# CLUSTER CATEGORIES AND THEIR RELATION TO CLUSTER ALGEBRAS, SEMI-INVARIANTS AND HOMOLOGY OF TORSION FREE NILPOTENT GROUPS

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Abstract. 1. Cluster categories will be defined and their basic properties stated as done by Buan, Marsh, Reineke, Reiten, T. 2. Cluster algebras were introduced by Fomin and Zelevinsky. Some known relations between cluster categories and combinatorics of cluster algebras will be stated, as well as some of the open questions. 3. Semi-invariants for quivers were studied by Schofield, Derksen and Weyman. Generalized semi-invariants will be defined and the theorems relating domains of such semi-invariants and the simplicial complexes associated to cluster categories will be given. 4. The same simplicial complexes associated to cluster categories are related to the Igusa-Orr pictures in the homology of nilpotent groups. Results, and many open questions in this direction will be stated.

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Dedicated to:
Maria Ines Platzeck for her 60th birthday
Hector Merklen for his 70th birthday

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## 1. Cluster Categories

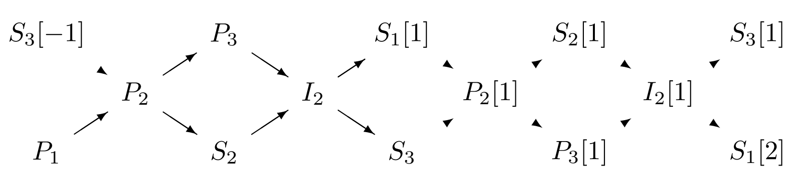
1.1. **A motivation.** Cluster categories were introduced as convenient categories for studying combinatorics of cluster algebras, (from a different point of view).

1.2. **Cluster categories.** We recall the standard definitions and notation:

- Let $Q$ be a quiver with $n$ vertices and no oriented cycles and $\Bbbk$ a field.
- Then $\Lambda = \Bbbk Q$, the associated path algebra is hereditary.
- Let $\operatorname{mod}\Lambda$ be the category of finitely generated $\Lambda$-modules, and
- $D^b := D^b(\operatorname{mod}\Lambda)$ the derived category of bounded complexes in mod$\Lambda$, which is easy to describe since $\Lambda$ is hereditary:
  $Objects$ in $D^b$ are $\cup_i\{(\operatorname{mod}\Lambda[i])\}$,
  $Morphisams$ in $D^b$ are given by:
  $\mathrm{Hom}_{D^b}(M,N) :=\mathrm{Hom}_\Lambda(M,N)$ for $M,N$ mod$\Lambda$,
  $\mathrm{Hom}_{D^b}(M,N[1]) :=\mathrm{Ext}_\Lambda(M,N)$ for $M,N$ mod$\Lambda$,
  $\mathrm{Hom}_{D^b}(M,N[i]) := 0$ for $M,N$ mod$\Lambda$ and all $i \neq 0,1$
  $\mathrm{Hom}_{D^b}(X[i],Y[i]) :=\mathrm{Hom}_{D^b}(X,Y)$ for $X,Y$ in $D^b$.
- Representatives of the orbits may be chosen in add(ind $\Lambda \cup \{P_i[1]\}_{i=1}^n$).
- $[1] : D^b \to D^b$ the suspension functor,
- $\tau : D^b \to D^b$ the Auslander-Reiten functor, and
- $F = \tau^{-1}[1] = [1]\tau^{-1} : D^b \to D^b$ the composition functor.
- Define the *cluster category* $C_\Lambda$:
  $Objects$ in $C_\Lambda$ are $F$-orbits,
  $Morphisams$ in $C_\Lambda$: Let $\tilde{X}$ and $\tilde{Y}$ be $F$-orbits of $X$ and $Y$. Then
  $Hom_{C_\Lambda}(\tilde{X},\tilde{Y}) := \coprod_i Hom_{D^b}(F^iX,Y)$.
- Representatives of the orbits may be chosen in add(ind $\Lambda \cup \{P_i[1]\}_{i=1}^n$).
- We will use the same symbol for the objects in $D^b$ and their orbits in $C_\Lambda$.

1.3. **An example.** Let $Q$ be the Dynkin diagram of type $A_3$ with the following orientation and labeling of the vertices: $3 \to 2 \to 1$.

- Let $S_1, S_2, S_3$ be the simple $Q$ representations, or simple $\Bbbk Q$-modules.
- Let $P_1, P_2, P_3$ be the projective $Q$ representations, or projective $\Bbbk Q$-modules.
- Let $I_1, I_2, I_3$ be the injective $Q$ representations, or injective $\Bbbk Q$-modules.
- Consider the Auslander-Reiten quiver in mod$\Lambda$ and $D^b$:




- Representatives for the $F$-orbits may be chosen to be the following 9 objects: $\{S_1, P_2, P_3, S_2, I_2, S_3, S_1[1], P_2[1], P_3[1]\}$

1.4. **Tilting objects and Tilting seeds.**

- Define $Ext_{C_\Lambda}(\tilde{X},\tilde{Y}) := \coprod_i Ext_{D^b}(F^iX,Y) = \coprod_i Hom_{D^b}(F^iX,Y[1])$.
- Define *Exceptional objects* in $C_\Lambda$ to be $X$ such that $Ext_{C_\Lambda}(X,X) = 0$.
- Define *Tilting objects (basic)* as maximal exceptional objects.
- Any basic tilting object has $n$ indecomposable summands.
- Let $T = T_1 \coprod \ldots \coprod T_n$ be a basic tilting object, with all $T_i$ indecomposable.

- Define *Tilting seed* as a pair $(T, Q_T)$, where $Q_T$ is the quiver of $\mathrm{End}_{C_\Lambda}(T)^{op}$

1.5. **Exchange pairs.** The following theorems give precise conditions when an indecomposable summand of a tilting object can be replaced by another indecomposable object, i.e. they "can be exchanged".

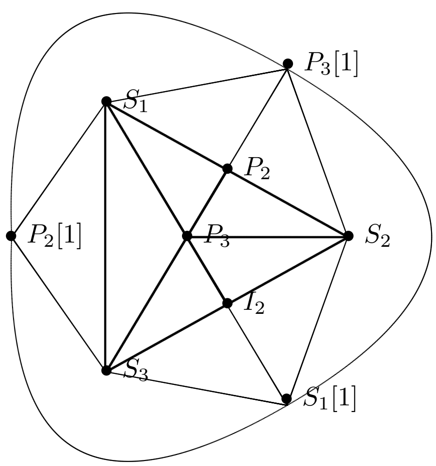
**Theorem 1.5.1** (BMRRT). *Let $T = M \coprod \bar{T}$ be a basic tilting object with $M$ indecomposable. Then, there exists exactly one object $M^*$ not isomorphic to $M$, such that $T' = M^* \coprod \bar{T}$ is also a basic tilting object.*

**Definition 1.5.2.** Using the notation from Theorem 1.5.1 we call the indecomposable objects $M$ and $M^*$ an *exchange pair* with respect to $T$ (or $T'$).

