# Characteristic varieties and homological finiteness properties

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# Reference

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# Homological finiteness properties

Let $G$ be a group, $k \geq 1$ an integer.

**Definition (C.T.C. Wall 1965)**

$G$ is of type $\mathrm{F}_k$ if there is a $K(G,1)$ with finite $k$-skeleton.

**Definition (J.-P. Serre 1971, R. Bieri 1976)**

$G$ is of type $\mathrm{FP}_k$ if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with $P_i$ finitely generated for all $i \leq k$.

- $\mathrm{F}_1 \Leftrightarrow \mathrm{FP}_1 \Leftrightarrow$ finitely generated.
- $\mathrm{F}_2 \Leftrightarrow$ finitely presented
- $\mathrm{F}_k \Rightarrow \mathrm{FP}_k$, but $\mathrm{FP}_k \not\Rightarrow \mathrm{F}_k$, $\forall k \geq 2$.
- $\mathrm{FP}_k$ & $\mathrm{F}_2 \Rightarrow \mathrm{F}_k$, $\forall k \geq 2$.
- $\mathrm{FP}_k \Rightarrow H_i(G,\mathbb{Z})$ finitely generated, $\forall i \leq k$.

# BNS invariant

$G$ f.g. group $\rightsquigarrow \mathcal{C}(G)$ Cayley graph.
$\chi\colon G \to \mathbb{R}$ homomorphism $\rightsquigarrow \mathcal{C}_\chi(G)$ induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

**Definition (Bieri, Neumann, Strebel 1987)**

$\Sigma^1(G) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}$

- $\Sigma^1(G)$ is open, conical subset of $\operatorname{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R})$.
- $\Sigma^1(G)$ does not depend on choice of generating set for $G$.

If $G = \pi_1(M)$, where $M$ is a closed 3-manifold:

- $\Sigma^1(G) = \bigcup_{F \text{ fibered face of Thurston norm unit ball}} \mathbb{R}_+ \cdot \overset{\circ}{F}$.
- $\Sigma^1(G) = -\Sigma^1(G)$.
- $M$ fibers over $S^1 \iff \Sigma^1(G) \neq \emptyset$.

# BNSR invariants

## Definition (Bieri, Renz 1988)

$\Sigma^q(G,\mathbb{Z}) = \{\chi \in \mathrm{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_q\}$

There is also a “homotopical" version, $\Sigma^q(G) \subseteq \Sigma^q(G,\mathbb{Z})$.

Properties:

- The BNSR invariants $\Sigma^q(G,\mathbb{Z})$ form a descending chain of *open* subsets of $\mathrm{Hom}(G,\mathbb{R}) \setminus \{0\}$.
- $\Sigma^q(G,\mathbb{Z}) \neq \emptyset \implies G$ is of type $\mathrm{FP}_q$.
- $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- $G$ of type $\mathrm{F}_k \implies \Sigma^q(G) = \Sigma^2(G) \cap \Sigma^q(G,\mathbb{Z})$, $\forall 2 \leq q \leq k$.

Importance of $\Sigma$-invariants: they control the finiteness properties of kernels of projections to abelian quotients.

### Theorem (Bieri, Neumann, Strebel/Bieri, Renz)

*Let $G$ f.g. group, $N \triangleleft G$ normal subgroup with $G/N$ is abelian. Set $S(G,N) = \{\chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \chi(N) = 0\}$. Then:*

1. *$N$ is of type $\mathrm{F}_k \iff S(G,N) \subseteq \Sigma^k(G)$.*
2. *$N$ is of type $\mathrm{FP}_k \iff S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$.*

In particular:

$$\ker(\chi\colon G \twoheadrightarrow \mathbb{Z}) \text{ is f.g.} \iff \{\chi, -\chi\} \subseteq \Sigma^1(G)$$

# Characteristic varieties

- $X$ connected CW-complex with finite $k$-skeleton ($k \geq 1$).
- $G = \pi_1(X)$.
- $\Bbbk$ field; Hom$(G, \Bbbk^\times)$ character variety.

