Which Kähler groups are 3-manifold groups?

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Realizing finitely presented groups

- Every finitely presented group G can be realized as
$$G = \pi_1(M),$$
for some smooth, compact, connected manifold M^n of dimension $n \geq 4$.

- M^n can be chosen to be orientable.

- M^n (n even) can be chosen to be symplectic (Gompf 1995).

- M^n (n even, $n \geq 6$) can be chosen to be complex (Taubes 1992).

- Requiring $n = 3$ puts severe restrictions on G, e.g.:
$$G \text{ abelian 3-manifold group} \iff$$
$$G \in \{\mathbb{Z}/m\mathbb{Z}\ (m \geq 1),\ \mathbb{Z},\ \mathbb{Z} \oplus \mathbb{Z}_2,\ \mathbb{Z}^3\}.$$

Kähler groups

If M is a compact Kähler manifold, $G = \pi_1(M)$ is called a ***Kähler group*** (or, ***projective group***, if M is actually a smooth projective variety).

This also puts strong restrictions on G, e.g.:

- $b_1(G)$ is even (Hodge theory)
- G is 1-formal, i.e., its Malcev Lie algebra is quadratic (Deligne–Griffiths–Morgan–Sullivan 1975)
- G cannot split non-trivially as a free product (Gromov 1989)

On the other hand:

- G finite $\implies$ G projective group (Serre 1958).
- $\Gamma = (V, E)$ finite simple graph $\rightsquigarrow$

$$G_\Gamma := \{v \in V \mid vw = wv \text{ if } \{v, w\} \in E\}$$

Then (Dimca-Papadima-S. 2005):

$$G_\Gamma \text{ Kähler group} \iff G_\Gamma \in \{\mathbb{Z}^n \mid n \text{ even}\}$$

Kähler & 3-manifold groups

Question (Donaldson–Goldman 1989, Reznikov 1993). Which 3-manifold groups are Kähler groups?

Clearly,

$$\{\text{Abelian, 3-manifold \& Kähler groups}\} \\ = \{\text{finite cyclic groups}\}$$

Partial answers (much harder):

Theorem (Reznikov 2002). *Let M be an irreducible, atoroidal 3-manifold. Suppose there is a homomorphism $\rho\colon \pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$ with Zariski dense image. Then $G = \pi_1(M)$ is not a Kähler group.*

Theorem (Hernández-Lamoneda 2001). *Let M be a geometrizable 3-manifold, with all pieces hyperbolic. Then $G = \pi_1(M)$ is not a Kähler group.*

In (Dimca-S. `arXiv:0709.4350`, to appear in JEMS), we answer the question for *all* 3-manifold groups:

Theorem. *Let G be the fundamental group of a compact, connected 3-manifold. If G is a Kähler group, then G is finite.*

By the 3-dim spherical space-form conjecture (proved by Perelman in 2003), M^3 has finite fundamental group iff it admits a metric of constant positive curvature, i.e., $M^3 = S^3/G$, where G is a finite subgroup of O(4), acting freely on S^3. Hence:

$$\{\text{Kähler groups}\} \cap \{\text{3-manifold groups}\}$$
$$= \{\text{finite subgroups of } \mathrm{O}(4)\}$$

By (Hopf 1925) and (Milnor 1957), these groups are:

$$1,\ D^*_{4m},\ O^*,\ I^*,\ D_{2^k(2m+1)},\ P'_{8\cdot 3^k},$$

and products of one of these with a cyclic group of relatively prime order.

Idea of proof

- Consider the cup-product map

  $$\cup_M \colon H^1(M,\mathbb{C}) \wedge H^1(M,\mathbb{C}) \to H^2(M,\mathbb{C})$$

  when M is either Kähler or 3-dim.

- Analyze the ***1-isotropic*** subspaces of $H^1(M,\mathbb{C})$: those subspaces W for which the restriction of $\cup_M$ to $W \wedge W$ has rank 1.

- Analyze the ***resonance varieties***

  $$R_1(M) \subset H^1(M,\mathbb{C}),$$

  and relate the extremal case $R_1(M) = H^1(M,\mathbb{C})$ to the 1-isotropicity of $H^1(M,\mathbb{C})$.

- The completely differing conclusions rule out G being both a Kähler group and a 3-manifold group, provided $b_1(G) > 0$.

- If $b_1(G) = 0$, use the different behavior of Kähler and 3-manifold groups wrt Kazhdan's property (T) to rule out everything except G finite.

Cohomology jumping loci

X a connected, finite-type CW-complex, $G = \pi_1(X)$.

$\rho\colon G \to \mathbb{C}^*$ character $\rightsquigarrow$ $\mathbb{C}_\rho$ rank 1 local system on X.

