CLUSTER CATEGORIES AND RELATED TOPICS I, II, III

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Abstract. The three talks were on the paper "Clusters and Seeds in Acyclic Cluster Algebras" by Aslak Buan, Robert Marsh, Idun Reiten and GT [BRMT].

O. Introduction

0.1. **Cluster algebras.** Cluster algebras were defined by Fomin and Zelevinsky. In this talk we will recall basic points in the definition, since we will need it in order to give some of the proofs.

$\mathcal{A}$ will be cluster algebra

$\mathbb{Q}(u_1, \dots, u_n)$ field of rational functions

$\mathcal{A}$ is subalgebra generated by cluster variables

{cluster variables} = ∪{elements of clusters}

{clusters} = {certain transcendence bases in $\mathbb{Q}(u_1, \dots, u_n)$ obtained from an initial cluster after applying sequences of mutations in every possible direction, as we will describe now}

$\underline{x} = \{x_1, \dots, x_n\}$ = initial cluster (a transcendence basis)

$(\underline{x}, Q)$ = initial cluster seed; $\underline{x}$ initial cluster, Q initial quiver (a matrix (b_{ij}))

Definition 0.1.1. Using the notation from above we define:

- *Cluster mutation of* x_i with respect to the cluster seed $(\underline{x}, Q)$ is the new cluster variable $(x_i)^*$ obtained as $x_i \cdot (x_i)^* = \prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b'_{ij}<0} x_i^{b'_{ij}}$.

  Notation $\mu(x_i) = (x_i)^*$.
- *Cluster mutation of the cluster* $\underline{x} = \{x_1, \dots, x_n\}$ at x_i is a new cluster $\underline{x}' = \{x_1, \dots, (x_i)^*, \dots x_n\} = \{x_1, \dots, \mu_i(x_i), \dots x_n\}$

  Notation: $\bar{\mu}_i(\underline{x}) = \{x_1, \dots, \mu_i(x_i), \dots x_n\} = \underline{x}'$
- *Cluster mutation of the cluster seed* $(\underline{x}, Q)$ *at* x_i is the new cluster seed $(\underline{x}', Q')$.

  Notation: $\tilde{\mu}_i(\underline{x}, Q) = (\bar{\mu}_i(\underline{x}), \vec{\mu}_i Q) = (\underline{x}', Q')$, where $\vec{\mu}_i(Q) = Q'$ new quiver obtained in quite a complicated way .

Theorem 0.1.2 (FZ). *A cluster algebra has finitely many cluster varibles, if and only if a Dynkin diagram appears as one of the quivers in the mutation process.*

0.2. **Cluster categories.** Recall the standard definition, notation and properties:

Q a quiver with n vertices and no oriented cycles, k a field

$\Lambda = kQ$ the path algebra, $\operatorname{mod}\Lambda$, $D^b(\operatorname{mod}\Lambda)$

$[1] : D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\Lambda)$ suspension functor

$\tau : D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\Lambda)$ Auslander-Reiten functor

$F := \tau^{-1}[1] = [1]\tau^{-1} : D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\Lambda)$ composition functor

C_Λ cluster category: $Ob(C_\Lambda)$=F-orbits, $Mor(C_\Lambda)$ given by Hom, Ext and shifts.

Representatives of the orbits may be chosen in $\mathrm{add}(\mathrm{ind}\,\Lambda \cup \{P_i[1]\}_{i=1}^n)$
Exceptional objects in C_Λ are X such that $Ext_{C_\Lambda}(X, X) = 0$
Tilting objects (basic) are maximal exceptional objects
$T = T_1 \coprod \ldots \coprod T_n$ basic tilting object has n indecomposable summands
Tilting seed is a pair (T, Q_T), where Q_T is the quiver of $\mathrm{End}_{C_\Lambda}(T)^{op}$

Theorem 0.2.1 (BMRRT)**.** *Let $T = M \coprod \bar{T}$ be a basic tilting object with M indecomposable. Then, there exists exactly one object M^* not isomorphic to M, such that $T' = M^* \coprod \bar{T}$ is also a basic tilting object.*

Definition 0.2.2. Using the notation from Theorem 0.2.1 we call the indecomposable objects M and M^* an *exchange pair* with respect to T (or T').

