

Lower central series and free resolutions of arrangements

Alex Suciu (Northeastern)

www.math.neu.edu/~suciu

joint work with

Hal Schenck (Texas A&M)

www.math.tamu.edu/~schenck

available at: math.AG/0109070

Special Session on Algebraic and
Topological Combinatorics

A.M.S. Fall Eastern Section Meeting

Williamstown, MA

October 13, 2001

Lower central series

G finitely-generated group.

- LCS: $G = G_1 \geq G_2 \geq \cdots, G_{k+1} = [G_k, G]$
- LCS quotients: $\text{gr}_k G = G_k / G_{k+1}$
- LCS ranks: $\phi_k(G) = \text{rank}(\text{gr}_k G)$

Hyperplane arrangements

$\mathcal{A} = \{H_1, \dots, H_n\}$ set of hyperplanes in \mathbb{C}^ℓ .

- Intersection lattice: $L(\mathcal{A}) = \{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$
- Complement: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$

Many topological invariants of $M = M(\mathcal{A})$ are determined by the combinatorics of $L(\mathcal{A})$. E.g.:

- Cohomology ring:
 $A := H^*(M, \mathbb{Q}) = E/I$ (Orlik-Solomon algebra)
- Betti numbers and Poincaré polynomial:
 $b_i := \dim H_i(M, \mathbb{Q}) = \sum_{X \in L_i(\mathcal{A})} (-1)^i \mu(X)$
 $P(M, t) := \text{Hilb}(A, t) = \sum_{i=0}^{\ell} b_i t^i$

$G = \pi_1(M)$ is not always combinatorially determined. Nevertheless, its LCS ranks are determined by $L(\mathcal{A})$.

Problem. Find an *explicit* combinatorial formula for the LCS ranks, $\phi_k(G)$, of an arrangement group G (at least for certain classes of arrangements).

LCS formulas

- **Witt formula**

$\mathcal{A} = \{n \text{ points in } \mathbb{C}\}$

$G = F_n$ (free group on n generators)

$\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}$, or:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = 1 - nt$$

- **Kohno [1985]**

$\mathcal{B}_\ell = \{z_i - z_j = 0\}_{1 \leq i < j \leq \ell}$ braid arrangement in \mathbb{C}^ℓ

$G = P_\ell$ (pure braid group on ℓ strings)

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \prod_{j=1}^{\ell-1} (1 - jt)$$

- **Falk-Randell LCS formula [1985]**

If \mathcal{A} fiber-type $[\iff L(\mathcal{A})$ supersolvable (Terao)]

with exponents d_1, \dots, d_ℓ

$$G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$$

$$\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i})$$

and so:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P(M, -t)$$

- **Shelton-Yuzvinsky [1997], Papadima-Yuz [99]**

If \mathcal{A} Koszul (i.e., $A = H^*(M, \mathbb{Q})$ is a Koszul algebra) then the LCS formula holds.

Remark. There are many arrangements for which the LCS formula fails. In fact, as noted by Peeva, there are arrangements for which

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k} \neq \text{Hilb}(N, -t),$$

for any graded-commutative algebra N .

LCS and free resolutions

We want to reduce the problem of computing $\phi_k(G)$ to that of computing the graded Betti numbers of certain free resolutions involving the OS-algebra $A = E/I$.

The starting point is the following (known) formula:

$$\prod_{k=1}^{\infty} (1 - t^k)^{-\phi_k} = \sum_{i=0}^{\infty} b_{ii} t^i$$

where $b_{ij} = \dim_{\mathbb{Q}} \text{Tor}_i^A(\mathbb{Q}, \mathbb{Q})_j$ is the i^{th} Betti number (in degree j) of a minimal free resolution of \mathbb{Q} over A :

$$\cdots \longrightarrow \bigoplus_j A^{b_{2j}}(-j) \longrightarrow A^{b_1}(-1) \xrightarrow{(e_1 \cdots e_{b_1})} A \longrightarrow \mathbb{Q} \rightarrow 0$$

Betti diagram:

$$\begin{array}{cccccc} 0 : & 1 & b_1 & b_{22} & b_{33} & \dots & \leftarrow \text{linear strand} \\ 1 : & . & . & b_{23} & b_{34} & \dots & \\ 2 : & . & . & b_{24} & b_{35} & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \end{array}$$

Formula follows from:

- Sullivan: M formal \implies assoc. graded Lie algebra of $G = \pi_1(M) \cong$ holonomy Lie algebra of $H_* = H_*(M, \mathbb{Q})$:

$$\begin{aligned} \text{gr } G &:= \bigoplus_{k \geq 1} G_k / G_{k+1} \otimes \mathbb{Q} \cong \\ &\mathfrak{g} := \mathbb{L}(H_1) / \text{im}(\nabla: H_2 \rightarrow H_1 \wedge H_1) \end{aligned}$$

- Poincaré-Birkhoff-Witt:

$$\prod_{k=1}^{\infty} (1 - t^k)^{-\phi_k} = \text{Hilb}(U(\mathfrak{g}), t)$$

- Shelton-Yuzvinsky:

$$U(\mathfrak{g}) = \overline{A}^!$$

- Priddy, Löfwall:

$$\overline{A}^! \cong \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})_i$$

Here $\overline{A} = E/I[2]$ is the quadratic closure of A , and $\overline{A}^!$ is its Koszul dual.

