

Clusters and Seeds

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Plan of the talk

- Recall very quickly definitions of Cluster algebras and Cluster categories.
- Discuss "Seed determines Cluster" conjecture of Fomin-Zelevinsky.
- Describe correspondences between "Cluster notions" and "Tilting notions".
- Give indications about the relation of this paper to a paper of Caldero and Keller.
- Proof of the Conjecture.

Cluster algebra \mathcal{A}

No coefficients (Fomin-Zelevinsky)

- \mathcal{A} is a subalgebra of a field of rational functions $\mathbb{Q}(u_1, \dots, u_n)$, \mathcal{A} is generated by cluster variables
- $\{\text{cluster variables}\} = \cup \{\text{elements of clusters}\}$
- a cluster = a transcendence basis obtained from an initial cluster after applying a sequences of mutations
- $\underline{x} = \{x_1, \dots, x_n\}$ = a cluster, i.e. a transcendence basis
- In order to define mutations, we also need a matrix B (equivalently a quiver Q). A pair (\underline{x}, Q) is called a cluster seed.

Cluster Mutations

- (\underline{x}, Q) a cluster seed: $\underline{x} = \{x_1, \dots, x_n\}$ a cluster, Q a quiver (a matrix (b_{ij}))
- Cluster mutation of x_i with respect to the cluster seed (\underline{x}, Q) is the new cluster variable $(x_i)^*$ obtained as

$$x_i \cdot (x_i)^* = \prod_{b_{ij} > 0} x_j^{b_{ij}} + \prod_{b'_{ij} < 0} x_j^{-b'_{ij}}.$$

- Cluster mutation of the cluster $\underline{x} = \{x_1, \dots, x_n\}$ at x_i is a new cluster $\underline{x}' = \{x_1, \dots, (x_i)^*, \dots, x_n\}$
- Cluster mutation of the cluster seed (\underline{x}, Q) at x_i is the new cluster seed (\underline{x}', Q') where \underline{x}' is the new cluster and Q' is a new quiver given by F-Z mutation formula.

Cluster determines Cluster Seed (Conjecture of Fomin-Zelevinsky)

- Start with an initial seed (\underline{x}, Q) . Perform all possible sequences of mutations.
- Conjecture: If two cluster seeds (\underline{x}', Q') and (\underline{x}'', Q'') have the same clusters, i.e. $\underline{x}' = \underline{x}''$ then they have the same quivers as well, i.e. $Q' = Q''$.
- Theorem [Caldero, Keller], [B, M, R, T] Cluster determines the cluster seed for acyclic cluster algebras with no coefficients.
- Remark: In both proofs, correspondences between cluster variables and indecomposable exceptional objects in cluster categories are used. So we recall...

Cluster category C_Λ

- Let Q be a quiver with n vertices and no oriented cycles, and \mathbb{k} a field.
- $\Lambda = \mathbb{k}Q$, the associated path algebra, $\text{mod } \Lambda$ the category of f.g. Λ -modules, $D^b := D^b(\text{mod } \Lambda)$ the derived category of bounded complexes in $\text{mod } \Lambda$.
- $[1] : D^b \rightarrow D^b$ the suspension functor,
 $\tau : D^b \rightarrow D^b$ Auslander-Reiten functor,
 $F = \tau^{-1}[1] = [1]\tau^{-1} : D^b \rightarrow D^b$ the composition functor.
- Define the cluster category C_Λ as F -orbit category. Representatives of the orbits may be chosen in

$$\text{add}(\text{ind } \Lambda \cup \{P_i[1]\}_{i=1}^n).$$

Tilting Mutations

- (T, Q_T) a tilting seed, $T = T_1 \oplus \cdots \oplus T_n$ a basic tilting object, Q_T the quiver of $\text{End}(T)^{op}$ (a matrix (b_{ij}))

- Tilting mutation of T_i with respect to the tilting seed (T, Q) is the new object T_i^* obtained from the "approximation triangles":

$$T_i^* \rightarrow B \rightarrow T_i \quad \text{and} \quad T_i \rightarrow B' \rightarrow T_i^*$$

- Tilting mutation of the tilting object $T = T_1 \oplus \cdots \oplus T_i \cdots \oplus T_n$ at T_i is a new tilting object $T' = T_1 \oplus \cdots \oplus T_i^* \cdots \oplus T_n$.
- Tilting mutation of the tilting seed (T, Q_T) at T_i is the new tilting seed $(T', Q_{T'})$.

Correspondences

- **Theorem (Fomin-Zelevinsky)** Let Q be a Dynkin diagram. Then there exists a one to one correspondence:

$$\{\textit{cluster variables}\} = \{\textit{almost positive roots}\}.$$

- **Theorem (BMRRT)** Let Q be a Dynkin diagram. Then there exists a 1 to 1 correspondence:

$$\{\textit{cluster variables}\} \rightarrow \{\textit{isoclasses ind. objects}\},$$

which induces a 1 to 1 correspondence:

$$\{\textit{clusters}\} \rightarrow \{\textit{basic tilting objects}\}.$$

- **Will consider more general case: simply laced acyclic case.**

Remark: In simply laced acyclic case, we may assume that the quiver Q is simply laced with no oriented cycles.