**Theorem 1.5.3** (BMRRT). *Two exceptional indecomposable objects $M$ and $M^*$ form an exchange pair if and only if* $\dim_{D_M} \mathrm{Ext}^1_{C_\Lambda}(M, M^*) = 1 = \dim_{D_{M^*}} \mathrm{Ext}^1_{C_\Lambda}(M, M^*)$.

1.6. **Tilting objects in the cluster category for $A_3$.**

- Vertices are labeled by representatives of indecomposable objects in the cluster category.
- Vertices are connected by a line if there are no extensions between the two objects.
- Each small triangle defines a tilting object and all tilting objects are given that way. There are 14 tilting objects in this example.




1.7. **Tilting mutations.** We talk about: mutation of an indecomposable object with respect to a tilting module or mutation of a tilting module at a particular indecomposable summand or mutation of a tilting seed at a particular indecomposable summand. Using the notation from Thm 1.5.1, we give the following definitions:

- $\nu(M) = M^*$ means: *Tilting mutation of $M$ with respect to $T$* is $M^*$.
- $\bar{\nu}(T) = \bar{\nu}(M \coprod \bar{T}) = \nu(M) \coprod \bar{T} = M^* \coprod \bar{T} = T'$ means: *Tilting mutation of the tilting object $T = M \coprod \bar{T}$ at $M$ is $T' = M^* \coprod \bar{T}$.*
- $\tilde{\nu}(T, Q_T) = (\bar{\nu}(T), Q_{\bar{\nu}(T)})$ means: *Tilting mutation of the tilting seed $(T, Q_T)$ at $M$ is the new seed $(T', Q_{T'})$.*

1.8. **Construction of exchange pairs.** We now describe construction of $M^*$ and state some of the facts and commutative diagrams from [BMRRT], which will be used in the proofs.

Let $T = M \coprod \bar{T}$ be a tilting object.

Let $f : B \to M$ be a minimal right $add\bar{T}$- approximation of $M$.

Let $M^* \to B \to M$ be the associated triangle in $D^b$.

1.9. **Some facts about exchange pairs.**

(1) $M^*$ is indecomposable exceptional.
(2) $M^* \to B$ is a minimal left $add\bar{T}$- approximation of $M^*$.
(3) $\dim_{D_M} \mathrm{Ext}^1_{C_\Lambda}(M, M^*) = 1 = \dim_{D_{M^*}} \mathrm{Ext}^1_{C_\Lambda}(M, M^*)$.
(4) $M$ and $M^*$ form an exchange pair with respect to $T$ (or $T'$).
(5) We may consider the AR-triangle for $M^*$ and get the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & M^* & \longrightarrow & A_{M^*} & \longrightarrow & \tau^{-1}M^* & \longrightarrow & & \\
 & {\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & & \\
\longrightarrow & M^* & \longrightarrow & B & \longrightarrow & M & \longrightarrow & & \\
 & & & & & \downarrow & & & \\
 & & & & & B' & & & \\
 & & & & & \downarrow & & & \\
 & & & & & \tau^{-1}M^*[1] = FM^* \cong_{\mathcal{C}_\Lambda} M^* & & &
\end{array}$$

(6) $\dim \mathrm{Hom}_{D^b}(\tau^{-1}M^*, M) = 1$
(7) $B' = Coker(h) \coprod Ker(h)[1] := L_h \coprod K_h[1]$
(8) $h$ is isomorphism if and only if $B' = 0$
(9) $M^* \cong \tau M$ if and only if $M^* \to B \to M$ is AR triangle if and only if $B' = 0$
(10) The vertical maps $M \to B'$ and $B' \to M^*$ are $add\bar{T}$-approx. of $M$ and $M^*$.
(11) Can also consider AR triangle for $M$ and get similar facts.

## 2. Cluster Algebras and Cluster Categories

### 2.1. **Cluster algebras.**

2.1.1. **Definition.** Cluster algebras were defined by Fomin and Zelevinsky. $\mathcal{A}$ will be the cluster algebra,

- $\mathbb{Q}(u_1, \ldots, u_n)$ field of rational functions
- $\mathcal{A}$ is subalgebra generated by cluster variables
- $\{$cluster variables$\} = \cup\{$elements of clusters$\}$
- $\{$clusters$\} = \{$certain transcendence bases in $\mathbb{Q}(u_1, \ldots, u_n)$ obtained from an initial cluster after applying sequences of mutations in every possible direction, as we will describe$\}$
- $\underline{x} = \{x_1, \ldots, x_n\}$ = initial cluster (a transcendence basis)
- $(\underline{x}, Q)$ = initial cluster seed;
- $\underline{x}$ initial cluster, $Q$ initial quiver (a matrix $(b_{ij})$)

**Definition 2.1.1.** Using the notation from above we define:

- *Cluster mutation of* $x_i$ with respect to the cluster seed $(\underline{x}, Q)$ is the new cluster variable $(x_i)^*$ obtained as $x_i \cdot (x_i)^* = \prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b'_{ij}<0} x_i^{b'_{ij}}$.
  Notation $\mu(x_i) = (x_i)^*$.
- *Cluster mutation of the cluster* $\underline{x} = \{x_1, \ldots, x_n\}$ at $x_i$ is a new cluster $\underline{x}' = \{x_1, \ldots, (x_i)^*, \ldots x_n\} = \{x_1, \ldots, \mu_i(x_i), \ldots x_n\}$
  Notation: $\bar{\mu}_i(\underline{x}) = \{x_1, \ldots, \mu_i(x_i), \ldots x_n\} = \underline{x}'$
- *Cluster mutation of the cluster seed* $(\underline{x}, Q)$ *at* $x_i$ is the new cluster seed $(\underline{x}', Q')$.
  Notation: $\tilde{\mu}_i(\underline{x}, Q) = (\bar{\mu}_i(\underline{x}), \vec{\mu}_i Q) = (\underline{x}', Q')$, where $\vec{\mu}_i(Q) = Q'$ new quiver obtained in quite a complicated way .

**Theorem 2.1.2** (FZ)**.** *A cluster algebra has finitely many cluster varibles, if and only if a Dynkin diagram appears as one of the quivers in the mutation process.*

2.2. **Acyclic cluster algebras. Definition:** A cluster algebra is called *acyclic*, if in the cluster mutation process a quiver with no oriented cycles appears.
**Remark:** We will only consider acyclic case, so we may assume that the quiver in the initial seed has no oriented cycles.

