

Quasi - projective
Bestvina-Brady groups

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Braids and their Ramifications
Cortona, Italy
May 23, 2007

Joint work with : • Alex Dimca
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* [DPS1] "Formality, Alexander invariants, and a question of Serre. math.AT/0512480

* [DPS2] "Quasi-Kähler Bestvina-Brady groups", to appear in J. Alg. Geometry ~2007
 math.AG/0603446

— . —
* [PS1] "Algebraic invariants for right-angled Artin groups", Math. Ann. 2006.

* [PS2] "Algebraic invariants for Bestvina-Brady groups", to appear in J. London. Math. Soc.
 math.GR/0603240

— . —
* [DPS3] "Non-finiteness properties of fundamental groups of smooth projective varieties"
 math.AG/0609456

Artin's braid groups (1926/1947)

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} : \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \end{array} \mid \begin{array}{l} 1 \leq i \leq n-2 \\ |i-j| > 1 \end{array} \rangle$$

$$1 \rightarrow P_n \rightarrow B_n \rightarrow S_n \rightarrow 1$$

Serre's problem (1958)

Which finitely-presented groups G are "quasi-projective", i.e.,

$$G = \pi_1(M)$$

M a smooth, connected, complex quasi-proj. variety?

Fox - Newirth (1962)

$$P_n = \pi_1(F(\mathbb{C}, n)), \text{ where}$$

$$F(\mathbb{C}, n) := \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \{z_i = z_j\}$$

$$B_n = \pi_1(C(\mathbb{C}, n)), \text{ where}$$

$$C(\mathbb{C}, n) := F(\mathbb{C}, n) / S_n = \mathbb{C}^n \setminus \underbrace{\{\Delta_n = 0\}}_{\substack{\text{Vieta} \\ \text{discriminant}}}$$

$\therefore P_n, B_n$ are quasi-projective

Artin groups

Let $\Gamma = (V, E, \ell)$ be an edge-labeled finite, simple graph

$$G_\Gamma = \langle v \in V : \underbrace{vwv \dots}_m = \underbrace{wvw \dots}_m \rangle$$

$$\forall e = \{v, w\} \in E, \ell(e) = m$$

$$W_\Gamma = G_\Gamma / \langle\langle v^2 = 1 : v \in V \rangle\rangle$$

$$1 \rightarrow P_\Gamma \rightarrow G_\Gamma \rightarrow W_\Gamma \rightarrow 1$$

Example $\Gamma = K_{n-1}$



$$\rightsquigarrow G_\Gamma = B_n$$


Example Right-angled Artin groups

$$\Gamma = (V, E) \quad \ell(e) = 2, \quad v \in E$$

$$G_\Gamma = \langle V : vw = wv, \quad \forall \{v, w\} \in E \rangle$$

• $\Gamma = K_n$  $\rightsquigarrow G_\Gamma = \mathbb{Z}^n$

• $\Gamma = \bar{K}_n$  $\rightsquigarrow G_\Gamma = F_n$

• $\Gamma = \Gamma' * \Gamma''$  $\rightsquigarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

• $\Gamma \cong \Gamma' \iff G_\Gamma \cong G_{\Gamma'}$

[Kim, Makar-Limanov, Neggels, Roush 1980]
[Droms 1987]

Brieskorn (1971) $\Gamma = (V, E, \ell)$

W_Γ finite $\Rightarrow G_\Gamma$ quasi-proj.

sketch: $A_\Gamma =$ reflection arrangement of type W_Γ (over \mathbb{C})

$$M_\Gamma = \mathbb{C}^n - \bigcup_{H \in A_\Gamma} H \quad (n = |V|)$$

$$P_\Gamma = \pi_1(M_\Gamma)$$

$$G_\Gamma = \pi_1(M_\Gamma/W_\Gamma) = \pi_1(\mathbb{C}^n - \underbrace{\{\Delta_\Gamma = 0\}}_{\text{discriminant}})$$

Kapovich - Millson (1998)