Existence of well defined maps

Theorem (Buan, Marsh, Reiten, T-) Suppose Q is simply laced, acyclic quiver. Then there exist well defined surjective maps:

$$\alpha : \{\text{cluster variables}\} \rightarrow \{\text{isocl.indec.except.obj.}\},$$

$$\bar{\alpha} : \{\text{clusters}\} \rightarrow \{\text{basic tilting objects}\},$$

$$\tilde{\alpha} : \{\text{cluster seeds}\} \rightarrow \{\text{tilting seeds}\}.$$

Main ingredients of the proof:

- Laurent phenomenon. (F-Z Theorem)
All cluster variables, when expressed in terms of the initial cluster variables, and reduced, are of the form $x = p/m$, where p is a polynomial and m is a monomial.

- **Positivity condition.** A polynomial f in $\{x_1, \dots, x_n\}$ is said to satisfy the positivity condition if $f(e_i) > 0$ for all $e_i = (1, \dots, 1, 0, 1, \dots, 1)$, 0 is at the i -th place.
- **Dimension condition.** A monomial $m = x_1^{k_1} \dots x_n^{k_n}$ is said to satisfy the dimension condition if there exists an indecomposable exceptional module M , such that $\dim M = (k_1, \dots, k_n)$. We denote $\delta(m) = M$.
- **(*) condition.** A cluster variable x is said to satisfy the (*) condition if it is either of the form $x = p/m$, or $x = px_i$ where p is a polynomial which satisfies the positivity condition and m is a monomial which satisfies dimension condition. In the first case we denote $\alpha(x) := \delta(m) = M$, and in the second $\alpha(x) := P_i[1]$.

- The map α is well defined on cluster variables satisfying $(*)$ condition, since exceptional objects are determined by their composition factors.
- We will show by induction that all cluster variables satisfy $(*)$ condition.
- $(\bar{*})$ condition. A cluster $\underline{x} = (x_1, \dots, x_n)$ is said to satisfy $(\bar{*})$ condition, if each cluster variable x_i satisfies $(*)$ and furthermore $\alpha(x_1) \oplus \dots \oplus \alpha(x_n)$ is a tilting object, denote it by $\bar{\alpha}(\underline{x})$.
- $(\tilde{*})$ condition. A cluster seed (\underline{x}, Q) is said to satisfy $(\tilde{*})$ condition, if the cluster \underline{x} satisfies $(\bar{*})$ and furthermore $Q = Q_{\bar{\alpha}(\underline{x})}$. Denote this tilting seed by $\tilde{\alpha}(\underline{x}, Q)$.

- First step for the induction. Cluster variables in the initial cluster seed satisfy $(*)$ condition, initial cluster satisfies $(\bar{*})$ condition, and the initial cluster seed satisfies $(\tilde{*})$ condition.
- Induction step. if all previous cluster seeds satisfy $(\tilde{*})$ condition, then after mutation, the new cluster seed satisfies $(\tilde{*})$ condition. (Main step.)
- New cluster variable corresponds to an indecomposable exceptional object. Hence, the map α is defined on that cluster variable.
- New Cluster corresponds to a Basic Tilting object. Hence $\bar{\alpha}$ defined on this cluster.

- **New Cluster seed corresponds to a Tilting seed. Hence $\tilde{\alpha}$ defined on this cluster seed.**
- **So all three maps α , $\bar{\alpha}$ and $\tilde{\alpha}$ are defined.**
- **The maps are onto.**

Proof: Cluster determines seed.

(Acyclic simply laced case)

- Initial cluster seed (\underline{x}, Q) and initial tilting seed $(\bigoplus_{i=1}^n P_i[1], Q)$ have the same quiver Q .
- The map $\tilde{\alpha} : \{\text{cluster seeds}\} \rightarrow \{\text{tilting seeds}\}$ commutes with mutations. (follows from Buan, Marsh, Reiten)
- The quiver Q in a tilting seed (T, Q) is the quiver of the endomorphism ring of the tilting object, i.e. $Q = Q_{\text{End}(T)^{op}}$, hence it is uniquely determined by the tilting object.

Bijection (Acyclic simply laced case)

- Caldero and Keller prove the existence of surjective map in the other direction:

$$\beta : \{isocl.ind.except.obj.\} \rightarrow \{cluster\ variables\}.$$

- The composition map is the identity:

$$\alpha\beta : \{isocl.ind.except.obj.\} \rightarrow \{isocl.ind.except.obj.\}$$

and preserves tilting objects.

- Summary (BCKMRT): There exists a bijection between cluster variables and isomorphism classes of indecomposable exceptional objects, taking clusters to basic tilting objects and cluster seeds to tilting seeds.