2.3. **Some relations between Cluster algebras and Cluster categories.** We now recall the following notions from the introduction:

$Q$ quiver

| | |
|---|---|
| $\mathcal{A}(Q)$ cluster algebra | $C_\Lambda$ cluster category, where $\Lambda = \Bbbk Q$ |
| $(\underline{x}, Q)$ initial cluster seed | $(P_1[1] \coprod \cdots \coprod P_n[1], Q)$ initial tilting seed |
| $(\underline{x}', Q')$ cluster seed | $(T, Q_T)$ tilting seed |
| $\underline{x}' = \{x'_1, \ldots, x'_n\}$ cluster | $T = T_1 \coprod \cdots \coprod T_n$ tilting object |
| $x'_i$ cluster variable | $M$ indecomposable exceptional object |

We want to discuss now what kind of correspondences hold and in which situations. First we will recall known results and then state and prove a new theorem from [BMRT].

**Theorem 2.3.1** (FZ,FMZ)**.** *Suppose $Q$ is a Dynkin diagram. Then:*
$|\{$ *cluster variables*$\}| = |\{$ *isomorphism classes of indecomposable objects* $\}|$.

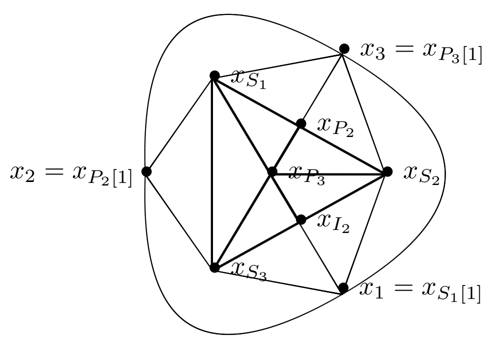
**Theorem 2.3.2** (BMRRT)**.** *Suppose $Q$ is a simply laced Dynkin diagram. Then there exist one to one correspondences (the second one being induced by the first):*
$\{$*cluster variables*$\} \to \{$*isomorphism classes of indecomposable objects*$\}$,
$\{$*clusters*$\} \to \{$*basic tilting objects*$\}$.

2.4. **General simply laced diagram. Conjecture**: [BMRRT] Suppose $Q$ is a simply laced diagram. Then there exist one to one correspondence (second one induced by the first):
$\{$cluster variables$\} \to \{$isomorphism classes of indecomposable exceptional objects$\}$,
$\{$clusters$\} \to \{$basic tilting objects$\}$.

The following theorem will be proved at the end of these lectures, however we need a few more definitions, and the main step of the proof will be the Proposition that we will prove in the next section.

**Theorem 2.4.1** (BMRT)**.** *Let $Q$ be simply laced, acyclic diagram.*
*(a) There exist well defined maps:*
$\alpha : \{$*cluster variables*$\} \to \{$*isoclasses of indecomp. exceptional objects*$\}$,
$\bar{\alpha} : \{$*clusters*$\} \to \{$*tilting objects*$\}$,
$\tilde{\alpha} : \{$*cluster seeds*$\} \to \{$*tilting seeds*$\}$.
*(b) All three maps: $\alpha, \bar{\alpha}, \tilde{\alpha}$ are onto.*

**An illustration on** $A_3$ Since $A_3$ is a Dynkin diagram, hence of finite type, all three maps $\alpha$, $\bar{\alpha}$ and $\tilde{\alpha}$ are in fact one-to-one correspondences.




- Vertices are labeled by cluster variables and also by representatives of indecomposable objects in the cluster category.
- Vertices are connected by a line if the compatibility degree between the cluster variables is 0 and if there are no extensions between the two objects.
- Each small triangle defines a cluster and a basic tilting object
- All clusters and all tilting objects are given that way.
- There are 14 clusters and 14 basic tilting objects in this example.
- The initial cluster $\{x_1, x_2, x_3\}$ is chosen in such a way, that the denominators of all other cluster variables are given by the dimension vectors of the corresponding $\Bbbk Q$-modules. (see 2.5.3)

2.5. **Monomials in the denominators of cluster variables.** First we state the famous "Laurent phenomenon" theorem of Fomin and Zelevinsky, and also some known results about the monomials which appear in the denominators of the cluster variables.

**Theorem 2.5.1** (Fomin-Zelevinsky)**.** *(Laurent phenomenon) Denominators of all cluster variables, when expressed in terms of the initial cluster, and then reduced, are monomials.*

Notation: Let $m = x_1^{k_1} \dots x_n^{k_n}$ be a monomial in variables $x_1, \dots, x_n$. Denote the exponent vector by $\epsilon(m) = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$.

**Theorem 2.5.2** (Fomin-Zelevinsky)**.** *Let $Q$ be a Dynkin diagram with alternating orientation. For each cluster variable in reduced form $f/m$, there is an indecomposable module $M$, such that $\epsilon(m) = \underline{dim} M$.*

**Theorem 2.5.3** (Caldero-Chapotlon-Schiffler)**.** *Let $Q$ be any Dynkin diagram. For each cluster variable in reduced form $f/m$, there is an indecomposable module $M$, such that $\epsilon(m) = \underline{dim} M$.* (Also [RT] and [Chapotlon, Keller] have the same result.)

2.6. **Positivity condition and Condition $(*)$.** These are essential notions for the proofs of the existence of the maps $\alpha$, $\bar{\alpha}$, $\tilde{\alpha}$, i.e. that the objects $\alpha(x)$, $\bar{\alpha}(\underline{x'})$, $\tilde{\alpha}((\underline{x'}, Q'))$ satisfy the desired conditions.

**Definition 2.6.1.** A polynomial $f$ in variables $\{x_1, \dots, x_n\}$ is said to satisfy the *positivity condition* if $f(e_i) > 0$ for all $e_i = (1, 1, \dots, 1, 0, 1, \dots, 1, 1)$, 0 is at the $i$-th place.

**Remark:** If $f$ satisfies the positivity condition, it does not have any non-constant monomial factors.

**Definition 2.6.2.** A cluster variable $x$ is said to satisfy *condition* $(*)$ if: either $x = f/m$ or $x = fx_i$, where $f$ satisfies positivity condition and if $x = f/m$, then there exists an indecomposable exceptional module $M$, such that $\epsilon(m) = \underline{dim} M$.

**Definition 2.6.3.** If a cluster variable $x$ satisfies condition $(*)$, we (can) define:

$$\alpha(x) = \begin{cases} M & \text{if } x = f/m \text{ and } \epsilon(m) = \underline{dim} M \\ P_i[1] & \text{if } x = fx_i. \end{cases}$$

**Remark:**

- We want to define $\alpha$ on all cluster variables.
- So far, we defined only on cluster variables satisfying $(*)$.
- So, we prove the following theorem.