- *Tilting mutation of M with respect to T is M^*.* Notation $\nu(M) = M^*$.
- *Tilting mutation of the tilting object $T = M \coprod \bar{T}$ at M is $T' = M^* \coprod \bar{T}$.* Notation: $\bar{\nu}(T) = \bar{\nu}(M \coprod \bar{T}) = \nu(M) \coprod \bar{T} = M^* \coprod \bar{T} = T'$.
- *Tilting mutation of the tilting seed (T, Q_T) at M* is the new seed $(T', Q_{T'})$. Notation: $\tilde{\nu}(T, Q_T) = (\bar{\nu}(T), Q_{\bar{\nu}(T)})$.

Theorem 0.2.3 (BMRRT)**.** *Two exceptional indecomposable objects M and M^* form an exchange pair if and only if* $\dim_{D_M} \mathrm{Ext}^1_{C_\Lambda}(M, M^*) = 1 = \dim_{D_{M^*}} \mathrm{Ext}^1_{C_\Lambda}(M, M^*)$.

0.3. **Some useful facts about exchange pairs.** We now describe construction of M^* and state some of the facts and commutative diagrams from [BMRRT], which will be used in the proofs.
Let $T = M \coprod \bar{T}$ be a tilting object.
Let $f : B \to M$ be a minimal right $add\bar{T}$- approximation of M.
Let $M^* \to B \to M$ be the associated triangle in D^b.
Some facts:

(1) M^* is indecomposable exceptional.
(2) $M^* \to B$ is a minimal left $add\bar{T}$- approximation of M^*.
(3) $\dim_{D_M} \mathrm{Ext}^1_{C_\Lambda}(M, M^*) = 1 = \dim_{D_{M^*}} \mathrm{Ext}^1_{C_\Lambda}(M, M^*)$.
(4) M and M^* form an exchange pair with respect to T (or T').
(5) We may consider the AR-triangle for M^* and get the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & M^* & \longrightarrow & A_{M^*} & \longrightarrow & \tau^{-1}M^* & \longrightarrow \\
 & {\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle h} & \\
\longrightarrow & M^* & \longrightarrow & B & \longrightarrow & M & \longrightarrow \\
 & & & & & \downarrow & \\
 & & & & & B' & \\
 & & & & & \downarrow & \\
 & & & & & \tau^{-1}M^*[1] = FM^* \cong_{C_\Lambda} M^* &
\end{array}$$

(6) $\dim \mathrm{Hom}_{D^b}(\tau^{-1}M^*, M) = 1$
(7) $B' = Coker(h) \coprod Ker(h)[1] := L_h \coprod K_h[1]$
(8) h is isomorphism if and only if $B' = 0$
(9) $M^* \cong \tau M$ if and only if $M^* \to B \to M$ is AR triangle if and only if $B' = 0$
(10) The vertical maps $M \to B'$ and $B' \to M^*$ are $add\bar{T}$-approx. of M and M^*.
(11) Can also consider AR triangle for M and get similar facts.

1. Relation between Cluster algebras and Cluster categories

1.1. **Definition:** A cluster algebra is called *acyclic*, if in the cluster mutation process a quiver with no oriented cycles appears.
Remark: We will only consider acyclic case, so we may assume that the quiver in the initial seed has no oriented cycles. We now recall the following notions from the introduction:

Q quiver

$\mathcal{A}(Q)$ cluster algebra	C_Λ cluster category, where $\Lambda = kQ$
$(\underline{x}, Q)$ initial cluster seed	$(P_1[1] \coprod \cdots \coprod P_n[1], Q)$ initial tilting seed
$(\underline{x}', Q')$ cluster seed	(T, Q_T) tilting seed
$\underline{x}' = \{x'_1, \dots, x'_n\}$ cluster	$T = T_1 \coprod \cdots \coprod T_n$ tilting object
x'_i cluster variable	M indecomposable exceptional object

We want to discuss now what kind of correspondences hold and in which situations. First we will recall known results and then state and prove a new theorem from [BMRT].