Remark. If A is a Koszul algebra, i.e.,

$$\text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})_j = 0, \quad \text{for } i \neq j,$$

then $A = \overline{A}$ and $\text{Hilb}(A^!, t) \cdot \text{Hilb}(A, -t) = 1$. This yields the LCS formula of Shelton-Yuzvinsky.

Change of rings spectral sequence

The idea now is to further reduce the computation to that of a (minimal) free resolution of A over E ,

$$\cdots \longrightarrow \bigoplus_j E^{b'_{2j}}(-j) \longrightarrow \bigoplus_j E^{b'_{1j}}(-j) \longrightarrow E \longrightarrow A \rightarrow 0$$

and its Betti numbers, $b'_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^E(A, \mathbb{Q})_j$.

This problem (posed by Eisenbud-Popescu-Yuzvinsky [1999]) is interesting in its own right.

Let

$$a_j = \#\{\text{minimal generators of } I \text{ in degree } j\}$$

Clearly, $a_2 + b_2 = \binom{b_1}{2}$. A 5-term exact sequence argument yields:

Lemma. $a_j = b'_{1j} = b_{2j}$, for all $j > 2$.

Betti diagram:

$$\begin{array}{cccccc}
 0 : & 1 & \cdot & \cdot & \cdot & \\
 1 : & \cdot & a_2 & b'_{23} & b'_{34} & \cdots \leftarrow \text{linear strand} \\
 2 : & \cdot & a_3 & b'_{24} & b'_{35} & \cdots \\
 \\
 \ell - 1 : & \cdot & a_\ell & b'_{2,\ell+1} & b'_{3,\ell+2} & \cdots \\
 \ell : & \cdot & \cdot & \cdot & \cdot &
 \end{array}$$

Key tool: Cartan-Eilenberg change-of-rings spectral sequence associated to the ring maps $E \twoheadrightarrow A \twoheadrightarrow \mathbb{Q}$:

$$\boxed{\mathrm{Tor}_i^A(\mathrm{Tor}_j^E(A, \mathbb{Q}), \mathbb{Q}) \implies \mathrm{Tor}_{i+j}^E(\mathbb{Q}, \mathbb{Q})}$$

$$\begin{array}{ccccccc}
 \mathrm{Tor}_2^E(A, \mathbb{Q}) & \leftarrow & \mathrm{Tor}_1^A(\mathrm{Tor}_2^E(A, \mathbb{Q}), \mathbb{Q}) & \leftarrow & \mathrm{Tor}_2^A(\mathrm{Tor}_2^E(A, \mathbb{Q}), \mathbb{Q}) & \leftarrow & \mathrm{Tor}_3^A(\mathrm{Tor}_2^E(A, \mathbb{Q}), \mathbb{Q}) \\
 & & & & \swarrow d_2^{2,1} & & \\
 \mathrm{Tor}_1^E(A, \mathbb{Q}) & \leftarrow & \mathrm{Tor}_1^A(\mathrm{Tor}_1^E(A, \mathbb{Q}), \mathbb{Q}) & \leftarrow & \mathrm{Tor}_2^A(\mathrm{Tor}_1^E(A, \mathbb{Q}), \mathbb{Q}) & \leftarrow & \mathrm{Tor}_3^A(\mathrm{Tor}_1^E(A, \mathbb{Q}), \mathbb{Q}) \\
 & & & & \swarrow d_2^{3,0} & & \swarrow d_3^{3,0} \\
 \mathbb{Q} & & \mathrm{Tor}_1^A(\mathbb{Q}, \mathbb{Q}) & & \mathrm{Tor}_2^A(\mathbb{Q}, \mathbb{Q}) & & \mathrm{Tor}_3^A(\mathbb{Q}, \mathbb{Q})
 \end{array}$$

The (Koszul) resolution of \mathbb{Q} as a module over E is linear, with

$$\dim \mathrm{Tor}_i^E(\mathbb{Q}, \mathbb{Q})_i = \binom{n+i-1}{i}$$

Thus, if we know $\mathrm{Tor}_i^E(A, \mathbb{Q})$, we can find $\mathrm{Tor}_i^A(\mathbb{Q}, \mathbb{Q})$, provided we can compute the differentials $d_r^{p,q}$.