**Theorem 2.6.4.** *(BMRT) All cluster variables satisfy condition $(*)$, if the cluster algebra is acyclic with no coefficients.*

*Proof.* This theorem will be proved after the Proposition 2.7.1 and Corollary 2.7.3, however the proof involves clusters and seeds as well, corresponding conditions on those: $(\bar{*})$,$(\tilde{*})$, and we prove a more general theorem about cluster variables, clusters and cluster seeds. ☐

2.7. **Properties: $(*)$, $(\bar{*})$, $(\tilde{*})$.** Definitions and properties.
$(*)$ is a property of a cluster variable $x_i'$ consisting of two parts:

- $x_i' = f/m$ or $x_i' = fx_i$, where $f$ satisfies positivity condition, and
- $\epsilon(m) = \underline{dim}M$, for some indecomposable exceptional module $M$.
- **Notice:** If $x_i'$ satisfies $(*)$, then $\alpha(x_i')$ is defined (i.e. it is an indecomposable exceptional object).

$(\bar{*})$ is a property of a cluster $\underline{x'} = \{x_1', \ldots, x_n'\}$ consisting of two parts:

- each cluster variable $x_i'$ satisfies $(*)$, and
- $\alpha(x_1') \coprod \cdots \coprod \alpha(x_n')$ is a tilting object.
- **Notice:** If $\underline{x}'$ satisfies $(\bar{*})$, then $\bar{\alpha}(\underline{x}')$ can be defined as $\bar{\alpha}(\underline{x}') = \alpha(x_1') \coprod \cdots \coprod \alpha(x_n')$, which is a tilting object.

$(\tilde{*})$ is a property of a cluster seed $(\underline{x}', Q')$ consisting of two parts:

- the cluster $\underline{x}'$ satisfies $(\bar{*})$, and
- $Q_{(\bar{\alpha}(\underline{x}'))} = Q'$.
- **Notice:** If $(\underline{x}', Q')$ satisfies $(\tilde{*})$, then $\tilde{\alpha}(\underline{x}', Q')$ can be defined as $\tilde{\alpha}(\underline{x}', Q') = (\bar{\alpha}(\underline{x}'), Q')$ which is a tilting seed.

**Proposition 2.7.1.** *Let $(\underline{x}', Q')$ be a cluster seed satisfying $(\tilde{*})$. Consider a cluster mutation. Then:*

$a$**:** *The new cluster variable satisfies $(*)$.*
$\bar{a}$**:** *The new cluster satisfies $(\bar{*})$.*
$\tilde{a}$**:** *The new cluster seed satisfies $(\tilde{*})$.*

**Corollary 2.7.2.** *Let $(\underline{x}', Q')$ be a cluster seed satisfying condition $(\tilde{*})$. Let $\tilde{\mu}$ be a cluster mutation and $\tilde{\nu}$ the corresponding tilting mutation. Then: $\tilde{\alpha} \circ \tilde{\mu} = \tilde{\nu} \circ \tilde{\alpha}$.*

*Proof.* This follows from the Definitions 2.1.1 and 1.5.2 and a theorem from [BMR]. □

**Corollary 2.7.3.** *Every cluster seed is $(\tilde{*})$, every cluster is $(\bar{*})$ and every cluster variable is $(*)$.*

*Proof.* The initial seed $(\{x_1 \ldots x_n\}, Q)$ satisfies $(\tilde{*})$ condition. Since every cluster seed can be reached by a finite number of cluster mutations, and by the proposition every cluster seed at each step is $(\tilde{*})$ it follows that all cluster seeds are $(\tilde{*})$. □

*Proof.* of the Theorem 2.4.1($a$): By the above corollary it follows that everything satisfies appropriate $(*)$, $(\bar{*})$, $(\tilde{*})$, and by the comments in the definitions of $(*)$, $(\bar{*})$, $(\tilde{*})$, all three maps $(\alpha)$, $(\bar{\alpha})$, $(\tilde{\alpha})$ are defined. □

*Proof.* of the Theorem 2.4.1($b$): We will show that $\tilde{\alpha}$ is onto.
Let $(T, Q_T)$ be a tilting seed.
$\exists$ a finite number of tilting mutations from the initial tilting seed to $(T, Q_T)$.
Consider the same sequence of cluster mutations from the initial cluster seed.
Use Corollary 2.7.2 □

2.8. **Fomin-Zelevinsky Conjecture: Cluster determines Seed.** The following was conjectured by Fomin and Zelevinsky (for any cluster algebra) and we prove it for the acyclic case with no coefficients.

**Theorem 2.8.1** (BMRT)**.** *Let* $(\underline{x}', Q')$ *and* $(\underline{x}', Q")$ *be cluster seeds for an acyclic cluster algebra with no coefficients. Then* $Q' = Q"$*.*

*Proof.* $\tilde{\alpha}(\underline{x}', Q') = (\bar{\alpha}(\underline{x}'), Q')$ and $\tilde{\alpha}(\underline{x}', Q") = (\bar{\alpha}(\underline{x}'), Q")$ are tilting seeds. But, $Q' = Q"$ since tilting seed is determined by its tilting object. □

## 3. Semi-invariants and Cluster Categories

We first recall the definition and some of the theorems about semi-invariants, as done by Kac, Schofield, Derksen-Weyman for non-negative integral vectors. After that we state theorems for "mixed signs" integral vectors as done in [IOTW]. In particular, in the Dynkin diagram case, we state the relation between the simplicial complex of the partial tilting objects and the domains of semi-invariants .

3.1. **Definitions; Representations and Semi-invariants.** Let $Q = (Q_0, Q_1)$ be a simply laced quiver, where $Q_0$ denotes the set of the vertices of $Q$, and $Q_1$ is the set of the arrows of $Q$. Assume $Q$ has no oriented cycles. Let $\Bbbk$ be an algebraically closed field. Let $n$ be the number of the vertices in $Q_0$. Recall that the *Euler matrix* $E$ is $n \times n$ matrix, with the diagonal entries equal to 1 and the entry $E_{i,j} =$ the number of arrows from $i$ to $j$.