Theorem 1.1.1 (FZ,FMZ)**.** *Suppose Q is a Dynkin diagram. Then:*
$|\{$ *cluster variables*$\}| = |\{$ *isomorphism classes of indecomposable objects* $\}|$.

Theorem 1.1.2 (BMRRT)**.** *Suppose Q is a simply laced Dynkin diagram. Then there exist one to one correspondences (the second one being induced by the first):*
$\{$*cluster variables*$\} \to \{$*isomorphism classes of indecomposable objects*$\}$,
$\{$*clusters*$\} \to \{$*basic tilting objects*$\}$.

1.2. **General simply laced diagram. Conjecture**: [BMRRT] Suppose Q is a simply laced diagram. Then there exist one to one correspondence (second one induced by the first):
$\{$cluster variables$\} \to \{$isomorphism classes of indecomposable objects$\}$,
$\{$clusters$\} \to \{$basic tilting objects$\}$.

The following theorem will be proved at the end of these lectures, however we need a few more definitions, and the main step of the proof will be the Proposition that we will prove in the next section.

Theorem 1.2.1 (BMRT)**.** *Let Q be simply laced, acyclic diagram.*
(a) There exist well defined maps:
$\alpha : \{$*cluster variables*$\} \to \{$*isoclasses of indecomp. exceptional objects*$\}$,
$\bar{\alpha} : \{$*clusters*$\} \to \{$*tilting objects*$\}$,
$\tilde{\alpha} : \{$*cluster seeds*$\} \to \{$*tilting seeds*$\}$.
(b) All three maps: $\alpha, \bar{\alpha}, \tilde{\alpha}$ are onto.

2. Denominators of Cluster variables

2.1. **Monomials in the denominators of cluster variables.** First we state the famous "Laurent phenomenon" theorem of Fomin and Zelevinsky, and also some known results about the monomials which appear in the denominators of the cluster variables.

Theorem 2.1.1 (Fomin-Zelevinsky). *(Laurent phenomenon) Denominators of all cluster variables, when expressed in terms of the initial cluster, and then reduced, are monomials.*

Notation: Let $m = x_1^{k_1} \dots x_n^{k_n}$ be a monomial in variables $x_1, \dots, x_n$. Denote the exponent vector by $\epsilon(m) = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$.

Theorem 2.1.2 (Fomin-Zelevinsky). *Let Q be a Dynkin diagram with alternating orientation. For each cluster variable in reduced form f/m, there is an indecomposable module M, such that $\epsilon(m) = \underline{dim} M$.*

Theorem 2.1.3 (Caldero-Chapotlon-Schiffler). *Let Q be any Dynkin diagram. For each cluster variable in reduced form f/m, there is an indecomposable module M, such that $\epsilon(m) = \underline{dim} M$.* (Also [RT] and [Chapotlon, Keller] have the same result.)

2.2. **Positivity condition and Condition ($*$).** These are essential notions for the proofs of the existence of the maps α, $\bar{\alpha}$, $\tilde{\alpha}$, i.e. that the objects $\alpha(x)$, $\bar{\alpha}(\underline{x'})$, $\tilde{\alpha}((\underline{x'}, Q'))$ satisfy the desired conditions.

Definition 2.2.1. A polynomial f in variables $\{x_1, \dots, x_n\}$ is said to satisfy the *positivity condition* if $f(e_i) > 0$ for all $e_i = (1, 1, \dots, 1, 0, 1, \dots, 1, 1)$, 0 is at the i-th place.