**Definition (Green–Lazarsfeld 1987, Beauville 1988, Simpson 1992, Libgober 1992, . . . )**

The *characteristic varieties* of $X$ (over $\Bbbk$):

$$\mathcal{V}^i_d(X, \Bbbk) = \{\rho \in \mathrm{Hom}(G, \Bbbk^\times) \mid \dim_\Bbbk H_i(X, \Bbbk_\rho) \geq d\},$$

for $0 \leq i \leq k$ and $d > 0$.

- For each $i$, get stratification $\mathrm{Hom}(G, \Bbbk^\times) \supseteq \mathcal{V}^i_1 \supseteq \mathcal{V}^i_2 \supseteq \cdots$
- If $\Bbbk \subseteq \mathbb{K}$ extension: $\mathcal{V}^i_d(X, \Bbbk) = \mathcal{V}^i_d(X, \mathbb{K}) \cap \mathrm{Hom}(G, \Bbbk^\times)$
- For $G$ of type $\mathrm{F}_k$, set: $\mathcal{V}^i_d(G, \Bbbk) := \mathcal{V}^i_d(K(G,1), \Bbbk)$
- If $X$ has finite 1-skeleton: $\mathcal{V}^1_d(X, \Bbbk) = \mathcal{V}^1_d(\pi_1(X), \Bbbk)$

Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover.

### Definition (Libgober 1992)

The *Alexander varieties* of $X$ (over $\Bbbk$):

$$\mathcal{W}^i_d(X, \Bbbk) = V(E_{d-1}(H_i(X^{\mathrm{ab}}, \Bbbk))),$$

the subvariety of $\operatorname{Spec}\Lambda = \operatorname{Hom}(G, \Bbbk^\times)$ defined by the ideal of codim $d-1$ minors of a presentation matrix for $H_i(X^{\mathrm{ab}}, \Bbbk)$, viewed as module over $\Lambda = \Bbbk H_1(X, \mathbb{Z})$.

Using the change-of-rings spectral sequence approach of [Dimca–Maxim 2007], we get:

### Proposition

$$\bigcup_{i=0}^{q} \mathcal{V}^i_1(X, \Bbbk) = \bigcup_{i=0}^{q} \mathcal{W}^i_1(X, \Bbbk), \quad \forall\, 0 \le q \le k$$

$$\Longrightarrow \quad \mathcal{V}^1_1(X, \mathbb{C}) \setminus \{1\} = \mathcal{W}^1_1(X, \mathbb{C}) \setminus \{1\} \quad \text{[E. Hironaka 1997]}$$

# Tangent cones and exponential tangent cones

The homomorphism $\mathbb{C} \to \mathbb{C}^{\times}$, $z \mapsto e^{z}$ induces

$$\exp\colon \operatorname{Hom}(G,\mathbb{C}) \to \operatorname{Hom}(G,\mathbb{C}^{\times}), \quad \exp(0) = 1$$

Let $W = V(I)$ be a Zariski closed subset in $\operatorname{Hom}(G,\mathbb{C}^{\times})$.

## Definition

- The *tangent cone* at 1 to $W$:

$$TC_1(W) = V(\operatorname{in}(I))$$

- The *exponential tangent cone* at 1 to $W$:

$$\tau_1(W) = \{z \in \operatorname{Hom}(G,\mathbb{C}) \mid \exp(tz) \in W,\ \forall t \in \mathbb{C}\}$$

Both types of tangent cones

- are homogeneous subvarieties of $\mathrm{Hom}(G, \mathbb{C})$
- are non-empty iff $1 \in W$
- depend only on the analytic germ of $W$ at 1
- commute with finite unions and arbitrary intersections

Moreover,

- $\tau_1(W) \subseteq TC_1(W)$
  - $=$ if all irred components of $W$ are subtori
  - $\neq$ in general
- $\tau_1(W)$ is a finite union of rationally defined subspaces

# Exponential tangent cone upper bound

Relate the BNSR invariants to the characteristic varieties:

**Theorem**

*Let $G$ be a group of type $F_n$. Then, for each $q \leq n$,*

$$\Sigma^q(G,\mathbb{Z}) \subseteq \Big( \bigcup_{i \leq q} \tau_1^{\mathbb{R}}(\mathcal{V}_1^i(G,\mathbb{C})) \Big)^{\complement} \qquad (*)$$

That is: each $\Sigma$-invariant is contained in the complement of a union of rationally defined subspaces.

