

**Lower central series, free
resolutions, and homotopy Lie
algebras of arrangements**

Alex Suciu

Northeastern University
www.math.neu.edu/~suciu

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References

- H. Schenck, A. Suciu, *Lower central series and free resolutions of hyperplane arrangements*, [arXiv:math.AG/0109070](https://arxiv.org/abs/math/0109070).
- S. Papadima, A. Suciu, *Homotopy Lie algebras, lower central series, and the Koszul property*, [arXiv:math.AT/0110303](https://arxiv.org/abs/math/0110303).

Hyperplane arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a set of hyperplanes in \mathbb{C}^ℓ (passing through 0).

- **Intersection lattice:**

$$L(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\}$$

ranked poset: $\leq \longleftrightarrow \supseteq$, rank \longleftrightarrow codim.

- **Complement:**

$$M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$$

a **formal** space, \simeq to an ℓ -dim CW-complex.

Main topological invariants associated to $M(\mathcal{A})$:

- **Cohomology ring:** $A := H^*(M, \mathbb{Q}) = E/I$
 E = exterior algebra, I = **Orlik-Solomon** ideal
—determined by $L(\mathcal{A})$
- **Fundamental group:** $G = \pi_1(M)$
—not always combinatorially determined, though
its LCS ranks are (since M formal).

Lower central series

Let G be a finitely-generated group. Set:

- **LCS series:** $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_k \triangleright \cdots$
where $G_{k+1} = [G_k, G]$
- **LCS quotients:** $\text{gr}_k G = G_k / G_{k+1}$
- **LCS ranks:** $\phi_k(G) = \text{rank}(\text{gr}_k G)$
- **Chen ranks:** $\theta_k(G) = \text{rank}(\text{gr}_k G / G'')$

Define **associated graded Lie algebra:**

$$L_*(G) := \text{gr}_* G \otimes \mathbb{Q} = \bigoplus_{k \geq 1} G_k / G_{k+1} \otimes \mathbb{Q}$$

with Lie bracket $[\cdot, \cdot]: L_i \times L_j \rightarrow L_{i+j}$ induced by the group commutator.

Problem. Find an *explicit* combinatorial formula for the LCS ranks $\phi_k(G)$ of an arrangement group G (at least for certain classes of arrangements).

Known LCS formulas

- **Witt formula**

$$\mathcal{A} = \{n \text{ points in } \mathbb{C}\}, \quad M(\mathcal{A}) \simeq \bigvee^n S^1$$

$$G = F_n \text{ (free group of rank } n)$$

$$\text{gr } G = \mathbb{L}_n \text{ (free Lie algebra of rank } n)$$

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = 1 - nt$$

$$\text{Hence: } \phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$$

$$\phi_1 = n, \quad \phi_2 = \binom{n}{2}, \quad \phi_3 = 2 \binom{n+1}{3}, \quad \phi_4 = \frac{n^2(n^2-1)}{4}.$$

- **Kohno formula** [1985]

$$\mathcal{A} = \{z_i - z_j = 0\}_{1 \leq i < j \leq \ell} \text{ braid arrangement in } \mathbb{C}^\ell$$

$$G = P_\ell \text{ (pure braid group on } \ell \text{ strings)}$$

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \prod_{j=1}^{\ell-1} (1 - jt)$$

- **Falk-Randell LCS formula** [1985]

\mathcal{A} **fiber-type** $\left(\overset{\text{Terao}}{\iff} L(\mathcal{A}) \text{ supersolvable} \right)$,

with exponents d_1, \dots, d_ℓ . Then:

$$G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$$

$$\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i})$$

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_M(-t)$$

where, for all \mathcal{A} :

$$P_M(t) := \sum_{i=0}^{\ell} b_i t^i, \quad b_i = \sum_{X \in L_i(\mathcal{A})} (-1)^i \mu(X)$$

- **Shelton-Yuzvinsky** [1997], **Papadima-Yuz** [99]

A Koszul algebra \implies LCS formula holds

Remark. (**Peeva**) There are arrangements for which

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k} \neq \text{Hilb}(N, -t),$$

for any graded commutative algebra N .

LCS and free resolutions

(joint work with Hal Schenck)

We want to reduce the problem of computing $\phi_k(G)$ to that of computing the graded Betti numbers of certain free resolutions involving the OS-algebra $A = E/I$.

Starting point:

$$\prod_{k=1}^{\infty} (1 - t^k)^{-\phi_k} = \sum_{i=0}^{\infty} b_{ii} t^i$$

where

$$b_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^A(\mathbb{Q}, \mathbb{Q})_j$$

i^{th} Betti number (in degree j) of a minimal free resolution of \mathbb{Q} over A :

$$\cdots \longrightarrow \bigoplus_j A^{b_{2j}}(-j) \longrightarrow \bigoplus_j A^{b_{1j}}(-j) \longrightarrow A \longrightarrow \mathbb{Q} \longrightarrow 0$$

with Betti diagram:

$$\begin{array}{cccccc} 0 : & 1 & b_1 & b_{22} & b_{33} & \dots & \leftarrow \text{linear strand} \\ 1 : & . & . & b_{23} & b_{34} & \dots & \\ 2 : & . & . & b_{24} & b_{35} & \dots & \end{array}$$

This (known) formula follows from:

- **Sullivan** Since M formal:

$$\text{gr } G \cong \mathfrak{g} := \mathbb{L}(H_1) / \text{im}(\nabla : H_2 \rightarrow H_1 \wedge H_1)$$

Holonomy Lie algebra of $H_* = H_*(M, \mathbb{Q})$

- **Poincaré-Birkhoff-Witt:**

$$\prod_{k=1}^{\infty} (1 - t^k)^{-\dim \mathfrak{g}_k} = \text{Hilb}(U(\mathfrak{g}), t)$$

- **Shelton-Yuzvinsky:**

$$U(\mathfrak{g}) = \overline{A}^!$$

- **Priddy, Löffwall:**

$$\overline{A}^! \cong \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})_i$$

Linear strand in Yoneda Ext-algebra

Here $\overline{A} = E / \langle I_2 \rangle$ is the quadratic closure of A , and $\overline{A}^!$ is its Koszul dual.

Koszul algebras

A^* connected, graded, graded-commutative \mathbb{Q} -algebra.

Definition. A is a **Koszul algebra** if

$$\text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})_j = 0, \quad \text{for } i \neq j.$$

i.e., Betti diagram = linear strand.

- **Necessary:** $A = T/I$, T gen in deg 1, I in deg 2.
- **Sufficient:** I has quadratic Gröbner basis.
- **Koszul duality:** $\text{Hilb}(A^!, t) \cdot \text{Hilb}(A, -t) = 1$
(\rightsquigarrow LCS formula of Shelton-Yuzvinsky.)
- **Arrangement interpretation (Shelton-Yuz.)**

$$\mathcal{A} \text{ fiber-type} \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \text{??} \end{array} H^*(M(\mathcal{A}), \mathbb{Q}) \text{ Koszul}$$

- **Topological interpretation (Pap-Yuz.)**

X finite-type CW, $X_{\mathbb{Q}}$ Bousfield-Kan rationalization.

$$H^*(X, \mathbb{Q}) \text{ Koszul} \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \text{if } X \text{ formal} \end{array} X_{\mathbb{Q}} \text{ aspherical}$$

Resolution of OS-algebra

Problem. (Eisenbud-Popescu-Yuzvinsky [1999])

Compute the (minimal) free resolution of A over E ,

$$\dots \longrightarrow \bigoplus_j E^{b'_{2j}}(-j) \longrightarrow \bigoplus_j E^{b'_{1j}}(-j) \longrightarrow E \longrightarrow A \rightarrow 0$$

and its Betti numbers, $b'_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^E(A, \mathbb{Q})_j$.

Let $a_j = \#\{\text{minimal generators of } I \text{ in degree } j\}$.

Clearly, $a_2 = \binom{b_1}{2} - b_2$.

Lemma. $a_j = b'_{1j} = b_{2j}$.

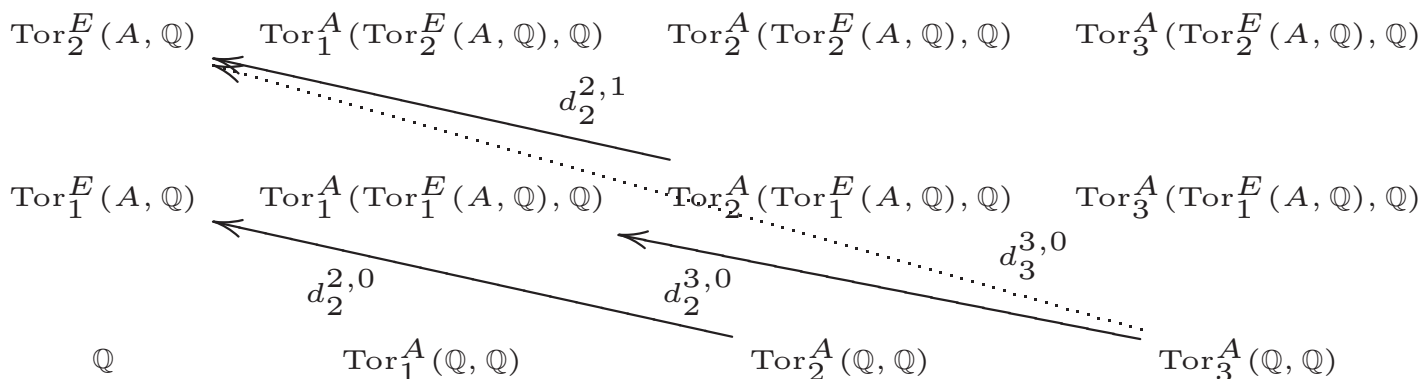
Thus, Betti diagram looks like:

$$\begin{array}{cccccc}
 0 : & 1 & \cdot & \cdot & \cdot & \dots \\
 1 : & \cdot & a_2 & b'_{23} & b'_{34} & \dots \leftarrow \text{linear strand} \\
 2 : & \cdot & a_3 & b'_{24} & b'_{35} & \dots \\
 \vdots & & & & & \\
 \ell - 1 : & \cdot & a_\ell & b'_{2,\ell+1} & b'_{3,\ell+2} & \dots
 \end{array}$$

Change of rings spectral sequence

Key tool (Cartan-Eilenberg):

$$\mathrm{Tor}_i^A(\mathrm{Tor}_j^E(A, \mathbb{Q}), \mathbb{Q}) \implies \mathrm{Tor}_{i+j}^E(\mathbb{Q}, \mathbb{Q})$$



The (Koszul) resolution of \mathbb{Q} as a module over E is linear, with $\dim \mathrm{Tor}_i^E(\mathbb{Q}, \mathbb{Q}) = \binom{n+i-1}{i}$.