3.1.1. **Representation space.** For a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{N}^n$, define the representation space $R(\alpha)$ and the group $Gl(\alpha)$ which acts on $R(\alpha)$:

$$R(\alpha) := \prod_{(i\to j)\in Q_1} Hom_{\Bbbk}(\alpha_i\Bbbk, \alpha_j\Bbbk) \qquad \text{and} \qquad Gl(\alpha) := \prod_{i\in Q_0} Gl(\alpha_i).$$

- If $X$ is an element of $R(\alpha)$, then $X$ is a collection of $\alpha_j \times \alpha_i$ matrices:
  $(M^X_{i,j}(\Bbbk))_{(i\to j)\in Q_1}$, together with vector spaces $\{X_i = \alpha_i \Bbbk\}_{i\in Q_0}$.
- For each vertex $i$ we denote by $P(i)$ the indecomposable projective.
- For each vector $\alpha \in \mathbb{N}^n$, define the projective representation:
$$P(\alpha) = \coprod_{i\in Q_0} \alpha_i P(i).$$

3.1.2. **Canonical projective presentation.** Let $\alpha \in \mathbb{N}^n$ and $X$ in $R(\alpha)$. Define:

$$0 \to \coprod_{(i\to j)\in Q_1} \alpha_i P(j) \xrightarrow{\pi(X)} \coprod_{j\in Q_0} \alpha_j P(j) \xrightarrow{f_X} X \to 0,$$

where $f_X$ maps $\alpha_j P(j)$ onto the generators of $X_j$, and $\pi(X)$ maps $\alpha_i P(j)$ to $\alpha_i P(i)$ by the negative inclusion map, and $\alpha_i P(j)$ to $\alpha_j P(j)$ by the mapping $M^X_{i,j}(\Bbbk)$ (which define the representation $X$).

- The above presentation is called *canonical projective presentation.*
- $\coprod_{(i\to j)\in Q_1} \alpha_i P(j) = P(\alpha - E^T\alpha)$,
- $\coprod_{j\in Q_0} \alpha_j P(j) = P(\alpha)$, and
- $0 \to P(\alpha - E^T\alpha) \to P(\alpha) \to M \to 0$ is the same canonical presentation. Consequently:
$$R(\alpha) = \mathrm{Hom}(P(\alpha - E^T\alpha), P(\alpha)).$$

3.1.3. **Definitions; General, Generic representations, Schur roots.**

- The *general representation* is the representation whose matrix coordinates are indeterminants. (This is actually a specific representation of the same quiver but with the coefficients in the function field of $Rep(\alpha)$.)
- A *generic representation* is an unspecified representation, which refers to a variable point in a Zariski open subset of $Rep(\alpha)$.
- Some justification for using both terms - in algebraically closed case, they are "the same".

- *Schur root* is a dimension vector for which the general representation is indecomposable.
- $ext(\alpha, \beta)$ is the minimum value of $dim\,\mathrm{Ext}(M, N)$ for all modules $M, N$ of dimensions $\alpha$ and $\beta$ respectively.

3.1.4. **Semi-invariants.** A polynomial function $\sigma : R(\alpha) \to \Bbbk$ is called (polynomial) *semi-invariant* if:

$$\sigma((g_i)X) = (detg_1)^{\beta_1} \dots (detg_n)^{\beta_n} \sigma(X)$$

for some $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, and for all $(g_i)$ in $Gl(\alpha)$, and all $\underline{X}$ in $R(\alpha)$. If this is the case, then $\beta = (\beta_1, \dots, \beta_n)$ is called the *weight* of the semi-invariant $\sigma$.

3.2. **Theorems of Kac, Schofield, Derksen and Weyman.** The following three theorems are proved in [K], [S], [DW], for $\alpha \in \mathbb{N}^n$.

3.2.1. **Semi-invariants given by determinants theorem.** [S] Let $\alpha \in \mathbb{N}^n$. Then:

- All semi-invariants of the representation space $R(\alpha)$ are generated by $\sigma_N : R(\alpha) \to \Bbbk$, for all indecomposable representations $N$ with $\alpha^T E\, \underline{\dim} N = 0$ and they are given by:
- $\sigma_N(X) := det\,\mathrm{Hom}(f_X, N)$, for the representations $X$ with $\dim(X) = \alpha$,
- $f_X : P_1 \to P_0$, the canonical projective presentation of $X = coker f_X$.

3.2.2. **Saturation theorem.** [DW] Let $\beta$ be a positive root. Then:

- $\{\alpha \in \mathbb{N}^n |\ R(\alpha) \text{has a semi-invariant of weight} \beta\} = D(\beta) \cap \mathbb{N}^n, \text{where}$
- $D(\beta) = \{\alpha \in \mathbb{R}^n \mid \alpha^T E \beta = 0\} \cap \bigcap_{\beta' \hookrightarrow \beta} \{\alpha \mid \alpha^T E \beta' < 0\}, \text{and}$
- $\beta' \hookrightarrow \beta$ means that the general representation of dimension $\beta$ has a subrepresentation of dimension $\beta'$.

3.2.3. **Generic decomposition theorem.** [K]

- Any $\alpha \in \mathbb{N}^n$ can be decomposed uniquely as a sum $\alpha = \sum \gamma_i$ of Schur roots $\gamma_i$, such that $ext(\gamma_i, \gamma_j) = 0$.
- The general representation of dimension $\alpha$ decomposes as a direct sum of general indecomposable representations of dimension $\gamma_i$.

3.3. **Definitions and Theorems for $\alpha \in \mathbb{Z}^n$.** We consider $\alpha \in \mathbb{Z}^n$, generalize the definitions and prove the theorems.

3.3.1. **Definition of Generalized representation space.** Let $\alpha \in \mathbb{Z}^n$. Then $\alpha = \alpha' - \underline{dim} Y$, with $\alpha' \in \mathbb{N}^n$ and $Y$ a projective representation. Furthermore, this decomposition is unique, providing it satisfies certain conditions (C1). Define *Generalized representation space* for $\alpha \in \mathbb{Z}^n$ to be:

$$R(\alpha) = \mathrm{Hom}(P(\alpha' - E^T \alpha') \coprod Y, P(\alpha')).$$

3.3.2. **Semi-invariants given by determinants theorem.** [IOTW] Let $\alpha \in \mathbb{Z}^n$. Then:

- All semi-invariants in the representation space $R(\alpha)$ are generated by $\sigma_N : R(\alpha) \to \Bbbk$, which are given by:
- $\sigma_N(X) := det\,\mathrm{Hom}(f_X, N)$, for the representations $X$ with $\dim(X) = \alpha$,
- $f_X : P_1 \to P_0$, the canonical projective presentation of $X = coker f_X$, and
- $N$ is an indecomposable representation with $\alpha^T E\, \underline{\dim} N = 0$.