Bound is sharp: If $G$ is a fin. gen. nilpotent group, then

$$\Sigma^q(G,\mathbb{Z}) = \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\}, \quad V_1^i(G,\mathbb{C}) = \{1\}, \quad \forall q, i$$

and so get equality in (*).

# Idea of proof: use Novikov homology

Let $\chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\}$. Write

$$\begin{tikzcd} G \arrow[r, two heads, "\xi"] \arrow[rr, bend right, "\chi"'] & \Gamma \arrow[r, hook, "\iota"] & \mathbb{R} \end{tikzcd},$$

with $\Gamma$ lattice. Get diagram:

$$\begin{tikzcd} \mathbb{Z}G \arrow[r, "\xi"] \arrow[d, hook] & \mathbb{Z}\Gamma \arrow[d, hook] \arrow[dr, hook] & \\ & & \mathcal{R}\Gamma_{\iota} \arrow[dl, hook'] \\ \widehat{\mathbb{Z}G}_{\chi} \arrow[r, "\hat{\xi}"] & \widehat{\mathbb{Z}\Gamma}_{\iota} & \end{tikzcd}$$

where

- $\mathbb{Z}\Gamma$ ring of Laurent polynomials
- $\widehat{\mathbb{Z}G}_{\chi}$ *Novikov-Sikorav completion* of $\mathbb{Z}G$:

$$\left\{\lambda \in \mathbb{Z}^{G} \mid \{g \in \operatorname{supp} \lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\right\}$$

- $\mathcal{R}\Gamma_{\iota}$ *rational Novikov ring*: localization of $\mathbb{Z}\Gamma$ at

$$\left\{p = \sum n_{\gamma}\gamma \mid \text{smallest element of } \iota(\operatorname{supp}(p)) \text{ is } 0\text{, and } n_0 = 1\right\}$$

Fact (Farber 2004): $\mathcal{R}\Gamma_\iota$ is a PID. Thus, we may define the *Novikov-Betti numbers* as

$$b_i(G,\chi) := \operatorname{rank}_{\mathcal{R}\Gamma_\iota} H_i(G,\mathcal{R}\Gamma_\iota)$$

Finally,

$$\chi \in \Sigma^q(G,\mathbb{Z}) \iff \operatorname{Tor}_i^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_{-\chi},\mathbb{Z}) = 0,\ \forall i \le q$$

(Sikorav 1987, Bieri 2007)

$$\implies b_i(G,\chi) = 0,\ \forall i \le q$$

(change of rings spectral sequence)

$$\iff \chi \notin \tau_1^{\mathbb{R}}\big(\mathcal{V}_1^i(G,\mathbb{C})\big),\ \forall i \le q$$

# BNSR invariants of spaces

## Definition (Farber, Geoghegan, Schütz 2008)

Let $X$ be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X)$.

$$\Sigma^q(X,\mathbb{Z}) := \{\chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\} \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \le q\}$$

- $G$ of type $\mathrm{FP}_k \implies \Sigma^q(G,\mathbb{Z}) = \Sigma^q(K(G,1),\mathbb{Z}), \forall q \le k$.
- If $X$ is a *finite* CW-complex, definition coincides with FGS's.

## Theorem

*If $X$ has finite $k$-skeleton, then, for every $q \le k$,*

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\tau_1^{\mathbb{R}}\big(\bigcup_{i\le q} \mathcal{V}_1^i(X,\mathbb{C})\big)\Big)^{\complement}.$$

# Σ-invariants, valuations, and characteristic varieties

- Connection between valuations and BNSR invariants (of metabelian groups) first explored by Bieri–Groves (1984).
- Delzant (2009) noticed a further connection to char. vars.
- We generalize Delzant's result (valid only for $X = K(G,1)$, $q = k = 1$, and $v$ a discrete valuation):

### Theorem

*Let $X$ be a connected CW-complex with finite $k$-skeleton ($k \geq 1$), and set $G = \pi_1(X)$. Assume:*

1. $\rho\colon G \to \Bbbk^\times$ *is a homomorphism such that* $\rho \in \bigcup_{i\leq q} \mathcal{V}^i_1(X, \Bbbk)$, *for some* $q \leq k$.
2. $v\colon \Bbbk^\times \to \mathbb{R}$ *is a valuation on* $\Bbbk$ *such that* $v \circ \rho \neq 0$.