\exists infinitely many $\Gamma = (V, E, \ell)$ s.t.
 G_Γ not quasi-proj

DPS1 (2005) $\Gamma = (V, E)$

G_Γ quasi-proj. $\Leftrightarrow \Gamma = \bar{K}_{n_1} * \dots * \bar{K}_{n_r} =: K_{n_1, \dots, n_r}$
complete multipartite graph

$$\Leftrightarrow G_\Gamma = F_{n_1} \times \dots \times F_{n_r}$$

Obstructions to quasi-projectivity

Deligne (~1972) M smooth, quasi-proj. variety
 $\Rightarrow H^*(M; \mathbb{C})$ has Mixed Hodge Structure (W, F)

Definition (Quillen 1969) Malcev Lie algebra of a group G :

$$E_G := \text{Prim}(\widehat{\mathbb{Q}G})$$

- a filtered Lie alg. with $\text{gr}(E_G) \cong \text{gr}(G) \otimes \mathbb{Q}$.

Morgan's test (1978) If G quasi-proj:

$$E_G \cong \widehat{\mathbb{L}}/\mathbb{J}$$

\mathbb{L} : free Lie alg. on gens in degree 1 & 2

\mathbb{J} : closed Lie ideal, gens in deg 2, 3, 4

Example $G = \langle x, y \mid [x, [x, [x, [x, y]]]] = 1 \rangle$ not q-proj.

Definition (Sullivan 1977)

A finitely presented group G is 1-formal if

$$E_G \cong \widehat{\mathbb{L}}/\mathbb{J}$$

\mathbb{L} free on gens in deg 1
 \mathbb{J} generated in deg 2

M formal space $\Rightarrow G = \pi_1(M)$ is 1-formal

\Uparrow Deligne-Griffiths
Morgan-Sullivan

M projective manifold

Kapovich - Millson :

- G quasi-proj \Rightarrow $\text{Hom}(G, \text{Lie group})$ has only certain types of singularities
- $\forall \Gamma = (V, E, \ell)$, G_Γ is 1-formal
- $\exists \Gamma$ s.t. G_Γ does not pass the K-M test

Characteristic varieties

$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_s \rangle$ fin. pres. group

$T_G = \text{Hom}(G, \mathbb{C}^*) = \text{Hom}(G_{ab}, \mathbb{C}^*)$

$T_G^\circ = \text{identity component} \cong (\mathbb{C}^*)^n$, $n = b_1(G)$

$V_1(G) := \{ \rho \in T_G : H_1(G; \mathbb{C}_\rho) \neq 0 \}$

$H_1(\mathbb{C}_*(\widetilde{K(G,1)}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho)$

Eg: $G = \mathbb{Z} = \langle t \rangle$, $K(G, 1) = S^1$, $T_G = \mathbb{C}^*$

$\mathbb{C}_*(\widetilde{K(G,1)}) : \mathbb{C}_1 \xrightarrow{\partial_1} \mathbb{C}_0$
 $\mathbb{Z}\mathbb{Z} \xrightarrow{t-1} \mathbb{Z}\mathbb{Z}$

$\mathbb{C}_*(\widetilde{K(G,1)}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho : \mathbb{C} \xrightarrow{\rho-1} \mathbb{C}$
 $H_1 \neq 0 \iff \rho = 1$: $V_1(\mathbb{Z}) = \{1\}$

For $\rho \in T_G^\circ$, $\rho \neq 1$:

$\rho \in V_1(G) \iff \text{rank} \left(\frac{\partial r_i}{\partial x_j} \right)^{ab}(\rho) < m-1$

$A_G = \text{Alexander matrix}$

Arapura's test (1997)

$$G \text{ quasi-proj.} \Rightarrow V_1(G) = \bigcup_{\alpha} \rho_{\alpha} \cdot T_{\alpha}$$

root of 1 subtorus of T_G^0

Resonance varieties

$$A = H^*(G; \mathbb{C})$$

$$a \in A^1 = \text{Hom}(G, \mathbb{C}) = \mathbb{C}^n \quad a \cdot a = 0$$

$$\leadsto (A, a) : 0 \rightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \rightarrow \dots$$

$$R_1(G) = \{ a \in A^1 : H^1(A, a) \neq 0 \}$$

$$= \{ a : \text{rank } A_G^{\text{lin}}(a) < n-1 \}$$

↑ linearized Alex. matrix
($t_i \mapsto 1 - a_i$, take linear part)

Resonance test (DPS 2005)

Let G be a quasi-proj., 1-formal group.