Remark: If f satisfies the positivity condition, it does not have any non-constant monomial factors.

Definition 2.2.2. A cluster variable x is said to satisfy *condition* ($*$) if: either $x = f/m$ or $x = f x_i$, where f satisfies positivity condition and if $x = f/m$, then there exists an indecomposable exceptional module M, such that $\epsilon(m) = \underline{dim} M$.

Definition 2.2.3. If a cluster variable x satisfies condition ($*$), we (can) define:

$$\alpha(x) = \begin{cases} M & \text{if } x = f/m \text{ and } \epsilon(m) = \underline{dim} M \\ P_i[1] & \text{if } x = f x_i. \end{cases}$$

Remark:

- We want to define α on all cluster variables.
- So far, we defined only on cluster variables satisfying ($*$).
- So, we prove the following theorem.

Theorem 2.2.4. *(BMRT) All cluster variables satisfy condition ($*$), if the cluster algebra is acyclic with no coefficients.*

Proof. This theorem will be proved after the Proposition 2.3.1 and Corollary 2.3.3, however the proof involves clusters and seeds as well, corresponding conditions on those: ($\bar{*}$),($\tilde{*}$), and we prove a more general theorem about cluster variables, clusters and cluster seeds. □

2.3. **Properties: ($*$), ($\bar{*}$), ($\tilde{*}$).** Definitions and properties.
($*$) is a property of a cluster variable x'_i consisting of two parts:

- $x'_i = f/m$ or $x'_i = f x_i$, where f satisfies positivity condition, and
- $\epsilon(m) = \underline{dim} M$, for some indecomposable exceptional module M.
- **Notice:** If x'_i satisfies ($*$), then $\alpha(x'_i)$ is defined (i.e. it is an indecomposable exceptional object).

($\bar{*}$) is a property of a cluster $\underline{x'} = \{x'_1, \dots, x'_n\}$ consisting of two parts:

- each cluster variable x'_i satisfies $(*)$, and
- $\alpha(x'_1)\coprod\cdots\coprod\alpha(x'_n)$ is a tilting object.
- **Notice:** If $\underline{x}'$ satisfies $(\bar{*})$, then $\bar{\alpha}(\underline{x}')$ can be defined as $\bar{\alpha}(\underline{x}')=\alpha(x'_1)\coprod\cdots\coprod\alpha(x'_n)$, which is a tilting object.

$(\tilde{*})$ is a property of a cluster seed $(\underline{x}',Q')$ consisting of two parts:

- the cluster $\underline{x}'$ satisfies $(\bar{*})$, and
- $Q_{(\bar{\alpha}(\underline{x}'))}=Q'$.
- **Notice:** If $(\underline{x}',Q')$ satisfies $(\tilde{*})$, then $\tilde{\alpha}(\underline{x}',Q')$ can be defined as $\tilde{\alpha}(\underline{x}',Q')=(\bar{\alpha}(\underline{x}'),Q')$ which is a tilting seed.

Proposition 2.3.1. *Let $(\underline{x}',Q')$ be a cluster seed satisfying $(\tilde{*})$. Consider a cluster mutation. Then:*

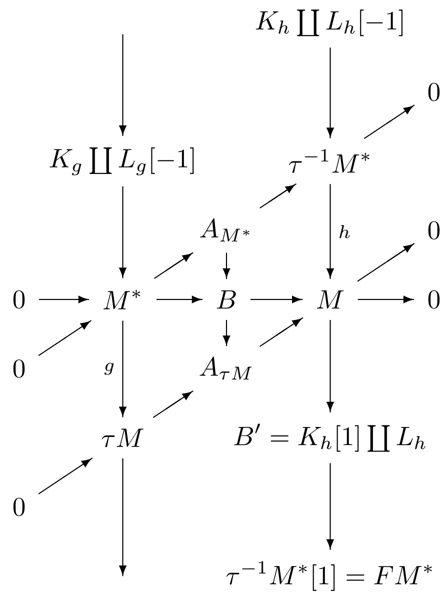
a: *The new cluster variable satisfies $(*)$.*
$\bar{a}$: *The new cluster satisfies $(\bar{*})$.*
$\tilde{a}$: *The new cluster seed satisfies $(\tilde{*})$.*