Thus, if we know $\mathrm{Tor}_i^E(A, \mathbb{Q})$, we can find $\mathrm{Tor}_i^A(\mathbb{Q}, \mathbb{Q})$, provided we can compute $d_r^{p,q}$.

We carry out part of this program—mostly in low degrees. As a result, we express ϕ_k , $k \leq 4$, solely in terms of the resolution of A over E .

Theorem. For an arrangement of n hyperplanes:

$$\phi_1 = n$$

$$\phi_2 = a_2$$

$$\phi_3 = b'_{23}$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4$$

where

$$a_2 = \#\{\text{generators of } I_2\} = \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)}{2}$$

$$b'_{23} = \#\{\text{linear first syzygies on } I_2\}$$

$$b'_{34} = \#\{\text{linear second syzygies on } I_2\}$$

$$\delta_4 = \#\{\text{minimal, quadratic, Koszul syzygies on } I_2\}$$

ϕ_1, ϕ_2 : elementary

ϕ_3 : recovers a formula of **Falk** [1988]

ϕ_4 : new

Decomposable arrangements

Let \mathcal{A} be an arrangement of n hyperplanes.

Recall:

$$\phi_1 = n, \quad \phi_2 = \sum_{X \in L_2(\mathcal{A})} \phi_2(F_{\mu(X)})$$

Falk [1989]:

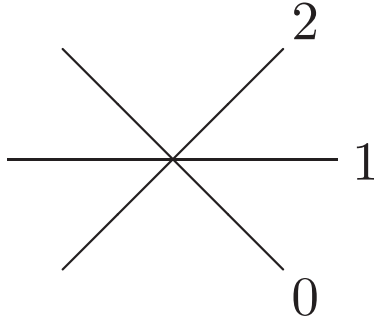
$$\phi_k \geq \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)}) \quad \text{for all } k \geq 3 \quad (*)$$

Definition. If the lower bound is attained for $k = 3$, \mathcal{A} is **decomposable** (or, *minimal linear strand*).

Conjecture (MLS LCS). If \mathcal{A} is decomposable, equality holds in (*), and so

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - t)^n \prod_{X \in L_2(\mathcal{A})} \frac{1 - \mu(X)t}{1 - t}$$

Example. $\mathcal{A} = \{H_0, H_1, H_2\}$ pencil of 3 lines in \mathbb{C}^2 .



$E =$ exterior algebra on e_0, e_1, e_2

$I =$ ideal generated by $\partial e_{012} = (e_1 - e_2) \wedge (e_0 - e_2)$

Minimal free resolution of A over E :

$$\begin{array}{ccccccc}
 0 & \leftarrow & A & \leftarrow & E & \xleftarrow{(\partial e_{012})} & E(-1) & \xleftarrow{(e_1 - e_2 \quad e_0 - e_2)} & E^2(-2) \\
 & & & & & & \left(\begin{array}{ccc} e_1 - e_2 & e_0 - e_2 & 0 \\ 0 & e_1 - e_2 & e_0 - e_2 \end{array} \right) & & \\
 & & & & & & \xleftarrow{\hspace{10em}} & E^3(-3) & \leftarrow \dots
 \end{array}$$

Thus,

$$b'_{i,i+1} = i \quad \text{for } i \geq 1$$

$$b'_{i,i+r} = 0 \quad \text{for } r > 1$$

More generally, if \mathcal{A} decomposable, we compute the entire linear strand of the resolution of A over E .

If moreover $\text{rank } \mathcal{A} = 3$, we compute all b'_{ij} from Möbius function.

Lemma. *For any arrangement \mathcal{A} :*

$$b'_{i,i+1} \geq i \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X) + i - 1}{i + 1}$$

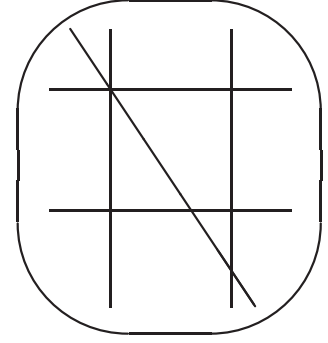
$$\delta_4 \leq \sum_{(X,Y) \in \binom{L_2(\mathcal{A})}{2}} \binom{\mu(X)}{2} \binom{\mu(Y)}{2}.$$

If \mathcal{A} is decomposable, then equalities hold.

Corollary. *The MLS LCS conjecture is true for $k = 4$:*

$$\phi_4 = \frac{1}{4} \sum_{X \in L_2(\mathcal{A})} \mu(X)^2 (\mu(X)^2 - 1)$$

Example. X_3 arrangement



res of residue field over OS alg

total: 1 6 25 92 325 1138

0: 1 6 24 80 240 672

1: . . 1 12 84 448

2: 1 18

res of OS alg over exterior algebra

total: 1 4 15 42 97 195 354 595 942 1422 2065

0: 1

1: . 3 6 9 12 15 18 21 24 27 30

2: . 1 9 33 85 180 336 574 918 1395 2035

We find: $b'_{i,i+1} = 3i$, $b'_{i,i+2} = \frac{1}{8}i(i+1)(i^2 + 5i - 2)$.