The ***characteristic varieties*** of X are the jump loci for cohomology with coefficients in such local systems:

$$\boxed{V^i_d(X) = \{\rho \in \operatorname{Hom}(G, \mathbb{C}^*) \mid \dim H^i(X, \mathbb{C}_\rho) \geq d\}.}$$

Now consider $H^*(X, \mathbb{C})$. For each $x \in H = H^1(X, \mathbb{C})$, get a cochain complex $(H^*(X, \mathbb{C}), x)$:

$$H^0(X, \mathbb{C}) \xrightarrow{x\cdot} H^1(X, \mathbb{C}) \xrightarrow{x\cdot} H^2(X, \mathbb{C}) \longrightarrow \cdots$$

The ***resonance varieties*** of X are the jump loci for the homology of this complex:

$$\boxed{R^i_d(X) = \{x \in H \mid \dim H^i(H^*(X, \mathbb{C}), x) \geq d\}.}$$

Note. $V_d^1(X)$ and $R_d^1(X)$ depend only on $G = \pi_1(X)$, so may write them as $V_d(G)$ and $R_d(G)$.

Note. $x \in H$ belongs to $R_d^1(X) \iff \exists$ subspace $W \subset H$ of dim $d+1$ such that $x \cup y = 0$, $\forall y \in W$.

Set $n = b_1(X)$, $m = b_2(X)$. Fix bases $\{e_1, \dots, e_n\}$ for $H^1(X, \mathbb{C})$ and $\{f_1, \dots, f_m\}$ for $H^2(X, \mathbb{C})$, and write

$$e_i \cup e_j = \sum_{k=1}^{m} \mu_{i,j,k} f_k.$$

Define an $m \times n$ matrix Δ of linear forms in variables $x_1, \dots, x_n$, with entries

$$\Delta_{k,j} = \sum_{i=1}^{n} \mu_{i,j,k} x_i.$$

Then:

$$R_d^1(X) = V(E_d(\Delta)),$$

where E_d is the ideal of $(n-d) \times (n-d)$ minors.

Remark. When $G = \pi_1(X)$ is a commutator-relators group, $\Delta = A^{\mathrm{lin}}$, the ***linearized Alexander matrix*** from (Cohen-S. 1999), (Matei-S. 2000).

Resonance varieties of 3-manifolds

Let M be a compact, connected, orientable 3-manifold. Fix an orientation $[M] \in H^3(M, \mathbb{Z}) \cong \mathbb{Z}$.

Then, $\cup_M \colon H^1(M, \mathbb{Z}) \wedge H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ determines (and is determined by) an alternating 3-form μ_M on $H^1(M, \mathbb{Z})$:

$$\mu_M(x, y, z) = \langle x \cup y \cup z, [M] \rangle,$$

where $\langle\, , \rangle$ is the Kronecker pairing.

Fix basis $\{e_1, \dots, e_n\}$ for $H^1(M, \mathbb{C})$.

Choose basis $\{e_1^\vee, \dots, e_n^\vee\}$ for $H^2(X, \mathbb{C})$, with $e_i^\vee$ the Kronecker dual of the Poincaré dual of e_i.

Then:

$$\mu_M(e_i, e_j, e_k) = \langle \sum_{1\le m\le n} \mu_{i,j,m} e_m^\vee, \mathrm{PD}(e_k) \rangle = \mu_{i,j,k}.$$

Recall the $n \times n$ matrix Δ with $\Delta_{k,j} = \sum_{i=1}^n \mu_{i,j,k} x_i$. Hence:

$$\mu_M \text{ alternating} \implies \Delta \text{ skew-symmetric}$$

Theorem 1. *Let M be a closed, orientable 3-manifold. Then:*

1. $H^1(M,\mathbb{C})$ *is not* 1-*isotropic.*
2. *If* $b_1(M)$ *is even, then* $R_1(M) = H^1(M,\mathbb{C})$.

Proof. (1) Suppose $\dim \operatorname{im}(\cup_M) = 1$. This means there is a hyperplane $E \subset H := H^1(M,\mathbb{C})$ such that $x \cup y \cup z = 0, \quad \forall\, x, y \in H,\ z \in E.$

Hence, the skew 3-form $\mu\colon \bigwedge^3 H \to \mathbb{C}$ factors through a skew 3-form $\bar{\mu}\colon \bigwedge^3 (H/E) \to \mathbb{C}$. Now,

$$\dim H/E = 1 \implies \bar{\mu} = 0 \implies \mu = 0,$$

a contradiction.

