longrightarrow 0.$$

The last term is finite-dimensional for all $n \geq 4$ by previous theorem, while the first term is finite-dimensional for all $n \geq 5$, by the nilpotency of the action of \mathbf{IA}'_n on F'_n/F''_n .