3.3.3. **Saturation theorem.** [IOTW] Let $\beta$ be a positive root. Then:

- $\{\alpha \in \mathbb{Z}^n | \ R(\alpha)$has a semi-invariant of weight$\beta\} = D(\beta) \cap \mathbb{Z}^n$, where
- $D(\beta) = \{\alpha \in \mathbb{R}^n \mid \alpha^T E \beta = 0\} \cap \{\bigcap_{\beta' \hookrightarrow \beta} \{\alpha \mid \alpha^T E \beta' < 0\}$, and
- $\beta' \hookrightarrow \beta$ means that the general representation of dimension $\beta$ has a subrepresentation of dimension $\beta'$.

3.3.4. **Generic decomposition theorem.** [IOTW] Let $\alpha \in \mathbb{Z}^n$. Then:

- Any $\alpha \in \mathbb{Z}^n$ can be decomposed uniquely as a sum $\alpha = \sum \gamma_i$ of Schur roots and "shifted projective roots" $\gamma_i$, such that $ext(\gamma_i, \gamma_j) = 0$.
- The general representation of dimension $\alpha$ decomposes as a direct sum of general indecomposable representations of dimension $\gamma_i$.

  More precisely, if $\gamma_i$ is a Schur root, the representation is given by the canonical projective presentation (or minimal projective presentation), and if $\gamma_i$ is shifted projective root, then the representation is given by the complex $P_i \to 0$, where $\gamma_i = (\underline{\dim} P_i)[1]$.

3.4. **Dynkin diagrams.** We show that the simplicial complex of the clusters can be identified with the domains of the generalized semi-invariants, which will also be identified with the Igusa-Orr pictures for the sets of simple roots 4.4.1.

3.4.1. **Simplicial complex of exceptional (or partial tilting) objects.** The *simplicial complex* of the Dynkin diagram is defined to be the simplicial complex $T(Q)$, with vertex set

$$\Phi_+ \cup \{\text{negative projective roots}\},$$

and faces are the spans of subsets of vertices, corresponding to the exceptional objets in the cluster category.

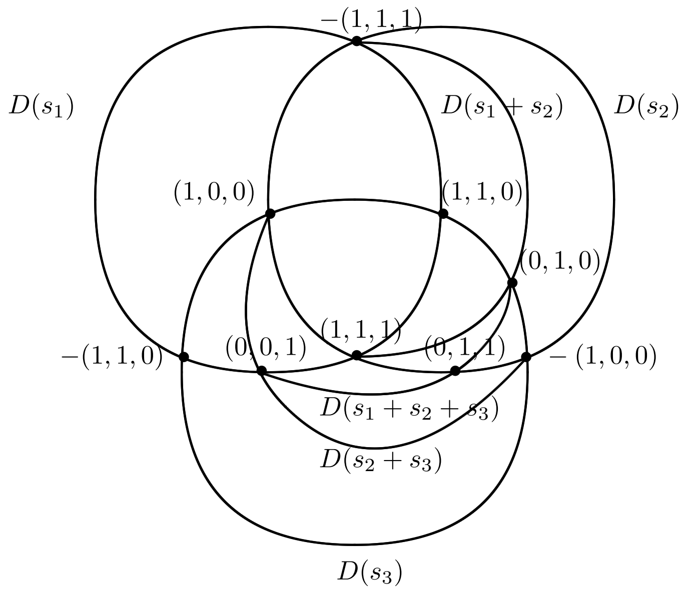
3.4.2. **Simplicial complex and domains of semi-invariants theorem.** [IOTW]. Let $Q$ be a Dynkin diagram of type $A, D, E$. Then:

- The simplicial complex of the exceptional objects is homeomorphic to the $n-1$ sphere.
- The image of the $n-2$ skeleton of $T(Q)$ in $S^{n-1}$ is the union:

$$\bigcup_{\beta \in \Phi_+} D(\beta) \cap S^{n-1}$$

3.4.3. **Illustration on the example of $A_3$.** The image of the 1 skeleton of $T(A_3)$ in $S^2$ is the union:

$$\bigcup_{\beta \in \Phi_+} D(\beta) \cap S^2.$$




- The quiver is $1 \leftarrow 2 \leftarrow 3$ with Euler matrix
$$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
- Circles and semicircles, labeled by $D(\beta)$, denote the domains of semiinvariants (with weight $\beta$).
- Vertices are labeled by the positive roots $\Phi_+$ and negative projective roots.
- Positive roots can also be viewed as dimension vectors of indecomposable representations by Gabriel's theorem, e.g. $(1,1,0) = \underline{\dim}(P_2)$.
- For example: $Rep((1,1,0))$ has semi-invariants of weights $s_1 = (1,0,0)$ and $s_3 = (0,0,1)$.
- For example: $Rep((1,1,0))$, $Rep((1,1,1))$, $Rep((0,0,1))$ ,$Rep(-(1,1,0))$, $Rep(-(1,1,1))$ all have semi-invariants of weights $s_1 = (1,0,0)$ since $Es_1 = (1,-1,0)$ making $\alpha^T E s_1 = 0$ iff $\alpha_1 = \alpha_2$.
- The semiinvariant $\sigma_{s_1}$ on a representation
$$V : V_1 \xleftarrow{f_{12}} V_2 \xleftarrow{f_{23}} V_3$$
is $\sigma_{s_1}(V) = \det f_{12}$. This is only well-defined if coordinates $\phi_i : V_i \xrightarrow{\approx} \alpha_i \Bbbk$ are chosen. If these are changed by $g_i \in GL(\alpha_i, \Bbbk)$ then $\det f_{12}$ becomes
$$\det(g_1 f_{12} g_2^{-1} = (\det g_1)(\det g_2)^{-1} \det f_{12}$$
So, this semiinvariant has weight $(1,-1,0) = Es_1$.
- In the case $\beta = s_1 + s_2 = (1,1,0)$ we have $s_1 \hookrightarrow s_1 + s_2$. So, $D(s_1 + s_2)$ does not contain points $\alpha$ where $\alpha^T E s_1 > 0$. So, $D(s_1 + s_2)$ contains no points inside the $D(s_1)$ circle.
- Furthermore, any integral vector belongs to one of the simplices, and the generic decomposition is given in terms of the vertices of that simplex.