Thus:

$$\phi_1 = n = 6$$

$$\phi_2 = a_2 = 3$$

$$\phi_3 = b'_{23} = 6$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4 = 3 + 9 - 3 = 9$$

Conjecture says: $\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - 2t)^3$

i.e.: $\phi_k(G) = \phi_k(F_2^3)$, though definitely $G \not\cong F_2^3$.

Graphic arrangements

$G = (\mathcal{V}, \mathcal{E})$ subgraph of complete graph K_ℓ . Set:

$$\kappa_s(G) = \#\{\text{complete subgraphs on } s + 1 \text{ vertices}\}$$

Graphic arrangement (in \mathbb{C}^{κ_0}) associated to G :

$$\mathcal{A}_G = \{\ker(z_i - z_j) \mid \{i, j\} \in \mathcal{E}\}$$

Theorem. (Stanley, Fulkerson-Gross)

$$\mathcal{A}_G \text{ is supersolvable} \iff G \text{ is chordal}$$

Lemma. (Cordovil-Forge [2001], S-S)

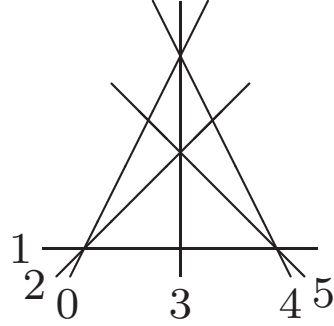
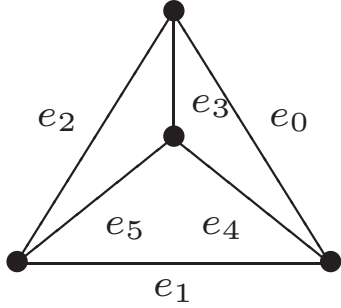
$$a_j = \#\{\text{chordless } j + 1 \text{ cycles}\}$$

Together with a previous lemma ($a_j = b_{2j}$), we get:

Corollary. \mathcal{A}_G supersolvable $\iff \mathcal{A}_G$ Koszul.

Example. Braid arrangement $\mathcal{B}_4 = \mathcal{A}_{K_4}$

$$\kappa_0 = 4, \kappa_1 = 6, \kappa_2 = 4, \kappa_3 = 1$$



Free resolution of A over E :

$$0 \leftarrow A \leftarrow E \xleftarrow{\partial_1} E^4(-2) \xleftarrow{\partial_2} E^{10}(-3) \leftarrow \dots$$

where $\partial_1 = (\partial e_{145} \quad \partial e_{235} \quad \partial e_{034} \quad \partial e_{012})$

and $\partial_2 =$

$$\begin{pmatrix} e_1 - e_4 & e_1 - e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3 - e_0 & e_2 - e_0 \\ 0 & 0 & e_2 - e_3 & e_2 - e_5 & 0 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_4 \\ 0 & 0 & 0 & 0 & e_0 - e_3 & e_0 - e_4 & 0 & 0 & 0 & e_1 - e_5 & e_2 - e_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_2 & e_3 - e_5 & e_4 - e_5 \end{pmatrix}$$

Non-local component in $\mathcal{R}_1(\mathcal{B}_4) \rightsquigarrow 2$ extra syzygies

Get: $b'_{i,i+1} = 5i, \quad \delta_4 = 0.$

Proposition. *For a graphic arrangement:*

$$b'_{i,i+1} = i(\kappa_2 + \kappa_3)$$

$$\delta_4 \leq \binom{\kappa_2}{2} - 6(\kappa_3 + \kappa_4)$$

Corollary. $\phi_1 = \kappa_1$

$$\phi_2 = \kappa_2$$

$$\phi_3 = 2(\kappa_2 + \kappa_3)$$

$$\phi_4 \geq 3(\kappa_2 + 3\kappa_3 + 2\kappa_4)$$

Moreover, if $\kappa_3 = 0$, equality holds for ϕ_4 .

Conjecture (Graphic LCS).

$$\phi_k = \frac{1}{k} \sum_{d|k} \sum_{j=1}^k \sum_{s=j}^k (-1)^{s-j} \binom{s}{j} \kappa_s \mu(d) j^{\frac{k}{d}}$$

or:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \prod_{j=1}^{\kappa_0-1} (1 - jt)^{\sum_{s=j}^{\kappa_0-1} (-1)^{s-j} \binom{s}{j} \kappa_s}$$

Resonance Conjectures

$$\mathcal{A} = \{H_1, \dots, H_n\}, \quad A = H^*(M(\mathcal{A}), \mathbb{C}), \quad G = \pi_1(M)$$

$$\mathcal{R}_1(\mathcal{A}) := \{\lambda \in \mathbb{C}^n \mid \dim_{\mathbb{C}} H^1(A, \cdot \sum \lambda_i e_i) > 0\}$$

Then: $\mathcal{R}_1(\mathcal{A}) = \bigcup_{i=1}^v L_i$, where L_i linear subspaces

Set $h_r = \#\{L_i \mid \dim L_i = r\}$.