We carry out this program, at least in low degrees. As a result, we express ϕ_k , $k \leq 4$, solely in terms of the resolution of A over E .

Theorem. For an arrangement of n hyperplanes:

$$\phi_1 = n$$

$$\phi_2 = a_2$$

$$\phi_3 = b'_{23}$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4$$

where

$$a_2 = \#\{\text{generators of } I_2\} = \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)}{2}$$

$$b'_{23} = \#\{\text{linear first syzygies on } I_2\}$$

$$b'_{34} = \#\{\text{linear second syzygies on } I_2\}$$

$$\delta_4 = \#\{\text{minimal, quadratic, Koszul syzygies on } I_2\}$$

ϕ_1, ϕ_2 : elementary

ϕ_3 : recovers a formula of Falk [1988]

ϕ_4 : new

Decomposable arrangements

Let \mathcal{A} be an arrangement of n hyperplanes.

Recall that $\phi_1 = n$, $\phi_2 = \sum_{X \in L_2(\mathcal{A})} \phi_2(F_{\mu(X)})$

Falk [1989]:

$$\boxed{\phi_k \geq \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})} \quad \text{for all } k \geq 3 \quad (*)$$

If the lower bound is attained for $k = 3$, we say that \mathcal{A} is *decomposable* (or *local*, or *minimal linear strand*).

Conjecture (MLS LCS). If \mathcal{A} decomposable, equality holds in (*), and so

$$\boxed{\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - t)^n \prod_{X \in L_2(\mathcal{A})} \frac{1 - \mu(X)t}{1 - t}}$$

Proposition. *The conjecture is true for $k = 4$:*

$$\phi_4 = \frac{1}{4} \sum_{X \in L_2(\mathcal{A})} \mu(X)^2 (\mu(X)^2 - 1)$$

If \mathcal{A} decomposable, we compute the entire linear strand of the resolution of A over E . If, moreover, $\text{rank } \mathcal{A} = 3$, we compute all b'_{ij} from Möbius function.

Example. $\mathcal{A} = \{H_0, H_1, H_2\}$ pencil of 3 lines in \mathbb{C}^2 .

OS-ideal generated by $\partial e_{012} = (e_1 - e_2) \wedge (e_0 - e_2)$.

Minimal free resolution of A over E :

$$\begin{array}{ccccccc}
0 & \leftarrow & A & \leftarrow & E & \xleftarrow{(\partial e_{012})} & E(-1) & \xleftarrow{(e_1 - e_2 \quad e_0 - e_2)} & E^2(-2) \\
& & & & & & \left(\begin{array}{ccc} e_1 - e_2 & e_0 - e_2 & 0 \\ 0 & e_1 - e_2 & e_0 - e_2 \end{array} \right) & & \\
& & & & & & \xleftarrow{\hspace{10em}} & E^3(-3) & \leftarrow \dots
\end{array}$$

Thus, $b'_{i,i+1} = i$, for $i \geq 1$, and $b'_{i,i+r} = 0$, for $r > 1$.

Lemma. *For any arrangement \mathcal{A} :*

$$\begin{aligned}
b'_{i,i+1} &\geq i \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X) + i - 1}{i + 1} \\
\delta_4 &\leq \sum_{(X,Y) \in \binom{L_2(\mathcal{A})}{2}} \binom{\mu(X)}{2} \binom{\mu(Y)}{2}.
\end{aligned}$$

If \mathcal{A} is decomposable, then equalities hold.

Lemma + Theorem \implies Proposition.

Example. X_3 arrangement (smallest non-LCS)

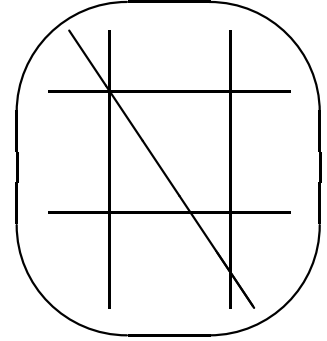
res of residue field over OS alg

total: 1 6 25 92 325 1138

0: 1 6 24 80 240 672

1: . . 1 12 84 448

2: 1 18



res of OS alg over exterior algebra

total: 1 4 15 42 97 195 354 595 942 1422 2065

0: 1

1: . 3 6 9 12 15 18 21 24 27 30

2: . 1 9 33 85 180 336 574 918 1395 2035

We find: $b'_{i,i+1} = 3i$, $b'_{i,i+2} = \frac{1}{8}i(i+1)(i^2 + 5i - 2)$.

Thus:

$$\phi_1 = n = 6$$

$$\phi_2 = a_2 = 3$$

$$\phi_3 = b'_{23} = 6$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4 = 3 + 9 - 3 = 9$$

Conjecture says: $\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - 2t)^3$

i.e.: $\phi_k(G) = \phi_k(F_2^3)$, though definitely $G \not\cong F_2^3$.