Proof. We need to set up a bit more detailed and precise notation.
Let $(\underline{x}',Q')$ be a cluster seed, where $\underline{x'}=\{x'_1,\dots,x'_n\}$ is a cluster.
Let μ a cluster mutation at x'_i.
Since x'_i satisfies $(*)$, there exists an object M such that $\alpha(x'_i)=M$.
Denote x'_i by x_M.
Let $(x_M)^*$ be the new cluster variable defined via cluster mutation μ.
Then $\mu(\underline{x}')=\mu(\{x'_1,\dots,x'_i\dots,x'_n\})=\mu(\{x'_1,\dots,x_M\dots,x'_n\})=\{x'_1,\dots,(x_M)^*\dots,x'_n\}$.

<u>Proof of *a:* WTS $(x_M)^*$ satisfies $(*)$.</u>
Let $T=T_1\coprod\cdots\coprod T_n=\alpha(x'_1)\coprod\cdots\coprod\alpha(x'_n)$.
It is a tilting object since $\underline{x}'$ satisfies $(\bar{*})$.
Consider tilting mutation of T at M, i.e. exchange M with another object M^*.
Then $T'=T_1\coprod\dots\coprod M^*\coprod\dots\coprod T_n$ is the new tilting object.
Let $M^*\to B\to M$ and $M\to B'\to M^*$ be the exchange triangles.
Notation and results from ([BMR2], 6.2) imply:
$x_M(x_M)^*=x_B+x_{B'}=\prod x_{B_i}+\prod x_{B'_i}$ if $B=\coprod B_i$ and $B'=\coprod B_i$.
At this point we brake the proof into several parts, depending on whether the representatives of M and M^* are modules or shifted projectives.

Case I: Both M and M^* are modules.

Step 1: $0\to M^*\to B\to M\to 0$ is an exact sequence of modules.
Step 2: Describe B', i.e. representatives of the summands of B' in the fundamental domain add(ind$\Lambda\cup\{P_i[1]\}_{i=1}^n$). Consider the following commutative diagram, together with the AR-triangles as mentioned at the beginning:



$\dim \mathrm{Ext}(M, M^*) = 1$ implies $\dim \mathrm{Hom}(\tau^{-1}M^*, M) = 1$ and $\tau^{-1}g = h$
$\tau^{-1}(K_g \coprod L_g[-1])[1] \cong_{D^b} (K_h[1] \coprod L_h)$
Let $L_g = I \coprod Z$ with I injective and Z with no injective summands.
Let $K_h = P \coprod Y$ with P iprojective and Y with no projective summands.
$\tau^{-1}K_g[1] \coprod \tau^{-1}I \coprod \tau^{-1}Z \cong_{D^b} (L_h \coprod P[1] \coprod Y[1])$

By comparing shifts of nonprojectives, shifts of projectives and actual modules, get the following isomorphisms in the derived category:
$\tau^{-1}K_g[1] \cong_{D^b} (Y[1])$, $P(SocI)[1] = \tau^{-1}I \cong_{D^b} P[1]$, $\tau^{-1}Z \cong_{D^b} (L_h)$
$\tau^{-1}K_g[1] \cong_{D^b} (Y[1])$ implies $FK_g \cong_{D^b} (Y[1])$ and so $K_g \cong_C (Y[1])$
Finally: $B' \cong_C (L_h \coprod K_g \coprod P[1])$