■ Chen Ranks & Linear Strand of OS-Resolution

$$\theta_k(G) = b'_{k-1,k} \quad \text{for } k \geq 2.$$

■ Resonance Formula for Chen Ranks

$$\theta_k(G) = \sum_{r \geq 2} h_r \theta_k(F_r) \quad \text{for } k \geq 4.$$

■ Resonance LCS Formula

If $\phi_4 = \theta_4 \longleftrightarrow \delta_4 = \binom{a_2}{2}$, then:

$$\phi_k(G) = \sum_{r \geq 2} h_r \phi_k(F_r) \quad \text{for } k \geq 4.$$

Rational homotopy groups and Koszul algebras

(joint work with Stefan Papadima)

Let $k \geq 1$ be an integer, and

- X connected space
- Y simply-connected space

(all spaces \simeq to a finite-type CW-complex)

Definition. Y is a k -**rescaling** of X (over R) if:

$$H^*(Y, R) \cong H^*(X, R)[k] \quad \text{as graded rings}$$

$$\text{i.e.: } H^i(Y, R) \cong \begin{cases} H^j(X, R) & \text{if } i = (2k + 1)j, \\ 0 & \text{otherwise,} \end{cases}$$

and isos are compatible with cup products.

(Abstracts a definition of

D.Cohen, F.Cohen, M.Xicoténcatl [2000])

Question. When does a homological rescaling pass to a homotopical rescaling?

Examples of rescalings (over $R = \mathbb{Z}$)

- $X = S^1$ $Y = S^{2k+1}$
- $X = \bigvee_1^n S^1$ $Y = \bigvee_1^n S^{2k+1}$
- $X = \prod_1^n S^1$ $Y = \prod_1^n S^{2k+1}$
- $X = \#_1^g S^1 \times S^1$ $Y = \#_1^g S^{2k+1} \times S^{2k+1}$
- $X = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$ $Y = \mathbb{C}^{(k+1)\ell} \setminus \bigcup_{i=1}^n H_i^{\times(k+1)}$

$\mathcal{A} = \{H_1, \dots, H_n\}$ hyperplane arrangement in \mathbb{C}^ℓ

$\mathcal{A}^k := \{H_1^{\times k}, \dots, H_n^{\times k}\}$ **redundant** subspace arr.

(Cohen-Cohen-Xico.)

- $X = S^3 \setminus \bigcup_{i=1}^n K_i$ $Y = S^{4k+3} \setminus \bigcup_{i=1}^n K_i^{\otimes k}$

$K = (K_1, \dots, K_n)$ link of oriented circles in S^3

$K^{\otimes k}$ link of S^{2k+1} 's in S^{4k+3} : **join** of K with k copies of Hopf_n (in sense of [Koschorke-Rolfen \[1989\]](#))

Rational rescalings

- Given X and k , there is a k -rescaling Y over \mathbb{Q} .

Take Y to be the realization of the Sullivan minimal model for dga $(H^*(X, \mathbb{Q})[k], d = 0)$. By construction, this rescaling is formal.

- In general, k -rescaling is not unique. Eg:

$$X = S^1 \vee S^1 \vee S^4$$

$$Y = S^3 \vee S^3 \vee S^{12}$$

$$Z = S_x^3 \vee S_y^3 \bigcup_{[x, [x, [x, [x, y]]]]} e^{12}$$

- Nevertheless, k -rescaling is unique $/\mathbb{Q}$, for k large:

Shiga-Yagita [1982]: If $H^{>d}(X, \mathbb{Q}) = 0$, then X has a unique k -rescaling (up to \mathbb{Q} -equivalence), for all $k > (d - 1)/2$.

- If rescaling Y is unique, it must be formal.

■ **Simple examples.**

- $X = S^1$ $Y = S^{2k+1}$
- $X = \bigvee_1^n S^1$ $Y = \bigvee_1^n S^{2k+1}$
- $X = \prod_1^n S^1$ $Y = \prod_1^n S^{2k+1}$
- $X = \#_1^g S^1 \times S^1$ $Y = \#_1^g S^{2k+1} \times S^{2k+1}$

X and Y formal, rescaling unique $/\mathbb{Q}$, for all k

■ **Arrangements.** $X = M(\mathcal{A})$, $Y = M(\mathcal{A}^{k+1})$

Arnol'd, Brieskorn: X formal

Yuzvinsky [2001]: Y formal

Y is unique k -rescaling $/\mathbb{Q}$, for $k > (\ell - 1)/2$

■ **Links.** $X = M(K)$, $Y = M(K^{\otimes k})$

X may not be formal, but Y is formal

(since $\dim X = 2$), so rescaling unique $/\mathbb{Q}$, for all k .

Rescaling Lie algebras

■ Lie algebras with grading

Graded \mathbb{Q} -vector space L_* , with $[\cdot, \cdot]: L_p \otimes L_q \rightarrow L_{p+q}$

Lie identities satisfied exactly. E.g.:

$$\text{gr}_*(G) \otimes \mathbb{Q}$$

■ Graded Lie algebras

L_* , with Lie identities satisfied up to sign. E.g.:

Homotopy Lie algebra

$$\pi_*(\Omega Y) \otimes \mathbb{Q} := \bigoplus_{r \geq 1} \pi_r(\Omega Y) \otimes \mathbb{Q}$$

Y simply-connected space, ΩY loop space

$[\cdot, \cdot] = \text{Samelson product}$

■ Rescaling

L_* Lie algebra with grading $\rightsquigarrow L[k]$ graded Lie algebra

$L[k]_{2kq} = L_q$ and $L[k]_p = 0$ otherwise,

and Lie bracket rescaled accordingly.