*Then*

$$v \circ \rho \notin \Sigma^q(X, \mathbb{Z})$$

We also prove a (partial) converse:

### Theorem

*Assume:*

1. $\chi\colon G \xrightarrow{\xi} \Gamma \xrightarrow{\iota} \mathbb{R}$ *is an additive character such that* $-\chi \in \Sigma^q(X, \mathbb{Z})$*, for some* $q \le k$*.*
2. $\rho\colon \Gamma \to \mathbb{k}^\times$ *is a character which is* not *an algebraic integer.*

*Then*

$$\rho \circ \xi \notin \bigcup_{i \le q} \mathcal{V}^i_1(X, \mathbb{k})$$

Here, $\rho$ is an *algebraic integer* if $\exists\ \Delta = \sum n_\gamma \gamma \in \mathbb{Z}\Gamma$ with

$$\Delta(\rho) = 0 \quad \text{and} \quad n_{\gamma_0} = 1,$$

where $\gamma_0$ is the greatest element of $\iota(\text{supp}(\Delta))$.

# Resonance varieties

Let $A = H^*(X, \Bbbk)$. If $\operatorname{char} \Bbbk = 2$, assume $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$. Get cochain complex

$$(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

### Definition (Falk 1997, Matei–S. 2000)

The *resonance varieties* of $X$ (over $\Bbbk$):

$$\mathcal{R}^i_d(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, \cdot a) \geq d\}$$

Homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$: $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots$

### Theorem (Libgober 2002)

$$TC_1(\mathcal{V}^i_d(X, \mathbb{C})) \subseteq \mathcal{R}^i_d(X, \mathbb{C})$$

Equality does not hold in general (Matei–S. 2002)

# Formality

### Definition (Quillen 1969)

A group $G$ is 1*-formal* if its Malcev Lie algebra, $\mathfrak{m}_G = \text{Prim}(\widehat{\mathbb{Q}G})$, is quadratic.

### Definition (Sullivan 1977)

A space $X$ is *formal* if its minimal model is quasi-isomorphic to $(H^*(X,\mathbb{Q}),0)$.

$X$ formal $\implies \pi_1(X)$ is 1-formal.

# Tangent cone theorem

## Theorem (Dimca, Papadima, S. 2009)

*If $G$ is 1-formal, then* $\exp\colon (\mathcal{R}^1_d(G,\mathbb{C}),0) \xrightarrow{\simeq} (\mathcal{V}^1_d(G,\mathbb{C}),1)$. *Hence*

$$\tau_1(\mathcal{V}^1_d(G,\mathbb{C})) = TC_1(\mathcal{V}^1_d(G,\mathbb{C})) = \mathcal{R}^1_d(G,\mathbb{C})$$

In particular, $\mathcal{R}^1_d(G,\mathbb{C})$ is a union of rationally defined subspaces in $H^1(G,\mathbb{C}) = \operatorname{Hom}(G,\mathbb{C})$.

## Example

Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1,x_2], [x_1,x_4][x_2^{-2},x_3], [x_1^{-1},x_3][x_2,x_4]\rangle$. Then

$$\mathcal{R}^1_1(G,\mathbb{C}) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$$

splits into subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, $G$ is *not* 1-formal.

## Example

- $X = F(\Sigma_g, n)$: the configuration space of $n$ labeled points of a Riemann surface of genus $g$ (a smooth, quasi-projective variety).
- $\pi_1(X) = P_{g,n}$: the pure braid group on $n$ strings on $\Sigma_g$.

Using computation of $H^*(F(\Sigma_g, n), \mathbb{C})$ by Totaro (1996), get

$$\mathcal{R}^1_1(P_{1,n}, \mathbb{C}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j < n \end{array} \right\}$$

For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll).
Hence, $P_{1,n}$ is not 1-formal.