## 4. Homology of Nilpotent, Torsion-free Groups

### 4.1. Homology of groups and Lie algebras.

4.1.1. **Homology of a group.** The homology of a group $G$ is defined by

$$H_k(G;\mathbb{Z}) := Tor_k^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$$

More explicitly, we need to choose a free $\mathbb{Z}G$ resolution $C_*(G)$ of $\mathbb{Z}$. Then $H_*(G;\mathbb{Z})$ is the homology of the complex $C_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}$. One standard complex is given by the *bar resolution* where $C_k(G)$ is the free $\mathbb{Z}G$ module generated by $G^k$ and the boundary map $d : C_k(G) \to C_{k-1}(G)$ is given by

$$d(g_1,\cdots,g_k) = g_1(g_2,\cdots,g_k)+\sum(-1)^i(g_1,\cdots,g_ig_{i-1},\cdots,g_k)+(-1)^k(g_1,\cdots,g_{k-1})$$

This chain complex is infinitely generated. In the case when $G$ is torsion-free, nilpotent, there is a smaller chain complex in which $C_k(G)$ is generated by the set of all $k'$-element subsets of the nilpotent basis $\mathcal{B}$ of $G$, see 4.2.7. There are various descriptions of this complex ([CP1], [IO]). One of the objectives of this work is to give a new canonical description of this smaller complex using root systems and semi-invariants in the case when $G$ is a monomial group (including groups of Dynkin type).

4.1.2. **Homology of a group - topological definition.** We recall, but will not use that the homology of the group $G$ is defined topologically by $H_k(G;\mathbb{Z}) = H_k(BG;\mathbb{Z})$ where $BG$ is the classifying space of the discrete group $G$, also known as the Eilenberg-MacLane space $BG = K(G,1)$.

4.1.3. **Rational homology of a group.** The rational homology of a group $G$ is $H_*(G;\mathbb{Q}) = H_*(G;\mathbb{Z}) \otimes \mathbb{Q}$.

4.1.4. **Homology of a Lie algebra.** The homology of a Lie algebra $L$, $H_*(L)$, is defined to be the homology of its Koszul complex $C_*(L)$ given by $C_k(L) := \Lambda^k L$ with boundary $d : C_k(L) \to C_{k-1}(L)$ given by

$$d(x_1\wedge\cdots\wedge x_k) = \sum_{i<j}(-1)^{i+j}[x_i,x_j]\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge\widehat{x_j}\wedge\cdots\wedge x_k$$

4.1.5. **Rational homology of a Lie algebra.** The rational homology of a Lie algebra $L$ is $H_*(L;\mathbb{Q}) = H_*(L) \otimes \mathbb{Q} \cong H_*(L\otimes\mathbb{Q})$.

4.1.6. **Lie algebra $L$ associated to a group $G$.** To every group $G$ there is an associated graded Lie algebra $L(G) = \bigoplus L_i(G)$ given by $L_i(G) = G_i/G_{i+1}$ with Lie bracket

$$[\ ,\ ] : L_i(G)\otimes L_j(G) \to L_{i+j}(G)$$

given by the commutator: $[gG_{i+1}, hG_{j+1}] := [g,h]G_{i+j+1}$.

### 4.2. Torsion-free Nilpotent groups.

4.2.1. **Definition of Torsion-free, Nilpotent groups.** Let $G$ be a group and $G = G_0 \supset G_1 \supset \cdots \supset G_i \supset G_{i+1} \supset \dots$ the chain of subgroups defined as $G_{i+1} = [G, G_i]$, where $[\ ,\ ]$ denotes the group commutator operation, i.e. $[g,h] = ghg^{-1}h^{-1}$ for all $g,h \in G$. The group $G$ is said to be:

- *Torsion free* if $G_i/G_{i+1}$ is finitely generated free abelian group for all $i$.
- *Nilpotent* if there exists an $s$ such that $G_s = 0$.

4.2.2. **Theorem of Nomizu.**

**Theorem 4.2.1** (Nomizu)**.** [N54] *The rational homology of a torsion-free nilpotent group $G$ is isomorphic to the homology of the Lie algebra $L(G) \otimes \mathbb{Q}$.*

4.2.3. **Remark.** Nomizu [N54] gave the first example of a nilpotent torsion free group whose integral homology is not isomorphic to that of the associated Lie algebra. Dwyer [D] points out that the upper triangular matrix group (the nilpotent group associated to the Dynkin diagram $A_n$) has $p$-torsion in its homology for all $p < n$.

4.2.4. **Koszul complex.** The Koszul complex gives both the integral and rational homology for any Lie algebra $L$ and in particular for the Lie algebra $L(G)$ and therefore, by Nomizu's theorem, it gives the rational homology of $G$ as well.

4.2.5. **Cenkl and Porter.** In [CP1] they give an algorithm for constructing the integral cochain complex for the integral cohomology of $G$. (The integral cohomology determines the integral homology.) In [CP2] they construct a nilpotent torsion free group with a prescribed Lie algebra (the reverse of the usual construction).

4.2.6. **A motivation from topology.** Kent Orr proved that all Milnor $\bar{\mu}$-invariants of links are represented by elements of $H_3$ of the fundamental group of the complement of the link. Igusa and Orr proved that $H_3$ has no torsion for nilpotent torsion free groups, using a particular complex in which boundaries are given by "pictures".

4.2.7. **Remark.** Each nilpotent torsion free group has a finite basis $\mathcal{B} = \{b_1, \dots, b_m\}$ which can be obtained as the union of preimages of the bases of the free abelian groups $G_i/G_{i+1}$ for $0 \le i \le s$. These elements form a basis in the sense that every element in the group can be expressed uniquely in the form $b_1^{k_1} b_2^{k_2} \cdots b_m^{k_m}$ where $k_1, \cdots, k_m$ are integers.

- Let $\{s_1, \dots, s_n\}$ be preimages of a set of basis elements of $G_0/G_1$.
- Then $\{s_1, \dots, s_n\}$ is a minimal set of generators for the group $G$, call these elements *simple generators.*
- The number $n$ does not depend on the choice of the basis.

4.2.8. **Igusa-Orr complex and "pictures".** This is a particular $\mathbb{Z}[G]$ resolution of $\mathbb{Z}$, which we will not describe precisely, but the following are some of the important facts about it:

- $C_k(G)$ is freely generated by the sets of k-element subsets of the basis $\mathcal{B}$.
- The boundary $d : C_k(G) \to C_{k-1}(G)$ is recursively defined for each subset of $\mathcal{B}$.
- Boundary is Koszul boundary plus additional terms.
- Boundary for each subset of $\mathcal{B}$ corresponds to a "picture", i.e. a cell decomposition of sphere, satisfying certain necessary and sufficient conditions.
- Left and right commutators in the group give the most efficient "collecting process".
- Different forms of commutators. The standard form of commutator is the left commutator $aba^{-1}b^{-1}$, the original Igusa-Orr pictures use the right commutator $a^{-1}b^{-1}ab$, and for the Cluster-Semiinvariant pictures it is more convenient to use the middle commutator $b^{-1}aba^{-1}$.