Step 3: All x_M, x_B, $x_{B'}$ satisfy condition $(*)$ by assumption.
$x_M = (f_M/m_M)$, $x_B = (f_B/m_B)$, $x_{K_g} = (f_{K_g}/m_{K_g})$, $x_{L_h} = (f_{L_h}/m_{L_h})$,
$x_{P[1]} = (c_{(P/rP)}f_{P[1]})$ where $\epsilon(c_{(P/rP)}) = \underline{\dim}(P/rP)$
f_M, f_B, f_{K_g}, f_{L_h}, $f_{P[1]}$ satisfy positivity condition
$\epsilon(m_M) = \underline{\dim}M$, $\epsilon(m_B) = \underline{\dim}B$, $\epsilon(m_{K_g}) = \underline{\dim}(K_g)$, $\epsilon(m_{L_h}) = \underline{\dim}(L_h)$.

Step 4: $(m_{K_g}m_{L_h})|m_B$ or $(m_{K_g}m_{L_h}) = (m_B/m')$ Reason:
$\underline{\dim}(K_g) + \underline{\dim}(L_h) + \underline{\dim}(-) + \underline{\dim}(-) = \underline{\dim}M^* + \underline{\dim}M = \underline{\dim}B$

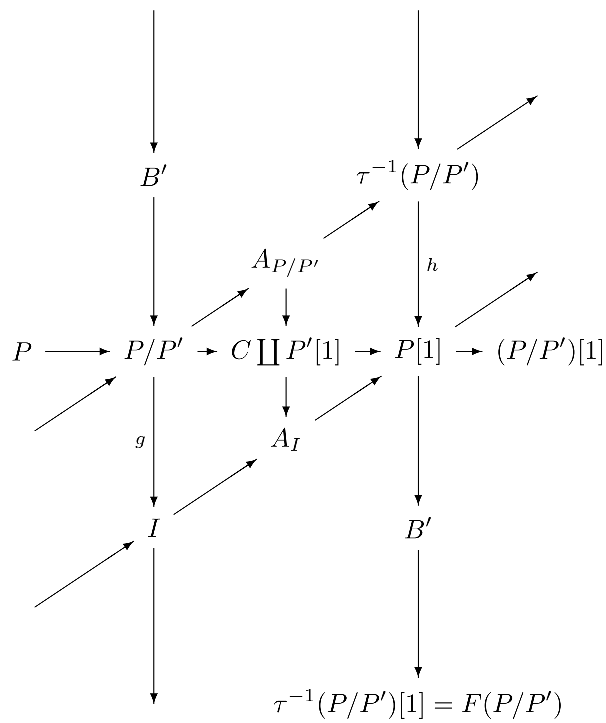
Step 5: Compute $(x_M)^*$.
$(x_M)^* = \frac{(x_B + x_{B'})}{(x_M)} = \frac{(f_B + f_{K_g} \cdot f_{L_h} \cdot m' \cdot c_{(P/rP)} \cdot f_{P[1]})(1/f_M)}{(m_B/m_M)}$

Step 6: $(x_M)^*$ satisfies condition $(*)$. Reason:
f_M divides the rest of the numerator (Laurent phenomenon).
Numerator is a polynomial which satisfies positivity condition.
Numerator does not have any non-constant monomial factors.
Denominator is monomial m_B/m_M, with $\epsilon(m_B/m_M) = \underline{\dim}M^*$.
M^* is an indecomposable exceptional module.

Step 7: $\alpha((x_M)^*) = M^*$.

Case II: M is not a module, i.e. $M = P[1]$.
First, we point out that in this case M^* must be a module, since $\mathrm{Ext}(M, M^*) \neq 0$. We will just set up the exchange triangles and commutative diagrams, and the rest is similar kind of analysis, but somewhat more complicated.

$M = P[1]$ implies that $B = C \coprod P'[1]$, wth C non-projective and P' projective.
$M^*[1] \cong P/P'[1]$ since $\dim \mathrm{Hom}(M, M^*[1]) = 1$



Case III: M a module, $M^* = P[1]$. Similar to the previous case.