Write $R_1(G) = \bigcup_{\alpha} R^{\alpha}$ (irred. components). Then:

(R1) R^{α} linear subspace

(R2) $R^{\alpha} \cap R^{\beta} = 0$, $\forall \alpha \neq \beta$

(R3) R^{α} is p -isotropic, of $\dim \geq 2p+2$ ($p=0$ or 1)

Right-angled Artin groups

$$\Gamma = (V, E) \rightsquigarrow G_\Gamma$$

$K(G_\Gamma, 1) =$ subcomplex of $(S^1)^{|V|}$ with k cells $\leftrightarrow k$ cliques of Γ
 $\mathbb{Z}_{\Delta(\Gamma)}(S^1, *)$ $\leftrightarrow (k+1)$ -simplices in $\Delta(\Gamma)$ flag complex

$C_*(\widehat{K(G_\Gamma, 1)})$: Salvetti complex

$$H^*(G_\Gamma) = \Lambda / J_\Gamma \quad \begin{array}{l} \Lambda: \text{exterior alg on } V^*, v \in V \\ J_\Gamma: \text{ideal } \langle v^*w^* : \{v, w\} \notin E_\Gamma \rangle \end{array}$$

G_Γ 1-formal (so passes Morgan's test)

$K(G_\Gamma, 1)$ formal [follows from Papadima-Turkivsky]

$$\text{Let } T_V = \text{Hom}(G_\Gamma, \mathbb{C}^*) = (\mathbb{C}^*)^{|V|}$$

$$V_1(G_\Gamma) = \bigcup_{W \subset V} T_W \quad \begin{array}{l} \text{[DPS1]} \\ T_W \text{ coord. subspace supported on } W \end{array}$$

Γ_W max disconnected $\rightarrow G_\Gamma$ passes Arapura's test

$$R_1(G_\Gamma) = \bigcup_{W \subset V} H_W \quad \begin{array}{l} \text{[PS1]} \\ H_W \text{ coord subspace supported on } W \end{array}$$

Γ_W max disc.

G_Γ passes resonance tests $\Leftrightarrow \Gamma = K_{n_1, \dots, n_r}$

Ex $\Gamma = \overset{1}{\bullet} - \overset{2}{\bullet} - \overset{3}{\bullet} - \overset{4}{\bullet}$ $R_1(G_\Gamma) = H_{134} \cup H_{124}$
intersect in $H_{12} \neq 0$

Artin groups

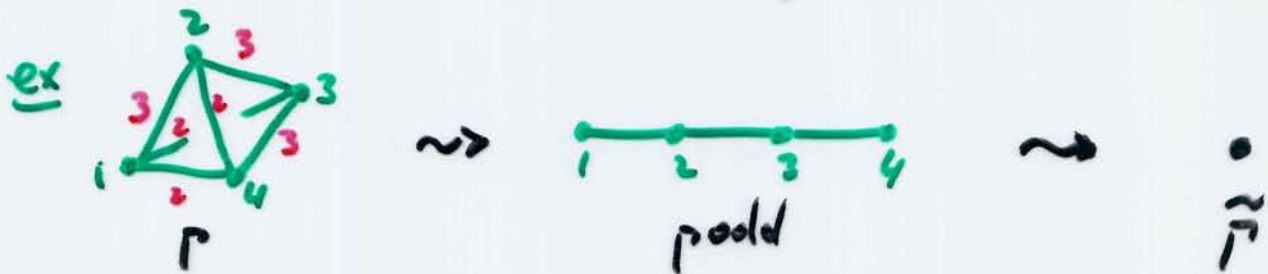
$$\Gamma = (V, E, \ell) \rightsquigarrow \Gamma^{\text{odd}} = (V, E^{\text{odd}})$$

$$\{e \in E; \ell(e) \text{ odd}\}$$

$$\rightsquigarrow \tilde{\Gamma} = (\tilde{V}, \tilde{E})$$

\tilde{V} = components of Γ^{odd}

$$\tilde{E} = \{(c, c') : \exists v \in c, v' \in c', \ell(v, v') \text{ odd}\}$$



Theorem (DPS11) For any $\Gamma = (V, E, \ell)$, TFAE:

- $E_{G_\Gamma} \cong E_G$, for some quasi-proj. G
- Resonance test (R3) satisfied by G_Γ
- $\tilde{\Gamma}$ complete multipartite
- $E_{G_\Gamma} \cong E_{F_{n_1} \times \dots \times F_{n_r}}$