## 4.3. **Monomial groups.**

4.3.1. **Definition of Monomial groups.** These are special torsion free monomial groups, for which Igusa-Orr pictures are related to the Cluster-Semi-invariant pictures. A group $G$ is called *monomial* if it is:

(1) Torsion free,
(2) Nilpotent and
(3) There exists a basis $\mathcal{B} = \{b_1, \ldots, b_m\}$ satisfying:
  **a:** $[b_i, b_j] \subset \mathcal{B} \cup \mathcal{B}^{-1} \cup [1]$
  **b:** For any three elements at least two commute
  **c:** For any $b_i, b_j \in \mathcal{B}$, $[b_i, b_j]$ commutes with both $b_i$ and $b_j$

4.3.2. **Maximal monomial groups.** These will be monomial groups which are maximal in the following sense (but we need to point out a few facts):

- The number of elements in $\{s_1 \ldots, s_n\}$ is an invariant of the group G.
- Monomial groups with $n$ simple generators, for a fixed $n$, are partially ordered by epimorphisms of groups, which send simple generators to simple generators (or their inverse).
- Maximal elements with respect to this partial order are called *Maximal monomial groups.*

4.3.3. **Dynkin diagrams and Maximal monomial groups.** To each simply laced Dynkin diagram $Q$ with the set of positive roots $\Phi_+$, we associate a group $G$ defined by:

- Generators: $\{\Theta(\alpha) | \alpha \in \Phi_+\}$ and
- Relations: $\{[\Theta(\alpha), \Theta(\beta)] = \Theta(\alpha + \beta)^{\epsilon(\alpha,\beta)}$ where $\epsilon(\alpha, \beta) = (-1)^{E(\alpha,\beta)}$ if $\alpha + \beta \in \Phi_+$ and 0 otherwise$\}$. Here $E(\alpha, \beta)$ denotes the Euler form.

We have the following facts, questions (and work in progress):

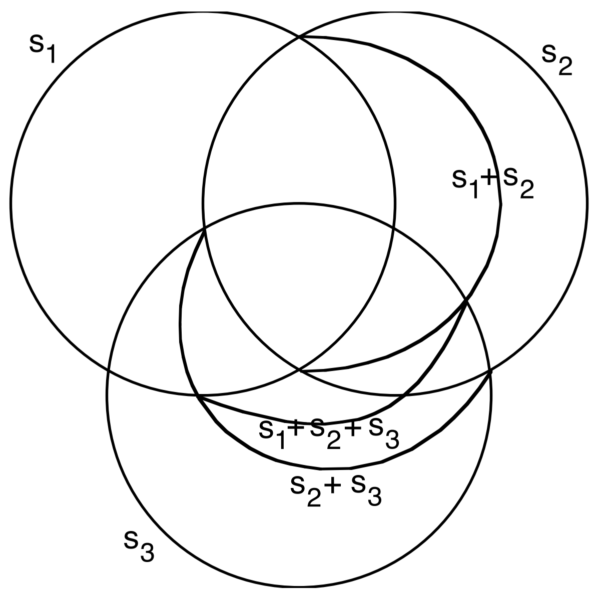
- Simply laced Dynkin diagrams define maximal monomial groups.
- Are all maximal monomial groups given by simply laced Dynkin diagrams (A,D,E) as above?
- Description of the groups (with appropriate modifications of the condition (3)), which will correspond to the other Dynkin diagrams (B,C,F,G).

## 4.4. **Pictures.**

4.4.1. **An $A_3$ example of Igusa-Orr picture.**

- Circles and semicircles are labeled by positive roots $\alpha \in \Phi_+$ for the Dynkin diagram $A_3$. Any positive root $\alpha$ is expressed as a sum of simple roots $\{s_1, s_2, s_3\}$
- For the middle-commutator I-O picture the same labels are used for the elements of the associated nilpotent group $G$, i.e. any positive root $\alpha$ actually stands for $\Theta(\alpha)$, see 4.3.3.
- A free $\mathbb{Z}G$ resolution of $\mathbb{Z}$ is given by $C_k$ = free module generated by subsets of $\Phi_+$ with $k$ elements, and with boundary given by the pictures.
- The following picture describes the boundary $d(s_1, s_2, s_3)$ as:
- $d(s_1, s_2, s_3) = -(s_1, s_2) + (s_1, s_3) - (s_2, s_3) - \Theta(s_2)\Theta(s_1 + s_2)(s_1, s_2) + \Theta(s_1)(s_2, s_3) + \Theta(s_3)(s_1, s_2 + s_3) - \Theta(s_3)\Theta(s_2 + s_3)(s_2, s_1 + s_2 + s_3) + \Theta(s_2)(s_1 + s_2, s_1 + s_2 + s_3) + \Theta(s_3)\Theta(s_2 + s_3)\Theta(s_1 + s_2 + s_3)(s_1, s_2)$

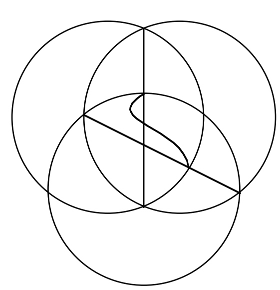
- Each term in the sum corresponds to a vertex, and is obtained by "reading" the picture.



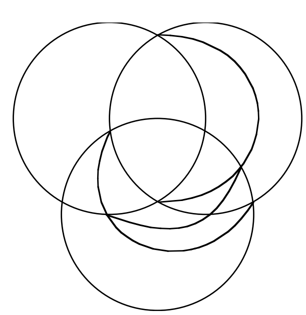

- Circles and semicircles are labeled by positive roots $\Phi_+$ for the Dynkin diagram $A_3$. Labeling agrees with the same labeling of the same picture by the domains of semiinvariants.

4.4.2. **I-O vs C-SI pictures.** The original, right commutator Igusa-Orr and Cluster-Semiinvariant pictures are given below for the example of Dynkin diagram $A_3$.

- I-O picture has 12 tri-angles, 1 quadr-angle, 1 bi-angle
- C-SI picture is a simplicial complex, triangulation of a shpere with 14 tri-angles.



I-O



C-SI

4.4.3. **Some facts.**

- Igusa-Orr pictures are Koszul boundary plus higher terms.
- Cluster-Semiinvariant pictures are modified Koszul boundary plus higher terms. The modification vanishes after tensoring with $\mathbb{Z}$.
- Cluster-Semiinvariant pictures are more natural - simplicial complex.
- Plan to re-do Igusa-Orr algorithm to construct all Cluster-Semiinvariant pictures.
- Cluster-Semiinvariant pictures give the middle commutator Igusa-Orr pictures for the particular subset consisting of the simple generators $\{s_1, \ldots, s_n\}$ of the group $G$, which actually correspond to the set of simple roots in $\Phi_+$.
- We believe that Cluster-Semiinvariant pictures can be used to construct, in a functorial way, all middle-commutator I-O pictures (for all subsets of $\Phi_+$.

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