This finishes the proof of a : the new cluster variable $(x_M)^*$ satisfies condition $(*)$ and also $\alpha((x_M)^*) = M^*$.

Proof of $\bar{a}$: The new cluster $\{x_1', x_2', \ldots, (x_M)^*, \ldots, x_{n-1}', x_n'\}$ satisfies $(\bar{*})$.

- Each cluster variable $\{x_1', x_2', \ldots, (x_M)^*, \ldots, x_{n-1}', x_n'\}$ satisfies $(*)$ condition by assumption and part a.
- $\alpha(x_1') \coprod \alpha(x_2') \ldots \alpha((x_M)^*) \ldots \alpha(x_{n-1}') \coprod \alpha(x_n') = T_1 \coprod T_2 \ldots M^* \ldots T_{n-1} \coprod T_n$ is a tilting object.

Proof of $\tilde{a}$: The new cluster seed $(\{x_1', x_2', \ldots, (x_M)^*, \ldots, x_{n-1}', x_n'\}, Q")$ satisfies $(\tilde{*})$.

- The new cluster $\{x_1', x_2', \ldots, (x_M)^*, \ldots, x_{n-1}', x_n'\}$ satisfies $(\bar{*})$ by $\bar{a}$.
- The quiver of $\mathrm{End}(T_1 \coprod T_2 \ldots M^* \ldots T_{n-1} \coprod T_n)^{op}$ is equal to $Q"$ by [BMR].

This finishes the proof of the proposition. □

Corollary 2.3.2. *Let* $(\underline{x}', Q')$ *be a cluster seed satisfying condition* $(\tilde{*})$*. Let* $\tilde{\mu}$ *be a cluster mutation and* $\tilde{\nu}$ *the corresponding tilting mutation. Then:* $\tilde{\alpha} \circ \tilde{\mu} = \tilde{\nu} \circ \tilde{\alpha}$.

Proof. This follows from the Definitions 0.1.1 and 0.2.2 and a theorem from [BMR]. □

Corollary 2.3.3. *Every cluster seed is* $(\tilde{*})$*, every cluster is* $(\bar{*})$ *and every cluster variable is* $(*)$*.*

Proof. The initial seed $(\{x_1 \dots x_n\}, Q)$ satisfies $(\tilde{*})$ condition. Since every cluster seed can be reached by a finite number of cluster mutations, and by the proposition every cluster seed at each step is $(\tilde{*})$ it follows that all cluster seeds are $(\tilde{*})$. □

Proof. of the Theorem 1.2.1(a): By the above corollary it follows that everything satisfies appropriate $(*)$, $(\bar{*})$, $(\tilde{*})$, and by the comments in the definitions of $(*)$, $(\bar{*})$, $(\tilde{*})$, all three maps (α), $(\bar{\alpha})$, $(\tilde{\alpha})$ are defined. □

Proof. of the Theorem 1.2.1(b): We will show that $\tilde{\alpha}$ is onto.
Let (T, Q_T) be a tilting seed.
$\exists$ a finite number of tilting mutations from the initial tilting seed to (T, Q_T).
Consider the same sequence of cluster mutations from the initial cluster seed.
Use Corollary 2.3.2 □

3. Fomin-Zelevinsky Conjecture: Cluster determines Seed

3.1. The following was conjectured by Fomin and Zelevinsky (for any cluster algebra) and we prove it for the acyclic case with no coefficients.

Theorem 3.1.1 (BMRT)**.** *Let* $(\underline{x}', Q')$ *and* $(\underline{x}', Q")$ *be cluster seeds for an acyclic cluster algebra with no coefficients. Then* $Q' = Q"$*.*

Proof. $\tilde{\alpha}(\underline{x}', Q') = (\bar{\alpha}(\underline{x}'), Q')$ and $\tilde{\alpha}(\underline{x}', Q") = (\bar{\alpha}(\underline{x}'), Q")$ are tilting seeds. But, $Q' = Q"$ since tilting seed is determined by its tilting object. □

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