## Proposition

*Suppose G is a 1-formal group. Then the BNS invariant of G is contained in the complement of a finite union of linear subspaces defined over $\mathbb{Q}$:*

$$\Sigma^1(G) \subseteq \mathcal{R}^1_1(G, \mathbb{R})^{\complement}.$$

This inclusion may be strict:

## Example

Let $G = \langle a, b \mid aba^{-1} = b^2 \rangle$ be the Baumslag-Solitar group. Then:

- $b_1(G) = 1$, thus $G$ is 1-formal.
- $\Sigma^1(G) = (-\infty, 0)$.
- $\mathcal{R}^1_1(G, \mathbb{R}) = \{0\}$.

## Corollary

*If G is 1-formal, and $\mathcal{R}^1_1(G, \mathbb{R}) = H^1(G, \mathbb{R})$, then $\Sigma^1(G) = \emptyset$.*

# 3-manifolds

Let $M$ be a closed, orientable 3-manifold.

## Proposition (Dimca–S. 2009)

*If $b_1(M)$ is even, then $\mathcal{R}^1_1(M,\mathbb{C}) = H^1(M,\mathbb{C})$.*

## Corollary

*If $b_1(M)$ is even, and $G = \pi_1(M)$ is 1-formal, then $\Sigma^1(G) = \emptyset$, and so $M$ does* not *fiber over $S^1$.*

Hence: $f\colon M \to S^1$ fibration, $b_1(M)$ even $\implies$ $M$ is not formal.

## Remark

A different bound for the BNS invariant in terms of the Alexander polynomial was obtained by McMullen (2002):

$$\|\phi\|_A \leq \|\phi\|_T, \quad \forall \phi \in H^1(M,\mathbb{Z})$$

# Toric complexes and right-angled Artin groups

## Definition

Let $L$ be simplicial complex on $n$ vertices. The associated *toric complex*, $T_L$, is the subcomplex of $n$-torus obtained by deleting the cells corresponding to the missing simplices of $L$.

- Special case of “generalized moment angle complex”.
- $\pi_1(T_L)$ is the *right-angled Artin group* associated to graph $\Gamma = L^{(1)}$:

$$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma)\rangle.$$

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where $\Delta_\Gamma$ is the *flag complex* of $\Gamma$.
  (Davis–Charney 1995, Meier–VanWyk 1995)
- $T_L$ is formal. (Notbohm–Ray 2005)

Using a result of Aramova, Avramov, Herzog (2000), we showed:

### Theorem (Papadima–S. 2009)

$$\mathcal{R}^i_d(T_L,\Bbbk) = \bigcup_{\substack{\mathsf{W}\subset\mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\Bbbk)\geq d}} \Bbbk^{\mathsf{W}},$$

*where $L_{\mathsf{W}}$ is the subcomplex induced by $L$ on $\mathsf{W}$, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex $\sigma$ in a subcomplex $K \subseteq L$.*

Using (1) resonance upper bound, and (2) computation of $\Sigma^k(G_\Gamma,\mathbb{R})$ by Meier, Meinert, VanWyk (1998), we get:

### Corollary

*For all $k \geq 0$,*

1. $\Sigma^k(T_L,\mathbb{Z}) \subseteq \Big(\bigcup_{i\leq k}\mathcal{R}^i_1(T_L,\mathbb{R})\Big)^{\complement}$.
2. $\Sigma^k(G_\Gamma,\mathbb{R}) = \Big(\bigcup_{i\leq k}\mathcal{R}^i_1(T_{\Delta_\Gamma},\mathbb{R})\Big)^{\complement}$.

# Kähler and quasi-Kähler manifolds

- A compact, connected, complex manifold $M$ is *Kähler* if there is a Hermitian metric $h$ such that $\omega = \mathfrak{Im}(h)$ is a closed 2-form.
- A manifold $X$ is called *quasi-Kähler* if $X = M \setminus D$, where $M$ is Kähler and $D$ is a divisor with normal crossings.