Graphic arrangements

$G = (\mathcal{V}, \mathcal{E})$ subgraph of the complete graph K_ℓ .

(Assume no isolated vertices, so that \mathcal{E} determines G .)

The *graphic arrangement* (in \mathbb{C}^ℓ) associated to G :

$$\mathcal{A}_G = \{\ker(z_i - z_j) \mid \{i, j\} \in \mathcal{E}\}$$

- $G = K_\ell \implies \mathcal{A}_G$ braid arrangement
- $G = A_{\ell+1}$ diagram $\implies \mathcal{A}_G$ Boolean arrangement
- $G = \ell$ -gon $\implies \mathcal{A}_G$ generic arrangement

Theorem. (Stanley, Fulkerson-Gross) \mathcal{A}_G is *supersolvable* $\iff \forall$ cycle in G of length > 3 has a chord.

Lemma. (Cordovil-Forge [2001], S-S)

$$a_j = \#\{\text{chordless } j + 1 \text{ cycles}\}$$

Together with a previous lemma ($a_j = b_{2j}$), this gives:

Corollary. \mathcal{A}_G supersolvable $\iff \mathcal{A}_G$ Koszul.

For arbitrary \mathcal{A} : \implies true (Shelton-Yuzvinsky)

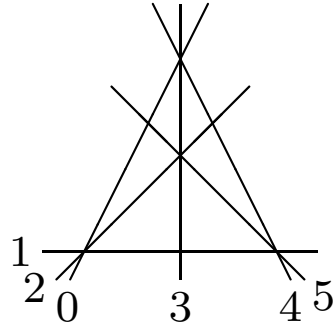
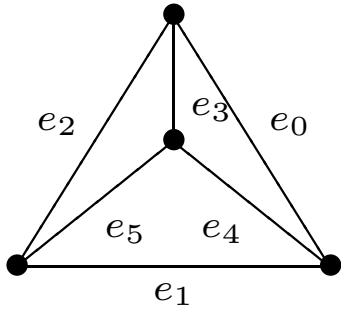
\iff open problem

Proposition. For a graphic arrangement:

$$b'_{i,i+1} = i(\kappa_3 + \kappa_4)$$

$$\delta_4 \leq \binom{\kappa_3}{2} - 6(\kappa_4 + \kappa_5)$$

Example. Braid arrangement $\mathcal{B}_4 = \mathcal{A}_{K_4}$



Free resolution of A over E :

$$0 \leftarrow A \leftarrow E \xleftarrow{\partial_1} E^4(-2) \xleftarrow{\partial_2} E^{10}(-3) \leftarrow \dots$$

$$\partial_1 = (\partial e_{145} \quad \partial e_{235} \quad \partial e_{034} \quad \partial e_{012})$$

$$\partial_2 = \begin{pmatrix} e_1 - e_4 & e_1 - e_5 & 0 & 0 & 0 & 0 & 0 & 0 & e_3 - e_0 & e_2 - e_0 \\ 0 & 0 & e_2 - e_3 & e_2 - e_5 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_4 \\ 0 & 0 & 0 & 0 & e_0 - e_3 & e_0 - e_4 & 0 & 0 & e_1 - e_5 & e_2 - e_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_2 & e_3 - e_5 & e_4 - e_5 \end{pmatrix}$$

The 2 non-local \longleftrightarrow 2-dim essential component in
linear syzygies the resonance variety $R_1(\mathcal{B}_4)$

Get: $b'_{i,i+1} = 5i$, $\delta_4 = 0$.

For a graph G , let

$$\kappa_s = \#\{\text{complete subgraphs on } s \text{ vertices}\}$$

From the Theorem, and the Proposition above, we get:

Corollary. *The LCS ranks of \mathcal{A}_G satisfy:*

$$\phi_1 = \kappa_2$$

$$\phi_2 = \kappa_3$$

$$\phi_3 = 2(\kappa_3 + \kappa_4)$$

$$\phi_4 \geq 3(\kappa_3 + 3\kappa_4 + 2\kappa_5)$$

Moreover, if $\kappa_4 = 0$, equality holds for ϕ_4 .

ϕ_3 : answers a question of Falk.

Conjecture (Graphic LCS).

$$\phi_k = \frac{1}{k} \sum_{d|k} \sum_{j=2}^{\kappa_2-1} \sum_{s=j}^{\kappa_2-1} (-1)^{s-j} \binom{s}{j} \kappa_{s+1} \mu(d) j^{\frac{k}{d}}$$

or

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \prod_{j=1}^{\kappa_2-1} (1 - jt)^{\sum_{s=j}^{\kappa_2-1} (-1)^{s-j} \binom{s}{j} \kappa_{s+1}}$$