The Rescaling Formula

Theorem A. *Let Y be a k -rescaling of X . If $H^*(X, \mathbb{Q})$ is a **Koszul** algebra, then:*

$$\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \text{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$$

as graded Lie algebras.

Theorem B. *Let Y be a **formal** k -rescaling of X . If $\text{Hilb}(\pi_*(\Omega Y) \otimes \mathbb{Q}, t) = \text{Hilb}(\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}, t^{2k})$, then:*

- 1. $H^*(X, \mathbb{Q})$ is a Koszul algebra.*
- 2. Y is a **coformal** space (i.e., its \mathbb{Q} -homotopy type is determined by its homotopy Lie algebra).*

The Homotopy LCS Formula

Set $\Phi_r := \text{rank } \pi_r(\Omega Y) = \text{rank } \pi_{r+1}(Y)$.

Theorem C. *Let Y be a k -rescaling of X .
If $H^*(X, \mathbb{Q})$ is Koszul, then:*

$$\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = P_X(-t^k)$$

and $\Phi_r = 0$, if $2k \nmid r$.

Hence:

$$\Omega Y \simeq_{\mathbb{Q}} \tilde{\prod}_{i \geq 1} K(\mathbb{Q}, 2ki)^{\Phi_{2ki}}$$

and so

$$P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$$

In fact, by Milnor-Moore:

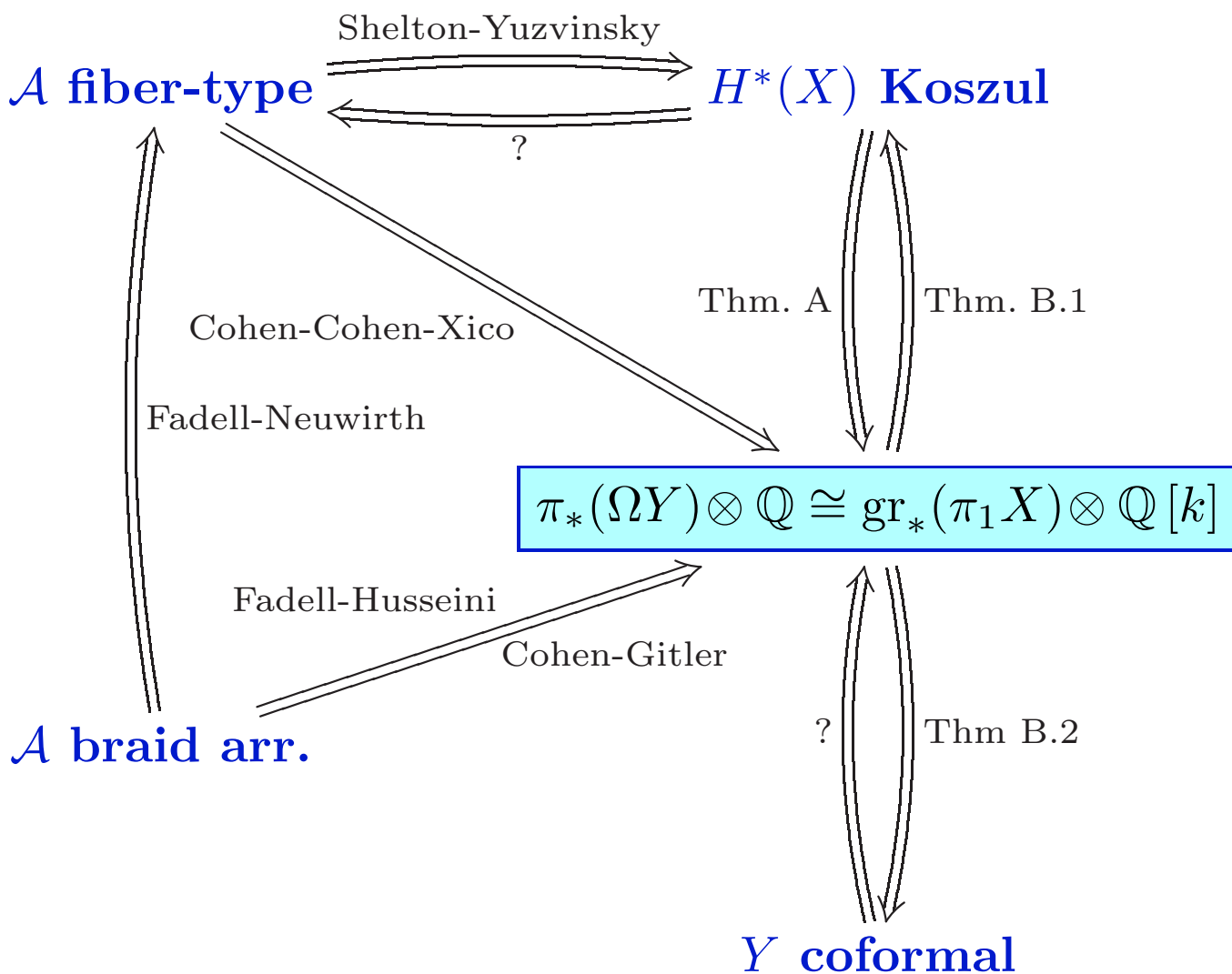
$$H_*(\Omega Y, \mathbb{Q}) \cong U(\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k])$$

(as Hopf algebras)

Rescaling arrangements

Let \mathcal{A} arrangement, $X = M(\mathcal{A})$, $Y = M(\mathcal{A}^{k+1})$.