(2) Since Δ is a skew-symmetric matrix of even size, it follows from (Buchsbaum–Eisenbud 1977) that

$$V(E_1(\Delta)) = V(E_0(\Delta)).$$

Since $x \cup x = 0$, $\forall x \in H$, we have $\Delta \cdot \vec{x} = 0$, where $\vec{x} = (x_1, \dots, x_n)^\top$, and so $\det \Delta = 0$. Hence,

$$R_1(M) = V(E_1(\Delta)) = V(E_0(\Delta)) = H.$$

□

Resonance varieties of Kähler manifolds

A *fibration* is a surjective morphism $f\colon M \to N$ with connected fibers between two compact complex manifolds M and N.

Two fibrations $f\colon M \to C$ and $f'\colon M \to C'$ over projective curves C and C' are *equivalent* if there is an iso $\phi\colon C \xrightarrow{\simeq} C'$ such that $f' = \phi \circ f$.

$$\mathcal{E}(M) := \{\text{equivalence classes of such fibrations, over curves of genus } g \geq 2\}$$

Let M be a compact Kähler manifold.

Theorem (Beauville 1992). *There is a bijection*

$$\mathcal{E}(M) \longleftrightarrow \{\textit{components of } V_1(M) \textit{ containing } 1\}$$

In particular, $\mathcal{E}(M)$ is finite.

Theorem (Catanese 1991). *For every maximal isotropic subspace $E \subset H^1(M,\mathbb{C})$ of dimension $g \geq 2$, there is a fibration $f\colon M \to C$ with $\operatorname{genus}(C) = g$, and a max isotropic subspace $E' \subset H^1(C,\mathbb{C})$ such that $E = f^*E'$.*

Theorem 2. *Let M be a compact Kähler manifold with $b_1(M) \neq 0$. If $R_1(M) = H^1(M, \mathbb{C})$, then $H^1(M, \mathbb{C})$ is 1-isotropic.*

Proof. Set $H := H^1(M, C)$. We have: $\dim H \geq 2$.

$R_1(M) = H$ means:

$$\forall x \in H \setminus \{0\}, \quad \exists y \in H \setminus \mathbb{C} \cdot x, \quad x \cup y = 0$$

$\therefore$ $E := \operatorname{span}\{x, y\}$ is 2-dim isotropic subspace, $x \in E$.

Let $U_x := \{\text{max isotropic subspace of } H \text{ with } x \in U_x\}$.

Then $\dim U_x \geq 2$.

By Catanese's Theorem, $\exists$ fibration $f_x \colon M \to C_x$ with

$$\operatorname{genus}(C_x) = \dim U_x \quad \text{and} \quad x \in f_x^*(H^1(C_x, \mathbb{C}))$$

Hence:

$$H^1(M, \mathbb{C}) = \bigcup_{[f] \in \mathcal{E}(M)} f^*(H^1(C_f, \mathbb{C})),$$

where $f = f_x$ for some $x \in H$, and $C_f := C_x$.

By Beauville's Theorem, this is a ***finite*** union of linear subspaces.

Hence, there exists a fibration $f_1 \colon M \to C_1$ with

$$H^1(M,\mathbb{C}) = f_1^*(H^1(C_1,\mathbb{C})). \qquad (*)$$

Consider the commuting diagram

$$\begin{tikzcd}
H^1(M,\mathbb{C}) \wedge H^1(M,\mathbb{C}) \arrow[r, "\cup_M"] & H^2(M,\mathbb{C}) \\
H^1(C_1,\mathbb{C}) \wedge H^1(C_1,\mathbb{C}) \arrow[r, "\cup_{C_1}"] \arrow[u, "f_1^* \wedge f_1^*"'] & H^2(C_1,\mathbb{C}) \arrow[u, "f_1^*"']
\end{tikzcd}$$

We have:

- Since f_1 fibration, $f_1^* \colon H^1(C_1,\mathbb{C}) \to H^1(M,\mathbb{C})$ is injective.
- By (*): f_1^* is surjective.
- Using the Hodge-Riemann bilinear relations: $f_1^* \colon H^2(C_1,\mathbb{C}) \to H^2(M,\mathbb{C})$ is injective.
- $\cup_{C_1}$ surjects onto $H^2(C_1,\mathbb{C}) = \mathbb{C}$.

Hence, $\dim \operatorname{im}(\cup_M) = 1$. □

An alternate way to prove Theorem 2:

Theorem (Beauville, Green–Lazarsfeld, Simpson, Campana). *If M is a compact Kähler manifold, then*

$$\boxed{V_d(M) = \bigcup_{\alpha} \rho_\alpha \cdot f_\alpha^* \operatorname{Hom}(\pi_1(C_\alpha), \mathbb{C}^*),}$$

whith $f_\alpha \colon M \to C_\alpha$ fibration, and $\operatorname{genus}(C_\alpha) > 0$.