Bestvina - Brady groups

$$N_\Gamma = \ker(\nu: G_\Gamma \rightarrow \mathbb{Z}) \quad \Gamma = (V, E)$$

$$\begin{matrix} \nu & \longmapsto & 1 \\ \nu & \longmapsto & 1 \end{matrix}$$

$$1 \rightarrow N_\Gamma \xrightarrow{\alpha} G_\Gamma \xrightarrow{\nu} \mathbb{Z} \rightarrow 1$$

• $\Gamma = \bar{K}_2 \dots \rightsquigarrow G_\Gamma = F_2 \rightsquigarrow N_\Gamma = F_\infty$

* Γ connected $\iff N_\Gamma$ finitely generated
 [Meier, Van Wyk 1995]

• $\Gamma = K_{2,2} \square \rightsquigarrow G_\Gamma = F_2 \times F_2 \rightsquigarrow N_\Gamma$ fin. gen.
 but not fin pres
 reason: $H_2(N_\Gamma) = \mathbb{Z}^\infty$ [Stallings 1963]

* $\pi_1(\Delta(\Gamma)) = 0 \iff N_\Gamma$ finitely presented
 [Bestvina - Brady 1997]

in this case:

$$N_\Gamma = \langle e \in E \mid ef = fe, ef = g \rangle$$



and $z(e) = uv^{-1}$ if $e = \{u, v\}$

[Dicks - Leary 1999]

Example Γ tree on n vertices $\implies N_\Gamma = F_{n-1}$

[$\Delta(\Gamma) = \Gamma \simeq *$, Γ has no triangles, Γ has $n-1$ edges]

In particular, N_Γ 's not classified by Γ 's

Theorem [DPS2]

N_Γ quasi-projective $\iff \Gamma$ is either

- a tree
- a complete multipartite graph

K_{n_1, \dots, n_r} with $\left\{ \begin{array}{l} \text{some } n_i = 1 \\ \text{all } n_i > 1 \text{ \& } r \geq 3 \end{array} \right.$

in which case, N_Γ belongs to precisely one of:

(1) \mathbb{Z}^r , $r \geq 0$

(2) $F_{n_1} \times \dots \times F_{n_r}$, $n_i > 1$

(3) $\mathbb{Z}^r \times F_{n_1} \times \dots \times F_{n_s}$, $r > 0$, $n_i > 1$

(4) $N_{K_{n_1, \dots, n_r}}$, $r \geq 3$, $n_i > 1$

N_Γ projective $\iff \Gamma = K_n$, n odd

$\iff N_\Gamma = \mathbb{Z}^{2r}$

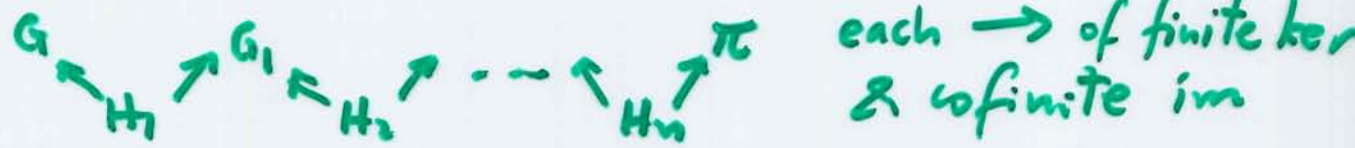
Example $N = N_{K_{2,2,2}}$ (Stallings group)

$N = \pi_1(\mathbb{C}^2 - \text{grid})$



Question (Kollar 1995)

Given a (quasi-) projective group G , is there a group π , commensurable to G up to finite kernels,



such that π admits a $K(\pi, 1)$ which is a smooth, (quasi-) projective variety?

Theorem [DPS 2]

Let N_r quasi-proj. Then:

- N_r of type 1, 2, or 3 \implies YES ✓
- N_r of type 4 \implies NO

since: type 4 $\implies \tilde{H}_2(\Delta(r)) \neq 0$
 $\implies N_r$ not FP_∞
 $\implies N_r$ not commens. (up to finite ker) to any FP_∞ gr.
 Bieri $\implies N_r$ " " " " any π
 $\implies N_r$ " " " " with finite $K(\pi, 1)$