Recall: both X, Y formal; Y rescaling of X .



Fiber-type arrangements

\mathcal{A} supersolvable, $\exp(\mathcal{A}) = \{d_1, \dots, d_\ell\}$

$X = M(\mathcal{A}), \quad Y = M(\mathcal{A}^{k+1})$

$\phi_r = \text{rank gr}_r(\pi_1 X), \quad \Phi_r = \text{rank } \pi_r(\Omega Y).$

- Classical LCS formula:

$$\prod_{r=1}^{\infty} (1 - t^r)^{\phi_r} = \prod_{i=1}^{\ell} (1 - d_i t)$$

- Homotopy LCS formula:

$$\prod_{r=1}^{\infty} (1 - t^{(2k+1)r})^{\Phi_{2kr}} = \prod_{i=1}^{\ell} (1 - d_i t^{2k+1})$$

- Poincaré series of loop space:

$$P_{\Omega Y}(t) = \prod_{i=1}^{\ell} (1 - d_i t^{2ki})^{-1}$$

Generic arrangements

If \mathcal{A} generic (and non-boolean), then:

- (a) $H^*(X, \mathbb{Q})$ not quadratic \implies not Koszul
- (b) Rescaling formula fails
- (c) Y not coformal

Failure of (b) and (c) is detected by higher-order Whitehead products.

Example. $\mathcal{A} = \{3 \text{ generic lines in } \mathbb{C}^2\}$

$X \simeq 2$ -skeleton of $S^1 \times S^1 \times S^1$

$Y \simeq 2(2k + 1)$ -skeleton of $S^{2k+1} \times S^{2k+1} \times S^{2k+1}$

Then:

$$\mathrm{gr}_*(\pi_1 X) = \mathbb{L}^{\mathrm{ab}}(x_1, x_2, x_3)$$

$\deg x_i = 1$

$$\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^{\mathrm{ab}}(y_1, y_2, y_3) * \mathbb{L}(w)$$

$\deg y_i = 2k, \deg w = 6k + 1.$

The Mal'cev Formula

Theorem D. *Let Y be a k -rescaling of X .*

Assume $H^(X, \mathbb{Q})$ is a Koszul algebra. Then:*

X is formal \iff there are filtered group isomorphisms

$$\begin{aligned} \text{Hom}^{\text{coalg}}(H_*(\Omega S^{2k+1}, \mathbb{Q}), H_*(\Omega Y, \mathbb{Q})) &\cong \\ [\Omega S^{2k+1}, \Omega Y]^\wedge &\cong \pi_1 X \otimes \mathbb{Q} \quad (\ddagger) \end{aligned}$$

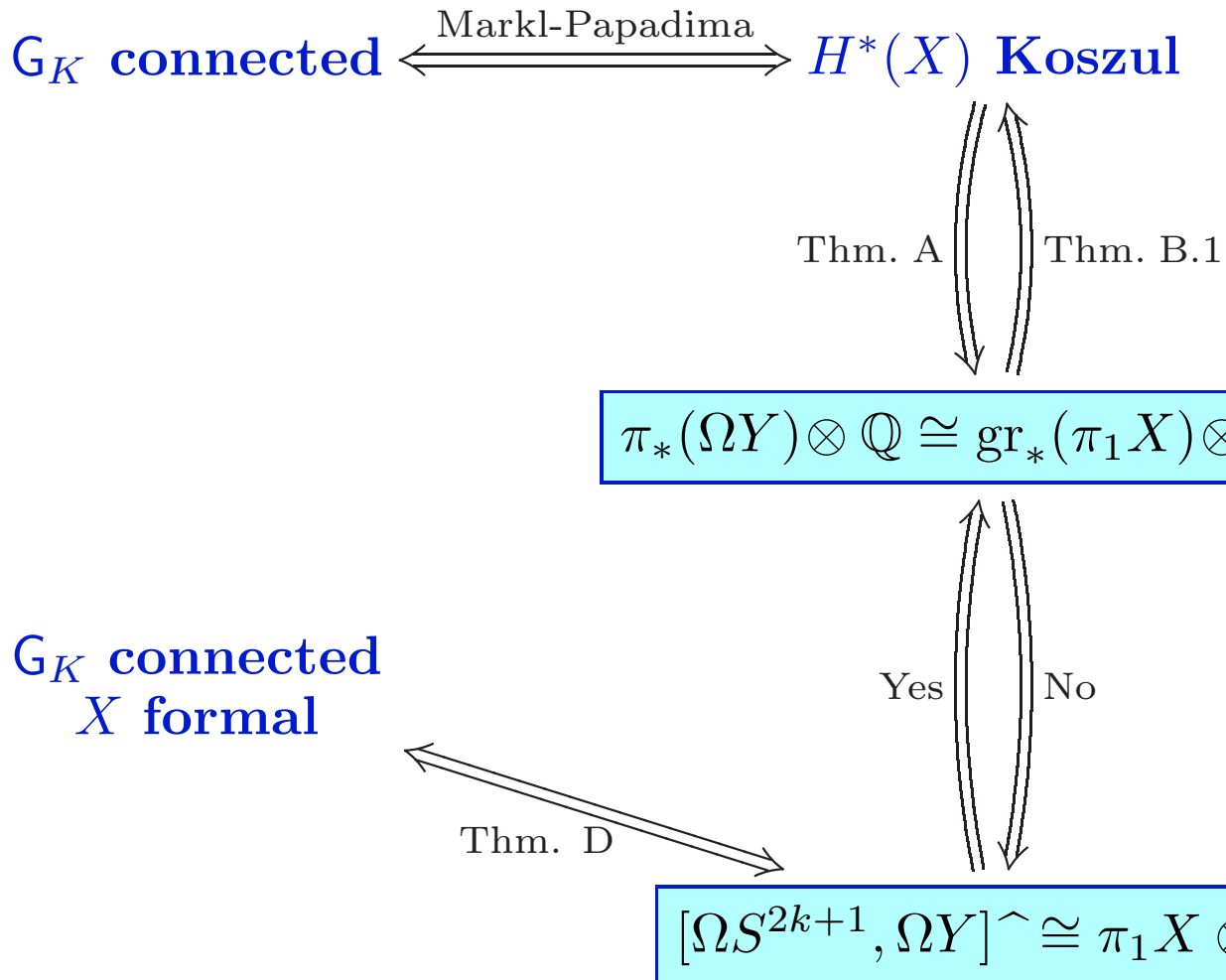
- Key ingredient in proof: A result of [H. Baues](#) [1981].
- Passing to associated graded Lie algebras:
Mal'cev Formula $(\ddagger) \rightsquigarrow$ Rescaling Formula (\dagger) .
- (\ddagger) holds for Koszul arrangements.
- For $X = \bigvee^n S^1$, $Y = \bigvee^n S^{2k+1}$:
recovers a result from [T. Sato's](#) thesis [2000].
- For $\mathcal{A} = \text{braid arr.}$, $X = M(\mathcal{A})$, $Y = M(\mathcal{A}^{k+1})$:
answers a question of [Cohen-Gitler](#) [2001].