Theorem (Dimca–Papadima–S. 2005). *Let G be a 1-formal group (e.g., a Kähler group). Then, $\forall d \geq 1$,*

$$\boxed{\exp\colon (R_d(G), 0) \xrightarrow{\simeq} (V_d(G), 1)}$$

is an iso of complex analytic germs. Consequently,

$$\boxed{\operatorname{TC}_1(V_d(G)) = R_d(G).}$$

Theorem (Dimca–Papadima–S. 2005). *Let M be a compact Kähler manifold. Then every positive-dimensional component of $R_1(M)$ is an 1-isotropic linear subspace of $H^1(M, \mathbb{C})$, of dimension at least 4.*

Kazhdan's property (T)

Definition. A discrete group G satisfies ***Kazhdan's property*** (T) if

$$\boxed{H^1(G, \mathcal{H}) = 0,}$$

for all unitary reps of G on a Hilbert space $\mathcal{H}$.

In particular, $b_1(G) \neq 0 \implies G$ not Kazhdan.

Theorem (Reznikov 2002). *Let G be a Kähler group. If G is not Kazhdan, then $b_2(G) \neq 0$.*

Theorem (Fujiwara 1999). *Let G be the fundamental group of a closed, orientable 3-manifold. If G is Kazhdan, then G is finite.*

Remark. Fujiwara assumes that each piece of the JSJ decomposition of M admits one of the 8 geometric structures in the sense of Thurston; this is now guaranteed by the work of Perelman.

Proof of the Main Theorem

Let G be the fundamental group of a compact, connected 3-manifold M.

Suppose G is a Kähler group, and G is not finite.

Step 1.

A finite-index subgroup of a Kähler group is again a Kähler group. Passing to the orientation double cover of M if necessary, we may assume M is orientable.

Step 2.

Since G is an infinite, orientable 3-manifold group,

G is not Kazhdan (by Fujiwara's Theorem).

Since G is Kähler and not Kazhdan,

$\boxed{b_2(G) \neq 0}$ (by Reznikov's Theorem).

Step 3.

By Hopf, $b_2(M) \geq b_2(G)$. Hence $b_2(M) \neq 0$.

By Poincaré duality, $b_1(M) = b_2(M)$.

But $b_1(G) = b_1(M)$. Hence:

$$\boxed{b_1(G) \neq 0.}$$

Step 4.

Since G is Kähler, $b_1(G)$ must be even.

Since M is a closed, orientable 3-manifold with $G = \pi_1(M)$, Theorem 1 gives

$$\boxed{R_1(G) = H^1(G, \mathbb{C}), \text{ and not 1-isotropic.}}$$

Since G is Kähler, Theorem 2 gives

$$\boxed{b_1(G) = 0.}$$

Our assumptions have led to a contradiction. Thus, the Theorem is proved. $\square$

Quasi-Kähler groups

A group G is ***quasi-Kähler*** (***quasi-projective***) if $G = \pi_1(M \setminus D)$, where M is a Kähler (projective) manifold and D is a divisor with normal crossings.

Examples:

- Free groups: $F_n = \pi_1(\mathbb{CP}^1 \setminus \{n + 1 \text{ points}\})$.
- Arrangement groups: $G = \pi_1(\mathbb{CP}^2 \setminus \{\ell_0, \dots, \ell_n\})$.

Theorem (Dimca–Papadima–S. 2005). *For a right-angled Artin group G_Γ:*

$$G_\Gamma \textit{ quasi-Kähler} \iff G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$$
$$\iff \Gamma = K_{n_1,\dots,n_r} \textit{ (complete multipartite)}$$

Theorem (Dimca–Papadima–S., JAG 2008). *For a Bestvina-Brady group $N_\Gamma := \ker(\chi\colon G_\Gamma \to \mathbb{Z}), \chi(v) = 1$:*

$$N_\Gamma \textit{ quasi-Kähler} \iff \Gamma \textit{ is a tree or } \Gamma = K_{n_1,\dots,n_r},$$

with some $n_i = 1$, or all $n_i > 1$ and $r \geq 3$.

Question. Which 3-manifold groups are quasi-Kähler?

We have partial results, including a complete answer in the case of *boundary manifolds* of line arrangements.

Theorem (Cohen–S., G&T, Monographs 2008, Dimca–Papadima–S., IMRN 2008)**.**
Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ *be a line arrangement in* $\mathbb{CP}^2$*. Let* M *be the boundary of a regular neighborhood of* $\mathcal{A}$*, and* $G = \pi_1(M)$*. The following are equivalent:*

- G *is* 1*-formal.*
- $\mathrm{TC}_1(V_d(G)) = R_d(G)$*.*
- G *is quasi-Kähler.*
- G *is quasi-projective.*
- $\mathcal{A}$ *is either a pencil or a near-pencil.*

In this case, $M = \sharp^n S^1 \times S^2$ *or* $M = S^1 \times \Sigma_{n-1}$*.*