Formality properties:

- $M$ Kähler $\Rightarrow$ $M$ is formal
  (Deligne, Griffiths, Morgan, Sullivan 1975)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal
  (Brieskorn 1973)
- $X$ quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal
  (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal
  (Kohno 1983)

# Cohomology jumping loci

Let $X$ be a quasi-Kähler manifold, $G = \pi_1(X)$.

## Theorem (Arapura 1997)

*All components of* $\mathcal{V}^i_d(X, \mathbb{C})$ *passing through* 1 *are subtori of* $\operatorname{Hom}(G, \mathbb{C}^\times)$*, provided*

1. $i = d = 1$*, or*
2. $X$ *is Kähler, or*
3. $W_1(H^1(X, \mathbb{C})) = 0$*.*

### Theorem (Dimca, Papadima, S. 2009)

*Let* $\{\mathcal{V}^{\alpha}\}_{\alpha}$ *be the irred components of* $\mathcal{V}^1_1(G)$ *containing* $1$*. Set* $\mathcal{T}^{\alpha} = TC_1(\mathcal{V}^{\alpha})$*. Then:*

1. *Each* $\mathcal{T}^{\alpha}$ *is a p-isotropic subspace of* $H^1(G, \mathbb{C})$*, of* $\dim \geq 2p+2$*, for some* $p = p(\alpha) \in \{0, 1\}$*.*
2. *If* $\alpha \neq \beta$*, then* $\mathcal{T}^{\alpha} \bigcap \mathcal{T}^{\beta} = \{0\}$*.*

*Assume further that G is* $1$*-formal. Let* $\{\mathcal{R}^{\alpha}\}_{\alpha}$ *be the irred components of* $\mathcal{R}^1_1(G)$*. Then:*

3. $\{\mathcal{T}^{\alpha}\}_{\alpha} = \{\mathcal{R}^{\alpha}\}_{\alpha}$.
4. $\mathcal{R}^1_d(G) = \{0\} \cup \bigcup_{\alpha:\ \dim \mathcal{R}^{\alpha} > d + p(\alpha)} \mathcal{R}^{\alpha}$.

# Σ-invariants

## Theorem

1. $\Sigma^1(G) \subseteq TC_1^{\mathbb{R}}(\mathcal{V}_1^1(G,\mathbb{C}))^{\complement}$.
2. *Suppose $X$ is Kähler, or $W_1(H^1(X,\mathbb{C})) = 0$. Then $\mathcal{R}_1^1(G,\mathbb{R})$ is a finite union of rationally defined linear subspaces, and $\Sigma^1(G) \subseteq \mathcal{R}_1^1(G,\mathbb{R})^{\complement}$.*

## Example

Assumption from (2) is necessary: Let $X$ be the complex Heisenberg manifold: bundle $\mathbb{C}^\times \to X \to (\mathbb{C}^\times)^2$ with $e = 1$. Then:

1. $X$ is a smooth quasi-projective variety;
2. $G = \pi_1(X)$ is nilpotent (and not 1-formal);
3. $\Sigma^1(G) = \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{R}_1^1(G,\mathbb{R}) = \mathbb{R}^2$.

Thus, $\Sigma^1(G) \not\subseteq \mathcal{R}_1^1(G,\mathbb{R})^{\complement}$.

For Kähler manifolds, we can say precisely when the resonance upper bound for $\Sigma^1$ is attained.

### Theorem

*Let $M$ be a compact Kähler manifold with $b_1(M) > 0$, and $G = \pi_1(M)$. The following are equivalent:*

1. $\Sigma^1(G) = \mathcal{R}^1_1(G, \mathbb{R})^{\complement}$.
2. *If $f\colon M \to C$ is an elliptic pencil (i.e., a holomorphic map onto an elliptic curve, with connected generic fiber), then $f$ has no fibers with multiplicity $> 1$.*

Proof uses results of Arapura (1997), DPS (2009), and Delzant (to appear).

# Hyperplane arrangements

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^n$, with complement $X = \mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H$.

## Theorem

*For all* $q \geq 0$,

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\bigcup_{i\leq q} \mathcal{R}^i_1(X,\mathbb{R})\Big)^{\complement}$$

## Question

For which arrangements $\mathcal{A}$ is the resonance upper bound attained?