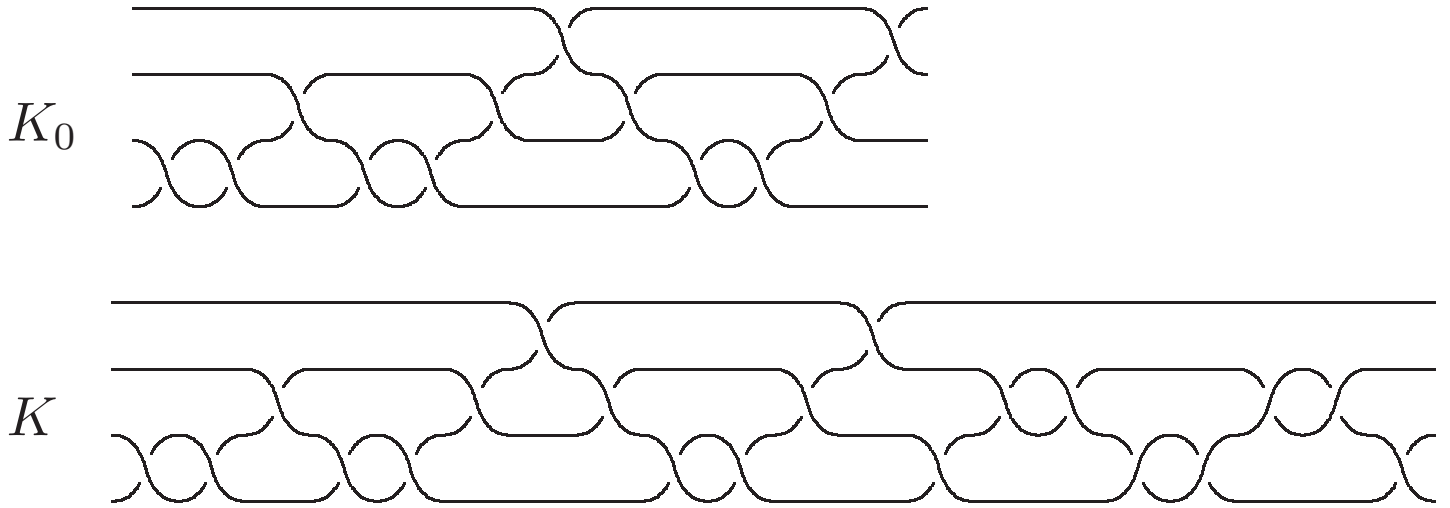
Rescaling links

Let K link, $X = M(K)$, $Y = M(K^{\otimes k})$

G_K **linking graph** of K

—vertices K_i , edges (K_i, K_j) if $\text{lk}(K_i, K_j) \neq 0$





$$K_0 = \widehat{\Delta_n^2} = \text{Hopf}_n, \quad n \geq 4$$

$$K = \widehat{\Delta_n^2} \beta, \quad \text{with } \beta \in (P_{n-1})_{r-1} \setminus (P_{n-1})_r, \quad r \geq 3$$

K_0 and K have:

- $G = K_n$, $\text{lk}_{ij} = 1$. Thus, $H^*(X_0, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$.
- Same Milnor $\bar{\mu}$ -invariants.
- Same Vassiliev invariants, up to order $r - 2$.
- $\text{gr}^*(\pi_1 X_0) \cong \text{gr}^*(\pi_1 X)$.

But X_0 **formal**, X **not formal**. So:

- Rescaling Formula holds for both K_0 and K .
- Mal'cev Formula **holds** for K_0 , **fails** for K .